KTH Mathematics

Examination in SF2980 Risk Management, June 3, 2013, 14:00–19:00.

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Allowed technical aids and literature: a calculator, the book *Risk and portfolio analysis: principles and methods*, the printed errata.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow. Only an answer, without an explanation, will give 0 points. You are supposed to make clever use of the figures presented below. The effects of interest rates over short time periods (a few days) may be ignored. Statements that one is asked to show in the exercises in the book can be taken as facts, i.e. you can use these statements without having to verify them.

GOOD LUCK!

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Problem 1

Figure 1 shows samples of size 300 from two distributions, A and B, say. Which of the p-quantiles $F_A^{-1}(p)$ and $F_B^{-1}(p)$ appears to be largest? Consider the values $p = 0.98$ and 0.999. (10 p)

Problem 2

Two future stock prices are believed to be $\mathrm{LN}(0, 1)$ and $\mathrm{LN}(0, 4)$, respectively, where $\mathrm{LN}(\mu, \sigma^2)$ denotes the lognormal distribution with expected value $e^{\mu+\sigma^2/2}$. It is further believed that the linear correlation coefficient for the pair of stock prices is -0.2. Is there a bivariate distribution for the pair of stock prices satisfying these conditions? (10 p)

Problem 3

(a) Let X and Y denote monthly log returns of two assets and suppose that X and Y are normally distributed with zero means and standard deviations 0.06 and 0.1, respectively. The vector (X, Y) has a Clayton copula and Kendall's tau correlation coefficient $\tau(X, Y) = 1/3$. An investor invests 10,000 dollars in each of the two assets. Let V_1^X and V_1^Y denote the values in one month of the positions in the two assets. Compute the (conditional) probability $\mathrm{P}(V_1^Y < a_Y \mid V_1^X < a_X)$, where a_X and a_Y are the 1%-quantiles of V_1^X and V_1^Y, respectively. (6 p)

(b) In what way do you believe that the value of the probability in (a) would change if (X, Y) had a Gaussian copula instead of a Clayton copula? (4 p)

Problem 4

(a) An insurer has a data set of 130 claims due to fire damages. The data are believed to be outcomes of independent Pareto distributed random variables with common distribution function $F(x) = 1 - x^{-\alpha}$ for $x > 1$. What is the probability that the empirical estimator takes a value that is more than 100% larger than the true, but unknown, 99%-quantile of the claim size distribution. (6 p)

(b) Give numerical answers for $\alpha = 1$ and $\alpha = 4$, and discuss the result. (4 p)

Problem 5

A bank has issued two European put options with strike price 100 dollars and one European call option with the same strike price on the value of one share of a non-dividend-paying asset in one year. The current spot price of the underlying asset is 100 dollars, the implied Black-Scholes volatility of the put option is 0.2 per year, and the current forward price for delivery of one share of the underlying asset in one year is $100e^{0.02}$. The daily log return of the underlying asset is assumed to be Student's t distributed with zero mean, standard deviation 0.015 and 6 degrees of freedom. The daily change in the option's implied volatility is assumed to be Student's t distributed with zero mean, standard deviation 0.02 and 5 degrees of freedom. The bank delta hedges the issued options. Moreover, the bank is required to put an additional amount of cash aside, corresponding to $\mathrm{VaR}_{0.05}$ of the net value tomorrow of the delta hedge portfolio and the issued options.

The Black-Scholes formula for the price of a European put option is given by

$$p(S, K, \sigma, r, T) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1),$$
$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

where Φ denotes the standard normal distribution function. Moreover,

$$\frac{\partial}{\partial S}p(S, K, \sigma, r, T) = \Phi(d_1) - 1, \quad \frac{\partial}{\partial \sigma}p(S, K, \sigma, r, T) = S\phi(d_1)\sqrt{T},$$

where ϕ denotes the standard normal density function. It holds that $t_6^{-1}(0.95) \approx 1.94$ and $t_5^{-1}(0.95) \approx 2.02$, where t_ν is the distribution function of the standard Student's t distribution with ν degrees of freedom.

Estimate the size of the buffer capital the bank is required to set aside. (10 p)

Hint: you may want to use the put-call parity

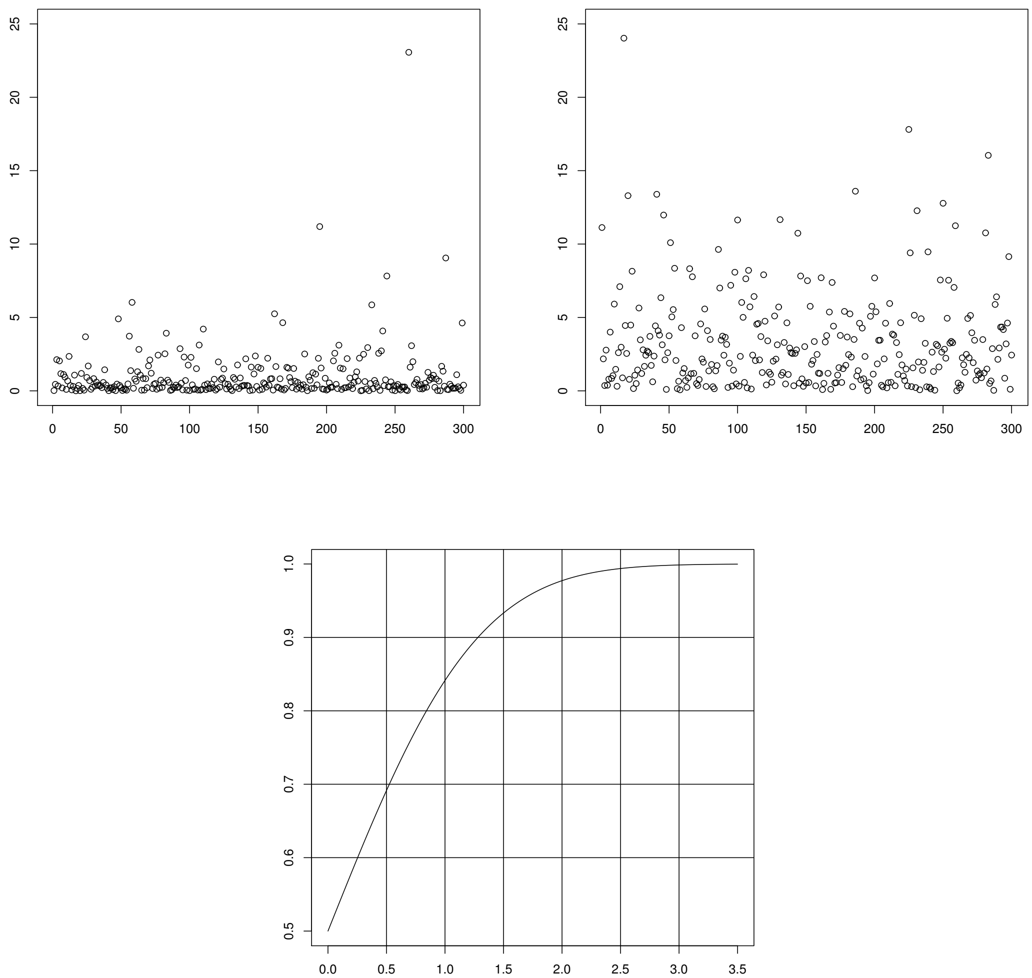

Figure 1: Upper plots: samples of size 300 from two distributions A (left) and B (right). Lower plot: the standard normal distribution function.

Problem 1

$F_{n,X}^{-1}(p) = X_{[n(1-p)]+1,n}$. Here, $[n(1-p)] + 1 = 7$ for $(n, p) = (300, 0.98)$ and $[n(1-p)] + 1 = 1$ for $(n, p) = (300, 0.999)$. The plots indicate that the empirical quantile is reasonably accurate for $(n, p) = (300, 0.98)$, making it plausible $F_A^{-1}(0.98) < F_B^{-1}(0.98)$, but far from accurate for $(n, p) = (300, 0.999)$. The plots indicate that A has a heavier right tail than B, and that quantile values are of similar size for $p \approx 0.99$. Therefore, it seems plausible that $F_A^{-1}(0.999) > F_B^{-1}(0.999)$.

Problem 2

Countermonotonicity yields the minimum linear correlation $\rho_{\min}$ and any negative correlation larger or equal to $\rho_{\min}$ is achievable. The pair of countermonotonic lognormal variables has the representation (e^{-Z}, e^{2Z}), where Z is standard normal.

$$\begin{aligned}
\operatorname{Cov}(e^{-Z}, e^{2Z}) &= \mathrm{E}[e^{Z}] - \mathrm{E}[e^{Z}]\,\mathrm{E}[e^{2Z}] = e^{1/2} - e^{1/2}e^{4/2} = e^{1/2} - e^{5/2} \\
\operatorname{Var}(e^{-Z}) &= \operatorname{Var}(e^{Z}) = \mathrm{E}[e^{2Z}] - \mathrm{E}[e^{Z}]^2 = e^2 - e^1 \\
\operatorname{Var}(e^{2Z}) &= \mathrm{E}[e^{4Z}] - \mathrm{E}[e^{2Z}]^2 = e^8 - e^4.
\end{aligned}$$

Thus,

$$\operatorname{Cor}(e^{-Z}, e^{2Z}) = \left(e^{1/2} - e^{5/2}\right) \Big/ \left((e^2 - e^1)(e^8 - e^4)\right)^{1/2} \approx -0.09 > -0.2.$$

So the answer is: no, there exists no bivariate distribution with linear correlation coefficient -0.2 and marginal distributions $\mathrm{LN}(0, 1)$ and $\mathrm{LN}(0, 4)$.

Problem 3

Notice that

$$\mathrm{P}(V_1^X \le x) = \mathrm{P}\left(V_0^X e^{\sigma_X Z} \le x\right) = \Phi\Big(\frac{1}{\sigma_X} \log\Big(\frac{x}{V_0^X}\Big)\Big),$$

where Z is standard normally distributed, $V_0^X = 10,000$, and $\sigma_X = 0.06$. Similarly for the distribution function of V_1^Y with $\sigma_Y = 0.1$. Since V_1^X and V_1^Y are strictly increasing and continuous functions of X and Y, respectively, it holds that (V_1^X, V_1^Y) has the same (Clayton) copula C as (X, Y). Since $1/3 = \tau(X, Y) = \theta/(2 + \theta)$ we find that the Clayton copula parameter $\theta = 1$. Therefore

$$\begin{aligned}
\mathrm{P}(V_1^X < a_X, V_1^Y < a_Y) &= C(0.01, 0.01) \\
&= (0.01^{-1} + 0.01^{-1} - 1)^{-1} \\
&= 1/199
\end{aligned}$$

so $\mathrm{P}(V_1^Y < a_Y \mid V_1^X < a_X) = 100/199 \approx 1/2$. A Gaussian copula does not have asymptotic dependence in the lower left tail so it is likely that the corresponding probability is smaller.

Problem 4

(a)

$$\begin{aligned}
\mathrm{P}(F_n^{-1}(p) > 2F^{-1}(p)) &= 1 - \mathrm{P}(F_n^{-1}(p) \le 2F^{-1}(p)) \\
&= 1 - \sum_{k=0}^{[n(1-p)]} \binom{n}{k} (1 - F(2F^{-1}(p)))^k F(2F^{-1}(p))^{n-k}.
\end{aligned}$$

Here, $F(2F^{-1}(p)) = 1 - (2F^{-1}(p))^{-\alpha} = 1 - 2^{-\alpha}(1-p)$, $n = 130$, $p = 0.99$ and $[np] = 1$:

$$\begin{aligned}\mathrm{P}(F_n^{-1}(p) > 2F^{-1}(p)) &= 1 - (1 - 2^{-\alpha}(1-p))^n - n2^{-\alpha}(1-p)(1 - 2^{-\alpha}(1-p))^{n-1} \\ &= 1 - (1 - 2^{-\alpha}(0.01))^{130} - 130 \cdot 2^{-\alpha}(0.01)(1 - 2^{-\alpha}(0.01))^{129}\end{aligned}$$

(b) For $\alpha = 1$ the probability is ≈ 0.138 and for $\alpha = 4$ the probability is ≈ 0.003. The heavier the right tail is, the less accurate is the empirical quantile estimator.

Problem 5

The net value tomorrow of the delta hedge and the issued put option is

$$V_1 = 2P_0 + C_0 + \Big(2\frac{\partial P_0}{\partial S_0} + \frac{\partial C_0}{\partial S_0}\Big)(S_1 - S_0) - 2P_1 - C_1,$$

where

$$P_1 \approx P_0 + \frac{\partial P_0}{\partial S_0}(S_1 - S_0) + \frac{\partial P_0}{\partial \sigma_0}(\sigma_1 - \sigma_0),$$

and similarly for C_1. The net value tomorrow of the delta hedge and the issued put option is approximately

$$\begin{aligned}V_1 &\approx 2P_0 + C_0 + \Big(2\frac{\partial P_0}{\partial S_0} + \frac{\partial C_0}{\partial S_0}\Big)(S_1 - S_0) - (2P_0 + C_0) - \Big(2\frac{\partial P_0}{\partial S_0} + \frac{\partial C_0}{\partial S_0}\Big)(S_1 - S_0) \\ &\quad - \Big(2\frac{\partial P_0}{\partial \sigma_0} + \frac{\partial C_0}{\partial \sigma_0}\Big)(\sigma_1 - \sigma_0) \\ &= \{\text{put-call parity}\} \\ &= -3\frac{\partial P_0}{\partial \sigma_0}(\sigma_1 - \sigma_0) \\ &= -3S_0\phi(d_1)\sqrt{T}(\sigma_1 - \sigma_0).\end{aligned}$$

Here, $S_0 = K = 100$, $r_0 = 0.02$, $\sigma_0 = 0.2$, $T = 1$, which yield $d_1 = 0.04/0.2 = 0.2$ and $\phi(d_1) = 0.3910427$. Therefore, $V_1 \approx 3 \cdot 39.10427 \cdot 0.02 \cdot \sqrt{0.6}W$ in distribution, where W has a standard Student's t distribution with 5 degrees of freedom. Thus, $\mathrm{VaR}_{0.05}(V_1) \approx 3 \cdot 39.10427 \cdot 0.02 \cdot \sqrt{0.6}t_5^{-1}(0.95) \approx 3.66$ dollars.