SOLUTIONS TO EXAMINATION IN SF2980 RISK MANAGEMENT, 2014-03-10, 08:00–13:00.

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Allowed technical aids: Everything except computers and communication devices. All books, notes, and similar are allowed. A calculator is necessary.

Any notation introduced must be explained and defined. Assumptions must be clearly stated. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

From the qq-plot we see that the monthly log-return seems to follow a normal distribution with mean zero. The standard deviation can be estimated from the slope and is approximately $\sigma = 0.14$. Let us assume that the monthly log-returns, $Z_1, \dots, Z_{12}$ are *independent* $N(0, \sigma^2)$. Then $Y = Z_1 + \cdots + Z_{12}$ has $N(0, 12\sigma^2)$-distribution and

$$V_{12} - V_0 R_0 = V_0 e^Y - V_0 e^{0.03} = 100(e^Y - e^{0.03}).$$

In particular,

$$\mathrm{ES}_{0.01}(V_{12} - V_0 R_0) = \{\text{see p. 187}\} = 100\Big(1 - \frac{\Phi(\Phi^{-1}(0.01) - 0.14\sqrt{12})e^{0.14^2 \cdot 12/2}}{0.01 e^{0.03}}\Big) = 73.1$$

Problem 2

Kendall's tau can be estimated as

$$\widehat{\tau} = \frac{1}{6}(4 - 2) = \frac{1}{3},$$

because there are four concordant pairs and two discordant pairs. Since the distribution is elliptical

$$\widehat{\varrho} = \sin\left(\frac{\pi}{2}\widehat{\tau}\right) = 0.5.$$

Problem 3

Since

$$P(L_1 > x) = 1 - e^{-x^{-\frac{1}{2}}} = 1 - \sum_{k=0}^{\infty}(-1)^k \frac{x^{-\frac{k}{2}}}{k!} = x^{-\frac{1}{2}} + \sum_{k=2}^{\infty}(-1)^k \frac{x^{-\frac{k}{2}}}{k!},$$

and

$$\lim_{t\to\infty} t^{\frac{1}{2}} \sum_{k=2}^{\infty} (-1)^k \frac{(tx)^{-\frac{k}{2}}}{k!} = 0,$$

for any $x > 0$, it follows that

$$\begin{aligned}
\lim_{t\to\infty} \frac{P(L_1 > tx)}{P(L_1 > t)} &= \lim_{t\to\infty} \frac{(tx)^{-\frac{1}{2}} + \sum_{k=2}^{\infty}(-1)^k \frac{(tx)^{-\frac{k}{2}}}{k!}}{(t)^{-\frac{1}{2}} + \sum_{k=2}^{\infty}(-1)^k \frac{t^{-\frac{k}{2}}}{k!}} \\
&= \lim_{t\to\infty} \frac{x^{-\frac{1}{2}} + t^{\frac{1}{2}} \sum_{k=2}^{\infty}(-1)^k \frac{(tx)^{-\frac{k}{2}}}{k!}}{1 + t^{\frac{1}{2}} \sum_{k=2}^{\infty}(-1)^k \frac{t^{-\frac{k}{2}}}{k!}} \\
&= x^{-\frac{1}{2}}.
\end{aligned}$$

That is, L_1 has a regularly varying tail with index $-1/2$. In particular, the subexponential property is satisfied and (as in Example 8.15 p. 259)

$$\lim_{x\to\infty} \frac{P(L_1 + L_2 > x)}{P(2L_1 > x)} = 2^{1-\frac{1}{2}} = \sqrt{2} > 1.$$

That is, the risk for the "diversified portfolio" with $L_1 + L_2$ is higher!

Problem 4

Here you can use the calculations in Section 9.3.4. Betting on $X < Y$ implies that $h_2 < 0$. We can compute the change in value as

$$\Delta V = h_2 S_0 \phi(d_1^1)(X - Y),$$

where $d_1^1 = 0.25$. Since $X - Y$ has a normal distribution with mean $\mu = -0.02$ and variance

$$v^2 = 2.3 \cdot 10^{-3} + 3.0 \cdot 10^{-3} - 2\sqrt{2.5 \cdot 3.0} \cdot 0.9 \cdot 10^{-3} = 5.7 \cdot 10^{-4},$$

it follows that

$$\mathrm{VaR}_{0.05}(\Delta V) = h_2 \Big(- S_0 \phi(d_1^1)\mu + h_2 S_0 \phi(d_1^1) v \Phi^{-1}(0.95) \Big).$$

To pick h_2 so that $\mathrm{VaR}_{0.05}(\Delta V) = 10$ we find that

$$h_2 = 10 \Big(- S_0 \phi(d_1^1)\mu + h_2 S_0 \phi(d_1^1) v \Phi^{-1}(0.95) \Big)^{-1} = -13.4.$$

The entire portfolio is then given by

$$h_0 = 407.5,\ h_1 = 1.6, B; h_2 = -13.4,\ h_3 = -9.9.$$

Problem 5

Consider, for simplicity, $n = 2$, and compute $P(I_1 = 1, I_2 = 1)$:

$$\begin{aligned}
P(I_1 = 1, I_2 = 1) &= C(p,p) \\
&= P(\Psi(-\frac{\log V_1}{X}) \leq p, \Psi(-\frac{\log V_2}{X}) \leq p) \\
&= P(\frac{\log V_1}{X} \leq -\Psi^{-1}(p), \frac{\log V_2}{X} \leq -\Psi^{-1}(p)) \\
&= P(V_1 \leq e^{-X\Psi^{-1}(p)}, V_2 \leq e^{-X\Psi^{-1}(p)}) \\
&= E[P(V_1 \leq e^{-X\Psi^{-1}(p)}, V_2 \leq e^{-X\Psi^{-1}(p)} \mid X)] \\
&= E[P(V_1 \leq e^{-X\Psi^{-1}(p)} \mid X)^2] \\
&= E[(e^{-X\Psi^{-1}(p)})^2].
\end{aligned}$$

For a Bernoulli mixture model the corresponding calculation is

$$P(J_1 = 1, J_2 = 1) = E[P(J_1 = 1, J_2 = 1 \mid Z)] = E[P(J_1 = 1 \mid Z)^2] = E[Z^2].$$

We see that we must take $Z = e^{-X\Psi^{-1}(p)}$. The calculation for arbitrary n is similar. Since X has Gamma$(1/\theta, 1)$-distribution and $\Psi^{-1}(p) = p^{-\theta} - 1$ the distribution of Z is

$$P(Z \leq z) = P(X \geq -\frac{\log z}{\Psi^{-1}(p)}),$$

and the density of Z is

$$f_Z(z) = \frac{1}{z\Psi^{-1}(p)} f_X(\frac{-\log z}{\Psi^{-1}(p)}),$$

where f_X is the density X.