KTH Mathematics

Examination in SF2980 Risk Management, January 14, 2015, 14:00–19:00.

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Allowed technical aids and literature: a pocket calculator, any written material on paper (books, notes, articles, etc. but not written material on electronic devices).

Any notation introduced must be explained and defined. Arguments and computations must be clearly presented and detailed so they are easy to follow.

GOOD LUCK!

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Problem 1

Consider two models for the joint distribution of ten consecutive one-day log returns of an asset, producing the outputs $(X_1, \dots, X_{10})$ and $(Y_1, \dots, Y_{10})$, respectively. $(X_1, \dots, X_{10})$ has a spherical distribution. The components of $(Y_1, \dots, Y_{10})$ are independent, $Y_k \stackrel{\mathrm{d}}{=} X_k$ for all k, and $\mathrm{P}(Y_1 < x)$ is regularly varying at $-\infty$ with index -4. Based on the model for the one-day log returns $(Y_1, \dots, Y_{10})$, the estimated probability of the ten-day log return taking a value < -0.1 is 0.005. Estimate the corresponding probability for the ten-day log return based on the model for the one-day log returns $(X_1, \dots, X_{10})$. (10 p)

Problem 2

Consider a homogeneous loan portfolio consisting of $1,000$ loans, to distinct borrowers, each of size $1,000,000$ dollars. If a borrower defaults within the next year, then no interest payments are made and the lender loses $300,000$ dollars of the loan. If a borrower does not default, then the lender receives an aggregate yearly interest rate payment of $20,000$ dollars. Operating costs of the lender are not considered. The expected yearly net result of the loan portfolio is $10,400,000$ dollars and the standard deviation is $10,000,000$ dollars. Determine the correlation for a pair of default indicators. (10 p)

Problem 3

An actuary claims to have formulated a bivariate model for dependent claim sizes in two lines of business, such that the two claim sizes are log normally distributed and have a nonzero coefficient of upper tail dependence. An outcome from the model has the stochastic representation

$$\Big(\exp\Big\{5 + \Phi^{-1}\Big(E_1/(E_0 + E_1)\Big)\Big\}, \exp\Big\{2 + 2\Phi^{-1}\Big(E_2/(E_0 + E_2)\Big)\Big\}\Big),$$

where Φ is the standard normal distribution function and E_0, E_1, E_2 are independent and standard exponentially distributed. Verify the actuary's claim or show that it is false. (10 p)

Problem 4

Use historical simulation based on the sample $\{0.001, -0.023, -0.009, 0.017\}$ of one-day log returns to compute the probability that the spot price three days from today is below 95 dollars if the current spot price is 100 dollars. (10 p)

Problem 5

Consider a portfolio consisting of 100 European put options with strike price 100 dollars on the value of one share of a non-dividend-paying asset in one year. The current spot price of the underlying asset is 100 dollars, the implied Black-Scholes volatility of the put option is 0.2 per year, and the current forward price for delivery of one share of the underlying asset in one year is $100e^{0.02}$. Figure 1 shows a scatter plot of a sample of size 30 of pairs $(\log S_t - \log S_{t-1}, \sigma_t - \sigma_{t-1})$ of spot price log returns and changes in the option's implied volatility over one-day periods. The corresponding bivariate empirical distribution is assumed to be representative of co-movements in the spot price and implied volatility during the next trading day.

The Black-Scholes formula for the price of a European put option is given by

$$p(S, K, \sigma, r, T) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1),$$
$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

where Φ denotes the standard normal distribution function. Moreover,

$$\frac{\partial}{\partial S}p(S, K, \sigma, r, T) = \Phi(d_1) - 1, \quad \frac{\partial}{\partial \sigma}p(S, K, \sigma, r, T) = S\phi(d_1)\sqrt{T},$$

where ϕ denotes the standard normal density function. Values of the standard normal distribution function are tabulated in Table 1.

Estimate $\mathrm{VaR}_{0.05}(V_1 - V_0)$, where V_0 and V_1 denote the portfolio value at time 0 (now) and 1 (tomorrow), respectively. (10 p)

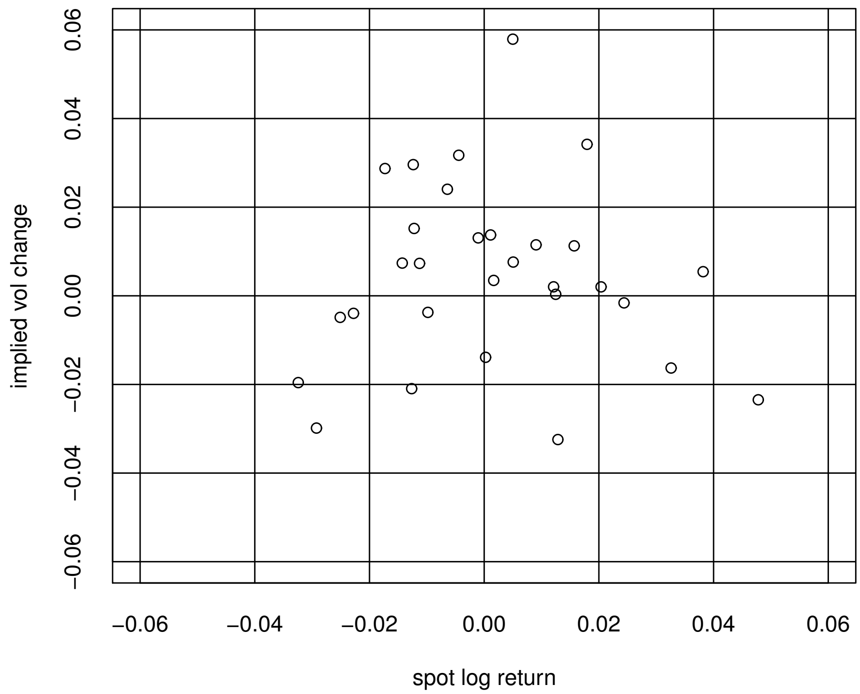

Figure 1: Pairs $(\log S_t - \log S_{t-1}, \sigma_t - \sigma_{t-1})$ of log returns of the underlying asset and changes in the option's implied volatility over one-day periods.

Table 1: The standard normal distribution function, $\Phi(x)$.

$x = 0.0$	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
$x = 0.1$	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
$x = 0.2$	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
$x = 0.3$	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
$x = 0.4$	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
$x = 0.5$	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
$x = 0.6$	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
$x = 0.7$	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
$x = 0.8$	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
$x = 0.9$	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
$x = 1.0$	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
$x = 1.1$	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
$x = 1.2$	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
$x = 1.3$	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
$x = 1.4$	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
$x = 1.5$	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
$x = 1.6$	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
$x = 1.7$	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
$x = 1.8$	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
$x = 1.9$	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Problem 1

Since $(X_1,\dots,X_{10})$ has a spherical distribution, $X_k \stackrel{d}{=} X_1$ for all k. Since $Y_k \stackrel{d}{=} X_k$ for all k, all Y_k and X_j are equally distributed. Using the fact that 0.005 is a small probability and the fact that the Y_ks are independent with a common regularly varying left tail,

$$0.005 \approx \mathrm{P}(Y_1+\dots+Y_{10} < -0.1) \approx 10\,\mathrm{P}(Y_1 < -0.1),$$

so $\mathrm{P}(Y_1 < -0.1) \approx 0.0005$. Using the fact that X has a spherical distribution,

$$\mathrm{P}(X_1+\dots+X_{10} < -0.1) = \mathrm{P}(10^{1/2}X_1 < -0.1).$$

Using the fact that $\mathrm{P}(X_1 < x)$ is regularly varying at $-\infty$ with index -4, and that $X_1 \stackrel{d}{=} Y_1$,

$$\mathrm{P}(10^{1/2}X_1 < -0.1) \approx (10^{-1/2})^{-4}\,\mathrm{P}(X_1 < -0.1) = 10^2\,\mathrm{P}(Y_1 < -0.1)$$

which gives $\mathrm{P}(X_1+\dots+X_{10} < -0.1) \approx 0.05$.

Problem 2

Let $n = 1000$, $c = 20,000$, $K = 1,000,000$, $\lambda = 300,000$. Let X be the net result in one year and let $N = I_1+\dots+I_n$ be the number of default. Set $p_1 = \mathrm{P}(I_1 = 1)$, $p_2 = \mathrm{P}(I_1 = I_2 = 1)$. Then

$$\mathrm{E}[N] = np_1, \quad \mathrm{E}[N^2] = np_1 + n(n-1)p_2, \quad \mathrm{var}(N) = np_1 + n(n-1)p_2 - n^2p_1^2$$

and

$$\begin{aligned} X &= (n-N)c - N\lambda K = nc - N(c+\lambda K), \\ \mathrm{E}[X] &= n(c - p(c+\lambda K)), \\ \mathrm{var}(X) &= \mathrm{var}(N)(c+\lambda K)^2 \end{aligned}$$

In particular,

$$\begin{aligned} p_1 &= \big(c - \mathrm{E}[X]/n\big)/\big(c+\lambda K\big) = 0.03, \\ p_2 &= \frac{1}{n(n-1)}\Big(\frac{\mathrm{var}(X)}{(c+\lambda K)^2} - np_1 + n^2p_1^2\Big) \approx 0.001848411 \end{aligned}$$

so

$$\mathrm{cor}(I_1, I_2) = \frac{p_2 - p_1^2}{p_1(1-p_1)} \approx 0.03259144.$$

Problem 3

Take $x \in (0,1)$, write $\mathrm{P}(E_1/(E_0+E_1) \le x) = \mathrm{E}[\mathrm{P}(E_1/(E_0+E_1) \le x \mid E_1)]$ and notice that

$$\mathrm{P}(E_1/(E_0+E_1) \le x \mid E_1) = \mathrm{P}(E_0 \ge E_1(1-x)/x \mid E_1) = e^{-E_1(1-x)/x}.$$

Hence,

$$\begin{aligned}\mathrm{P}(E_1/(E_0+E_1) \le x) &= \mathrm{E}[e^{-E_1(1-x)/x}] \\ &= \int_0^\infty e^{-z(1-x)/x} e^{-z} dz \\ &= \int_0^\infty e^{-z/x} dz \\ &= \big[-xe^{-z/x}\big]_0^\infty \\ &= x.\end{aligned}$$

Hence, $E_1/(E_0+E_1)$ and $E_2/(E_0+E_2)$ are uniformly distributed on $(0,1)$. If U is uniformly distributed on $(0,1)$, then $\mu+\sigma\Phi^{-1}(U)$ is normally distributed with mean μ and standard deviation σ (the quantile transform). Hence, the model produces log normally distributed claim sizes. To check upper tail dependence, we compute

$$\begin{aligned}&\mathrm{P}(E_1/(E_0+E_1) > x, E_2/(E_0+E_2) > x) \\ &\quad = \mathrm{E}[\mathrm{P}(E_1/(E_0+E_1) > x, E_2/(E_0+E_2) > x \mid E_0)] \\ &\quad = \mathrm{E}[\mathrm{P}(E_1/(E_0+E_1) > x \mid E_0)\,\mathrm{P}(E_2/(E_0+E_2) > x \mid E_0)].\end{aligned}$$

Notice that

$$\mathrm{P}(E_1/(E_0+E_1) > x \mid E_0) = \mathrm{P}(E_1 > E_0 x/(1-x) \mid E_0) = e^{-E_0 x/(1-x)}$$

so

$$\begin{aligned}&\mathrm{P}(E_1/(E_0+E_1) > x, E_2/(E_0+E_2) > x) \\ &\quad = \mathrm{E}[e^{-2E_0 x/(1-x)}] \\ &\quad = \int_0^\infty e^{-z(1+x)/(1-x)} dz \\ &\quad = \big[-(1-x)/(1+x)e^{-z(1+x)/(1-x)}\big]_0^\infty \\ &\quad = (1-x)/(1+x).\end{aligned}$$

Hence,

$$\frac{\mathrm{P}(E_1/(E_0+E_1) > x, E_2/(E_0+E_2) > x)}{\mathrm{P}(E_2/(E_0+E_2) > x)} = \frac{1-x}{(1+x)(1-x)} \to \frac{1}{2} > 0$$

as $x \uparrow 1$. The actuary's claim is verified.

Problem 4

Notice that $\exp\{-3 \cdot 0.023\} \approx 0.933$, $\exp\{-2 \cdot 0.023 - 0.009\} \approx 0.946$ and that any other, not necessarily distinct, three log returns from the original sample gives a three-day log return x with $\exp\{x\} > 0.95$. According to the historical simulation procedure, we draw independently with replacement three indexes I, J, K from the index set $\{1,2,3,4\}$ and form the three-day log return as the sum of the log returns corresponding to the drawn indexes. This procedure is repeated a large number m times. Only the four index triplets $(2,2,2), (3,2,2), (2,3,2), (2,2,3)$ produce a spot price three days from today less than SEK 95. There are in total $4^3 = 64$ equally likely index triplets. By the LLN,

$$\frac{1}{m}\sum_{l=1}^{m} I\{\exp\{z_{I_l} + z_{J_l} + z_{K_l}\} < 0.95\} \to \mathrm{E}[I\{\exp\{z_I + z_J + z_K\} < 0.95\}]$$
$$= \mathrm{P}(\exp\{z_I + z_J + z_K\} < 0.95) = \frac{4}{64} = \frac{1}{16}$$

with probability one as $m \to \infty$.

Problem 5

Write $(Z_1, Z_2) = (\log(S_1/S_0), \sigma_1 - \sigma_0)$. The one-day period is short and the Z_k-values are small, so linearisation is justified. Similarly to the analysis on page 296 in Section 9.3.3,

$$V_1 - V_0 \approx 100(\Phi(d_1) - 1)S_0 Z_1 + 100\phi(d_1)S_0 Z_2,$$

where $d_1 = (r_0 + \sigma_0^2/2)/\sigma_0 = 0.2$. Hence, $\Phi(d_1) \approx 0.5793$, $\Phi(d_1) - 1 \approx -0.4207$ and $\phi(0.2) \approx 0.3910$. Therefore, $V_1 - V_0 \approx -4207Z_1 + 3910Z_2$. The three smallest portfolio values are due to the (Z_1, Z_2)-pairs:

$$\begin{array}{lll} (0.047, -0.023) & \text{which gives} & V_1 - V_0 \approx -288 \\ (0.032, -0.018) & \text{which gives} & V_1 - V_0 \approx -205 \\ (0.014, -0.032) & \text{which gives} & V_1 - V_0 \approx -184. \end{array}$$

With $L = -X = -(V_1 - V_0)$, $\widehat{\mathrm{VaR}}_{0.05}(V_1 - V_0) = L_{[30\cdot 0.05]+1,30} = L_{2,30}$ so the estimate is $l_{2,30} \approx 205$.