KTH Mathematics

Examination in SF2980 Risk Management, December 13, 2012, 8:00–13:00.

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Allowed technical aids and literature: a calculator, the book *Risk and portfolio analysis: principles and methods*, the printed errata.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow. Only an answer, without an explanation, will give 0 points. You are supposed to make clever use of the figures presented below. The effects of interest rates over short time periods (a few days) may be ignored. Statements that one is asked to show in the exercises in the book can be taken as facts, i.e. you can use these statements without having to verify them.

GOOD LUCK!

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Problem 1

The yearly credit losses of two business units of a bank are believed to be $\mathrm{LN}(0,1)$ and $\mathrm{LN}(0,4)$, respectively, where $\mathrm{LN}(\mu,\sigma^2)$ denotes the lognormal distribution with expected value $e^{\mu+\sigma^2/2}$. It is further believed that the linear correlation coefficient for the pair of yearly credit losses is 0.6. Is there a bivariate distribution for the pair of yearly credit losses satisfying these conditions? (10 p)

Problem 2

Let X and Y denote monthly log returns of two assets and suppose that X and Y are normally distributed with zero means and standard deviations 0.1 and 0.06, respectively. The vector (X,Y) has a Clayton copula and Kendall's tau correlation coefficient $\tau(X,Y) = 1/3$. An investor invests 10,000 dollars in each of the two assets. Let V_1^X and V_1^Y denote the values in one month of the positions in the two assets. Compute the probability $\mathrm{P}(V_1^X < 9,000, V_1^Y < 9,000)$. (10 p)

Problem 3

Let $X = 2 - Y$, where Y is standard Exponentially distributed, be the net value of a portfolio one day from today. Suppose that there is a sample of size 150 from the distribution of X and let $\widehat{\mathrm{VaR}}_{0.01}(X)$ be the empirical estimator of $\mathrm{VaR}_{0.01}(X)$ based on this sample. Compute the probability

$$\mathrm{P}\left(\widehat{\mathrm{VaR}}_{0.01}(X) > 2\,\mathrm{VaR}_{0.01}(X)\right).$$

(10 p)

Problem 4

A bank has issued a European put option with strike price 100 dollars on the value of one share of a non-dividend-paying asset in one year. The current spot price of the underlying asset is 100 dollars, the implied Black-Scholes volatility of the put option is 0.2 per year, and the current forward price for delivery of one share of the underlying asset in one year is $100e^{0.02}$. The daily log return of the underlying asset is assumed to be Student's t distributed with zero mean, standard deviation 0.02 and 3.9 degrees of freedom. The daily change in the option's implied volatility is assumed to be Student's t distributed with zero mean, standard deviation 0.02 and 5 degrees of freedom. The bank delta hedges the issued put option. Moreover, the bank is required to put an additional amount of cash aside, corresponding to $\mathrm{VaR}_{0.05}$ of the net value tomorrow of the delta hedge portfolio and the issued put option.

The Black-Scholes formula for the price of a European put option is given by

$$p(S,K,\sigma,r,T) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1),$$
$$d_1 = \frac{\log(S/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

where Φ denotes the standard normal distribution function. Moreover,

$$\frac{\partial}{\partial S}p(S,K,\sigma,r,T) = \Phi(d_1) - 1, \quad \frac{\partial}{\partial \sigma}p(S,K,\sigma,r,T) = S\phi(d_1)\sqrt{T},$$

where ϕ denotes the standard normal density function. It holds that $t_{3.9}^{-1}(0.95) \approx 2.14765$ and $t_5^{-1}(0.95) \approx 2.015048$, where t_ν is the distribution function of the standard Student's t distribution with ν degrees of freedom.

Estimate the amount of dollars held as cash for the delta hedge and the additional buffer capital. (10 p)

Problem 5

Consider the sample shown in Figure 1. The sample is of size 200 and simulated from a distribution function F with a regularly varying right tail. The 40 largest simulated values are:

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6.926 4.271 3.354 2.849 2.618 2.501 2.155 2.129 2.033 1.968 1.767
1.763 1.586 1.532 1.511 1.369 1.352 1.345 1.284 1.267 1.266 1.216
1.203 1.149 1.082 1.040 0.951 0.885 0.840 0.834 0.808 0.786 0.764
0.743 0.731 0.729 0.721 0.721 0.705 0.695
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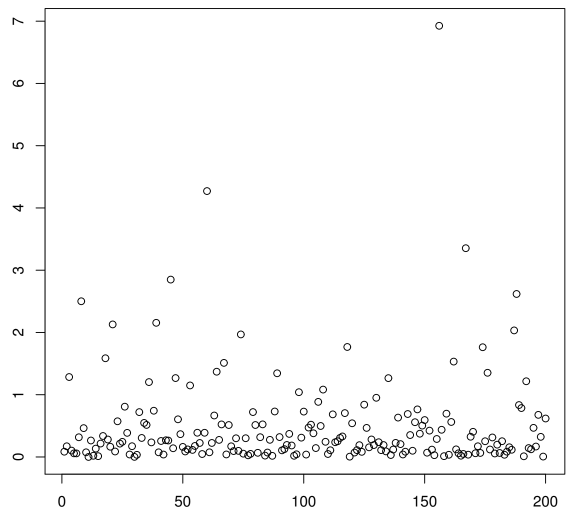
(a) Suggest an estimator of the tail index $\alpha > 0$. (5 p)

(b) Estimate $1 - F(5)$. (5 p)



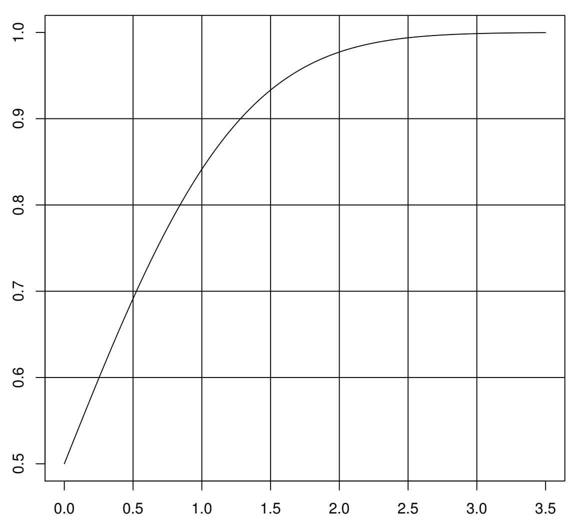


Figure 1: Upper plot: a simulated sample of size 200 from a distribution with a regularly varying right tail. Lower plot: the standard normal distribution function.

Problem 1

Comonotonicity yields the maximum linear correlation $\rho_{\max}$ and any positive correlation smaller or equal to $\rho_{\max}$ is achievable. The pair of comonotonic lognormal variables has the representation (e^Z, e^{2Z}), where Z is standard normal.

$$\begin{aligned}\operatorname{Cov}(e^Z, e^{2Z}) &= \mathrm{E}[e^{3Z}] - \mathrm{E}[e^Z]\,\mathrm{E}[e^{2Z}] = e^{9/2} - e^{1/2}e^{4/2} = e^{9/2} - e^{5/2}\\ \operatorname{Var}(e^Z) &= \mathrm{E}[e^{2Z}] - \mathrm{E}[e^Z]^2 = e^2 - e^1\\ \operatorname{Var}(e^{2Z}) &= \mathrm{E}[e^{4Z}] - \mathrm{E}[e^{2Z}]^2 = e^8 - e^4.\end{aligned}$$

Thus,

$$\operatorname{Cor}(e^Z, e^{2Z}) = \Big(e^{9/2} - e^{5/2}\Big)\Big/\Big((e^2 - e^1)(e^8 - e^4)\Big)^{1/2} \approx 0.6657553 > 0.6.$$

So the answer is: yes, there exists a bivariate distribution with linear correlation coefficient 0.6 and marginal distributions $\mathrm{LN}(0,1)$ and $\mathrm{LN}(0,4)$.

Problem 2

Notice that

$$\mathrm{P}(V_1^X \le x) = \mathrm{P}\left(V_0^X e^{\sigma_X Z} \le x\right) = \Phi\Big(\frac{1}{\sigma_X}\log\Big(\frac{x}{V_0^X}\Big)\Big),$$

where Z is standard normally distributed, $V_0^X = 10{,}000$, and $\sigma_X = 0.1$. Similarly for the distribution function of V_1^Y with $\sigma_Y = 0.06$. Since V_1^X and V_1^Y are strictly increasing and continuous functions of X and Y, respectively, it holds that (V_1^X, V_1^Y) has the same (Clayton) copula C as (X,Y). Since $1/3 = \tau(X,Y) = \theta/(2+\theta)$ we find that the Clayton copula parameter $\theta = 1$. Therefore

$$\begin{aligned}\mathrm{P}(V_1^X < 9{,}000, V_1^Y < 9{,}000) &= C(\mathrm{P}(V_1^X \le 9{,}000), \mathrm{P}(V_1^Y \le 9{,}000))\\ &= \Big(\Phi\Big(0.1^{-1}\log(0.9)\Big)^{-1} + \Phi\Big(0.06^{-1}\log(0.9)\Big)^{-1} - 1\Big)^{-1}\\ &\approx 0.032.\end{aligned}$$

Problem 3

Set $L = -X = Y - 2$.

$$\begin{aligned}\mathrm{P}\left(\widehat{\mathrm{VaR}}_p(X) > 2\,\mathrm{VaR}_p(X)\right) &= \mathrm{P}(F_{n,L}^{-1}(1-p) > 2F_L^{-1}(1-p))\\ &= 1 - \mathrm{P}(F_{n,L}^{-1}(1-p) \le 2F_L^{-1}(1-p))\\ &= 1 - \sum_{k=0}^{[np]}\binom{n}{k}(1 - F_L(2F_L^{-1}(1-p)))^k F_L(2F_L^{-1}(1-p))^{n-k}.\end{aligned}$$

It holds that $F_L(l) = 1 - e^{-(l+2)}$, $l > -2$, and $F_L^{-1}(u) = -2 - \log(1-u)$. In particular, $F_L(2F_L^{-1}(1-p)) = 1 - e^2p^2$ if $2F_L^{-1}(1-p) > -2$ and 0 otherwise. Here, with $n = 150$, $p = 0.01$ and $[np] = 1$:

$$\begin{aligned}&\mathrm{P}\left(\widehat{\mathrm{VaR}}_{0.01}(X) > 2\,\mathrm{VaR}_{0.01}(X)\right)\\ &= 1 - (1 - e^2\cdot 0.01^2)^{150} - 150(e^2\cdot 0.01^2)^1(1 - e^2\cdot 0.01^2)^{149}\\ &\approx 0.005674.\end{aligned}$$

Problem 4

The net value tomorrow of the delta hedge and the issued put option is

$$V_1 = P_0 - \frac{\partial P_0}{\partial S_0} S_0 + \frac{\partial P_0}{\partial S_0} S_1 - P_1,$$

where

$$P_1 \approx P_0 + \frac{\partial P_0}{\partial S_0}(S_1 - S_0) + \frac{\partial P_0}{\partial \sigma_0}(\sigma_1 - \sigma_0).$$

The net value tomorrow of the delta hedge and the issued put option is approximately

$$\begin{aligned} V_1 &\approx P_0 - \frac{\partial P_0}{\partial S_0} S_0 + \frac{\partial P_0}{\partial S_0} S_1 - P_0 - \frac{\partial P_0}{\partial S_0}(S_1 - S_0) - \frac{\partial P_0}{\partial \sigma_0}(\sigma_1 - \sigma_0) \\ &= -S_0 \phi(d_1)\sqrt{T}(\sigma_1 - \sigma_0). \end{aligned}$$

Here, $S_0 = K = 100$, $r_0 = 0.02$, $\sigma_0 = 0.2$, $T = 1$, which yield $d_1 = 0.04/0.2 = 0.2$ and $\phi(d_1) = 0.3910427$. Therefore, $V_1 \approx 39.10427 \cdot 0.02 \cdot \sqrt{0.6}W$ in distribution, where W has a standard Student's t distribution with 5 degrees of freedom. Thus, $\mathrm{VaR}_{0.05}(V_1) \approx 39.10427 \cdot 0.02 \cdot \sqrt{0.6} t_5^{-1}(0.95) \approx 1.220718$. The value of the cash position of the delta hedge is

$$\begin{aligned} P_0 - \frac{\partial P_0}{\partial S_0} S_0 &= 100(e^{-0.02}\Phi(0) - \Phi(-0.2)) - (\Phi(0.2) - 1)100 \\ &\approx 6.935905 + 42.07403 \approx 49.00993. \end{aligned}$$

The total value of the cash position is therefore 50.23065.

Problem 5

$$\begin{aligned} \mathrm{P}(X > u + y) &= \mathrm{P}(X > u + y \mid X > u)\,\mathrm{P}(X > u) = \frac{\mathrm{P}(X > u(1 + y/u))}{\mathrm{P}(X > u)}\,\mathrm{P}(X > u) \\ &\approx (1 + y/u)^{-\alpha}\overline{F}_n(u). \end{aligned}$$

Consider the GPD $H_\alpha(y) = 1 - (1 + y/u)^{-\alpha}$, $y > 0$, and its density $h_\alpha(y) = \alpha(1 + y/u)^{-\alpha-1}/u$. Given a sample $y_1, \dots, y_k$ of excesses above the threshold u, consider the loglikelihood function

$$\begin{aligned} \log L(y_1, \dots, y_k; \alpha) &= \sum_{j=1}^{k} \Big(\log \frac{\alpha}{u} - (\alpha + 1)\log(1 + y_j/u) \Big) \\ &= k\log\alpha - k\log u - (\alpha + 1)\sum_{j=1}^{k} \log(1 + y_j/u), \end{aligned}$$

Computing the derivative wrt α and equating to zero yields the estimator

$$\widehat{\alpha} = \Big(\frac{1}{k} \sum_{j=1}^{k} \log \frac{u + y_j}{u} \Big)^{-1}.$$

If $u = x_{k+1,n}$, then

$$\widehat{\alpha}(k) = \Big(\frac{1}{k}\sum_{j=1}^{k}\log\frac{x_{j,n}}{x_{k+1,n}}\Big)^{-1}.$$

Estimates for $k = 9, \dots, 39$ are

```
2.46 2.11 2.31 1.99 2.00 2.10 1.84 1.91 2.01 1.94 1.99 2.09 2.02 2.07 1.97 1.83
1.77 1.58 1.47 1.41 1.45 1.43 1.42 1.41 1.39 1.40 1.44 1.46 1.50 1.48 1.49
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An estimate of $\overline{F}(5)$ is therefore

$$\Big(\frac{5}{x_{k+1,n}}\Big)^{-\widehat{\alpha}(k)}\frac{k}{n}$$

Estimates for $k = 9, \dots, 39$, multiplied by 100, are

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0.46 0.56 0.50 0.61 0.61 0.57 0.69 0.65 0.61 0.64 0.62 0.56 0.60 0.58 0.64 0.73
0.78 0.94 1.06 1.13 1.09 1.11 1.12 1.14 1.16 1.15 1.10 1.08 1.02 1.04 1.03
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