KTH Mathematics

Examination in SF2980 Risk Management, April 7, 2015, 08:00–13:00.

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Allowed technical aids and literature: a pocket calculator, any written material on paper (books, notes, articles, etc. but not written material on electronic devices).

Any notation introduced must be explained and defined. Arguments and computations must be clearly presented and detailed so they are easy to follow.

Good luck!

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Problem 1

Consider a portfolio consisting of a long position in a non-dividend-paying stock whose 1-day log returns are believed to be independent and Student's t distributed with zero mean, standard deviation 0.01 and 4 degrees of freedom.

(a) Estimate the probability of the portfolio losing more than 10% of its value in 100 days. (5 p)

(b) Estimate the probability of the portfolio losing more than 10% of its value in 3 days. (5 p)

Problem 2

Consider a homogeneous loan portfolio consisting of $1,000$ loans, to distinct borrowers, each of size $1,000,000$ dollars. If a borrower defaults within the next year, then no interest payments are made and the lender loses $300,000$ dollars of the loan. If a borrower does not default, then the lender receives an aggregate yearly interest rate payment of $20,000$ dollars. Operating costs of the lender are not considered. The yearly net result X of the loan portfolio has expected value $10,400,000$ dollars and standard deviation $10,000,000$ dollars. Express X as a random variable of the form $a_0 + a_1 Y$, where Y takes values in $[0, 1]$, and suggest a model for Y. (10 p)

Problem 3

Consider an investment of 100 dollars in long positions in each one of two stocks. The current share prices are 5 and 20 dollars, respectively. The pair of 1-day log returns (X_1, X_2) has distribution function

$$F(x_1, x_2) = \Big(\Phi(100x_1)^{-2} + t_4(100x_2)^{-2} - 1\Big)^{-1/2}.$$

Determine the probability that, over the next day, the investment in the first stock looses more than 2 dollars and the investment in the second stock looses more than 4 dollars. (10 p)

Problem 4

An actuary studies the claim size distribution by making a q-q plot of recorded claim sizes against the quantiles of the standard exponential distribution, see Figure 1. The plot does not look linear but more like the graph of the function $x \mapsto c_0 e^{c_1 x}$.

Suggest a distribution function for the claim sizes. (10 p)

Problem 5

A company has a liability consisting of 100 European put options with strike price 100 dollars on the value of one share of a non-dividend-paying asset in one year. The current spot price of the underlying asset is 100 dollars, the implied Black-Scholes volatility of the put option is 0.2 per year, and the current forward price for delivery of one share of the underlying asset in one year is $100e^{0.02}$.
The company wants to hedge its liability today until tomorrow by forming a hedging portfolio made up of cash and a position in the underlying asset, with the aim of minimizing the variance of the hedging error X. The pair (Y, Z) of spot price log return and change in the option's implied volatility over a one-day period is assumed to have a bivariate Student's t distribution with 4 degrees of freedom and linear correlation coefficient -0.5. Both Y and Z are assumed to have zero mean and standard deviation 0.01. Estimate $\mathrm{VaR}_{0.05}(X)$. (10 p)

The Black-Scholes formula for the price of a European put option is given by

$$p(S, K, \sigma, r, T) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1),$$
$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

where Φ denotes the standard normal distribution function. Moreover,

$$\frac{\partial}{\partial S}p(S, K, \sigma, r, T) = \Phi(d_1) - 1, \quad \frac{\partial}{\partial \sigma}p(S, K, \sigma, r, T) = S\phi(d_1)\sqrt{T},$$

where ϕ denotes the standard normal density function.

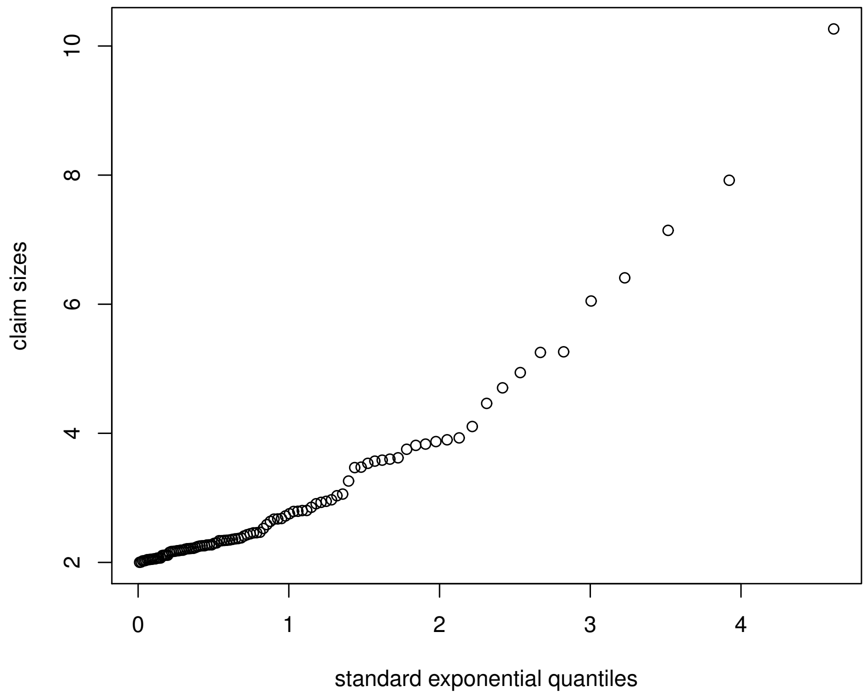

Figure 1: A q-q plot of 100 claim sizes against quantiles of the standard exponential distribution $F(x) = 1 - e^{-x}$.

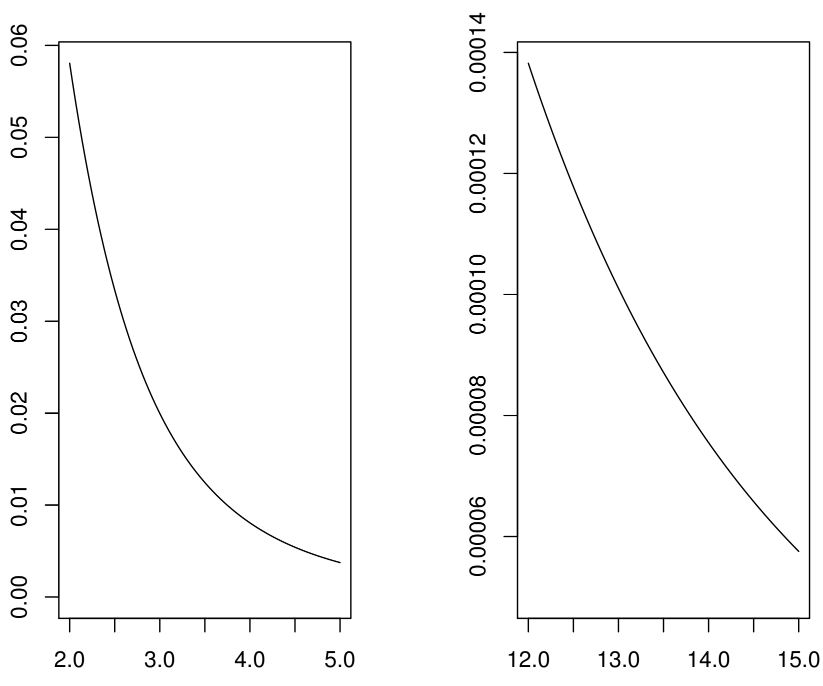

Figure 2: The graph of the function $x \mapsto 1 - t_4(x)$, where t_4 denotes the standard Student's t distribution with 4 degrees of freedom.

Table 1: The standard normal distribution function, $\Phi(x)$.

$x = 0.0$	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
$x = 0.1$	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
$x = 0.2$	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
$x = 0.3$	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
$x = 0.4$	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
$x = 0.5$	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
$x = 0.6$	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
$x = 0.7$	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
$x = 0.8$	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
$x = 0.9$	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
$x = 1.0$	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
$x = 1.1$	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
$x = 1.2$	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
$x = 1.3$	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
$x = 1.4$	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
$x = 1.5$	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
$x = 1.6$	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
$x = 1.7$	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
$x = 1.8$	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
$x = 1.9$	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
$x = 2.0$	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
$x = 2.1$	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
$x = 2.2$	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
$x = 2.3$	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916

Problem 1

The portfolio value in t days is

$$hS_t = hS_0 \frac{S_t}{S_0} = hS_0 \exp\Big\{ \sum_{j=1}^{t} Z_j \Big\},$$

where $Z_1, \dots, Z_t$ are t one-day log returns, and the event that the portfolio looses more than 10% of its value in t days can be expressed as

$$\sum_{j=1}^{t} Z_j < \log 0.9.$$

Notice that $Z_j \stackrel{\mathrm{d}}{=} Y/(100\sqrt{2})$, where Y has the standard Student's t distribution $t_4 = t_4(0,1)$ with zero mean and variance $\sqrt{2}$.
(a) The probability of a drop of at least $10/100\% = 0.1\%$ of the share price over a one-day period corresponds to the event $Y < \log(0.999)100\sqrt{2} \approx -0.14$. This is not a rare event. An approximation of the probability $\mathrm{P}(\sum_{j=1}^{100} Z_j < \log 0.9)$ using the CLT seems appropriate:

$$\mathrm{P}\Big(\sum_{j=1}^{100} Z_j < \log 0.9\Big) = \mathrm{P}\Big(\frac{\sum_{j=1}^{100} Z_j}{\sqrt{0.01}} < \frac{\log 0.9}{\sqrt{0.01}}\Big) \approx \Phi(10 \log 0.9) \approx \Phi(-1.05) \approx 0.15.$$

(b) The probability of a drop of $10/3\% \approx 3.33\%$ of the share price over a one-day period corresponds approximately to the event $Y < \log(0.967)100\sqrt{2} \approx -4.8$. This is a rare event. An approximation of the probability $\mathrm{P}(\sum_{j=1}^{3} Z_j < \log 0.9)$ using the asymptotics of tail probabilities for sums of subexponential random variables seems appropriate:

$$\begin{aligned}\mathrm{P}\Big(\sum_{j=1}^{3} Z_j < \log 0.9\Big) &\approx 3\,\mathrm{P}\Big(Z_1 < \log 0.9\Big) = 3\,\mathrm{P}(Y < \log(0.9)100\sqrt{2}) \\ &\approx 3t_4(-14.9) \approx 3 \cdot 5.9 \cdot 10^{-5} = 1.77 \cdot 10^{-4}.\end{aligned}$$

Problem 2

Let $n = 1000$, $c = 20,000$, $K = 1,000,000$, $\lambda = 0.3$. Let X be the net result in one year and let $N = I_1 + \cdots + I_n$ be the number of default. Set $p_1 = \mathrm{P}(I_1 = 1)$, $p_2 = \mathrm{P}(I_1 = I_2 = 1)$. Then

$$\mathrm{E}[N] = np_1, \quad \mathrm{E}[N^2] = np_1 + n(n-1)p_2, \quad \mathrm{var}(N) = np_1 + n(n-1)p_2 - n^2p_1^2$$

and

$$\begin{aligned} X &= (n-N)c - N\lambda K = nc - N(c + \lambda K), \\ \mathrm{E}[X] &= n(c - p(c + \lambda K)), \\ \mathrm{var}(X) &= \mathrm{var}(N)(c + \lambda K)^2 \end{aligned}$$

In particular,

$$p_1 = \bigl(c - \mathrm{E}[X]/n\bigr)/\bigl(c + \lambda K\bigr) = 0.03,$$
$$p_2 = \frac{1}{n(n-1)}\Bigl(\frac{\mathrm{var}(X)}{(c+\lambda K)^2} - np_1 + n^2 p_1^2\Bigr) \approx 0.001848411$$

and

$$\mathrm{cor}(I_1, I_2) = \frac{p_2 - p_1^2}{p_1(1-p_1)} \approx 0.03259144.$$

Set $Y = N/n$. The distribution of Y may be well approximated by a Beta distribution (see section in the book). For a Beta Mixture Model, $\mathrm{P}(I_k = 1 \mid Z) = Z$, where Z is Beta(a, b)-distributed. In particular,

$$p_1 = \mathrm{E}[\mathrm{P}(I_k = 1 \mid Z)] = \mathrm{E}[Z] = \frac{a}{a+b},$$
$$p_2 = \mathrm{E}[\mathrm{P}(I_j = I_k = 1 \mid Z)] = \mathrm{E}[Z^2] = \frac{a(a+1)}{(a+b)(a+b+1)}$$

which gives

$$b = \frac{(p_1 - p_2)(1-p_1)}{p_2 - p_1^2} \approx 28.79241,$$
$$a = b\frac{p_1}{1-p_1} \approx 0.890487.$$

Problem 3

The considered event is

$$(100e^{X_1}, 100e^{X_2}) < (98, 96), \quad \text{or equivalently } (X_1, X_2) < (\log 0.98, \log 0.96).$$

The probability of this event is

$$\begin{aligned}
&\Bigl(\Phi(100\log 0.98)^{-2} + t_4(100\log 0.96)^{-2} - 1\Bigr)^{-1/2} \\
&\approx \Bigl(\Phi(-2.02)^{-2} + t_4(-4.08)^{-2} - 1\Bigr)^{-1/2} \\
&\approx \Bigl((0.0217)^{-2} + (0.0075)^{-2} - 1\Bigr)^{-1/2} \\
&\approx 0.0071.
\end{aligned}$$

Problem 4

From the q-q plot we see that $F_n^{-1}(p) \approx h(F^{-1}(p))$, where $h(x) = c_0 e^{c_1 x}$ and $F^{-1}(p) = -\log(1-p)$ (the quantile function of standard exponential). Solving the equation

$$x = h(-\log(1-p)) = c_0(1-p)^{-c_1}$$

for p gives $p = 1 - (x/c_0)^{-1/c_1}$. From the q-q plot we find that $h(0) \approx 2$, suggesting $c_0 = 2$, and $h(2) \approx 4$, $h(3) \approx 6$, $h(4) \approx 8$, all suggesting $c_1 \approx 1/3$. These findings suggest that

$$F_n(x) \approx 1 - \left(\frac{x}{2}\right)^{-3}, \quad x > 2,$$

is a suitable model for the claim size distribution.

Problem 5

Write $(Y, Z) = (\log(S_1/S_0), \sigma_1 - \sigma_0)$. The one-day period is short and the Z_k-values are small, so linearisation is justified.

$$P_t \approx P_0 + \frac{\partial P_0}{\partial S_0}(S_t - S_0) + \frac{\partial P_0}{\partial \sigma_0}(\sigma_t - \sigma_0) \approx P_0 + \frac{\partial P_0}{\partial S_0} S_0 Y + \frac{\partial P_0}{\partial \sigma_0} Z.$$

where

$$\begin{aligned} d_1 &= (r_0 + \sigma_0^2/2)/\sigma_0 = 0.2, d_2 = 0, \\ \frac{\partial P_0}{\partial S_0} S_0 &= S_0(\Phi(d_1) - 1) \approx -42.07, \\ \frac{\partial P_0}{\partial \sigma_0} &= S_0 \phi(d_1)\sqrt{T} \approx 39.10. \end{aligned}$$

The portfolio weights (h_0, h_1) is determined by minimizing $\mathrm{E}[X^2]$, where X is the hedging error $X = h_0 + h_1 S_t - 100 P_t$. Since $\mathrm{Var}(S_t) \approx S_0^2 \,\mathrm{Var}(Y) = 100^2 100^{-2} = 1$,

$$\begin{aligned} h_1 &= 100 \frac{\mathrm{Cov}(S_t, P_t)}{\mathrm{Var}(S_t)} \\ &\approx 100 \,\mathrm{Cov}(S_0 Y, (\partial P_0/\partial S_0) S_0 Y + (\partial P_0/\partial \sigma_0) Z) \\ &= 100 S_0^2 (\partial P_0/\partial S_0) \,\mathrm{Var}(Y) + 100 S_0 (\partial P_0/\partial \sigma_0) \sqrt{\mathrm{Var}(Y)\,\mathrm{Var}(Z)}\,\mathrm{Cor}(Y, Z) \\ &= 100(\Phi(d_1) - 1) - 50\phi(d_1) \approx -61.6. \end{aligned}$$

Moreover, $P_0 \approx 6.935905$ so $h_0 \approx 100 P_0 - h_1 S_0 \approx 6856.207$.

$$\begin{aligned} X &= h_0 + h_1 S_t - P_t \\ &\approx h_0 + h_1 S_0(1 + Y) - 100\Big(P_0 + \frac{\partial P_0}{\partial S_0} S_0(1 + Y) + \frac{\partial P_0}{\partial \sigma_0} Z\Big) \\ &= -5000\phi(d_1)(Y + 2Z). \end{aligned}$$

Notice that $Y + 2Z = (\mathbf{a}^{\mathrm{T}} \Sigma \mathbf{a})^{1/2} V$, where V has distribution function t_4 and

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \Sigma = 100^{-2} 2^{-1} \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}.$$

$\mathbf{a}^{\mathrm{T}} \Sigma \mathbf{a} = 3/2 \cdot 100^{-2}$. Hence, $\mathrm{VaR}_{0.05}(X) \approx 50\sqrt{3/2}\phi(d_1) t_4^{-1}(0.95) \approx 51.1$ dollars.