KTH Matematik

EXAMINATION IN SF2980 RISK MANAGEMENT, 2016-01-13, 14:00–19:00.

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Allowed technical aids: Everything except computers and communication devices. All books, notes, old exams and similar are allowed. A calculator is necessary.

Any notation introduced must be explained and defined. Assumptions must be clearly stated. Arguments and computations must be detailed so that they are easy to follow.

Good luck!

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Problem 1

In Figure 1 (on the next page) the empirical distribution function of 20 insurance claims is plotted. The insurance claims are assumed to be independent and identically distributed. Construct an exact confidence interval, with confidence level as close as possible to 0.95, for the 0.80-quantile of the insurance claim distribution. State the confidence interval and the exact confidence level.
Hint: A table of the Binomial distribution is given at the end of the exam. (10 p)

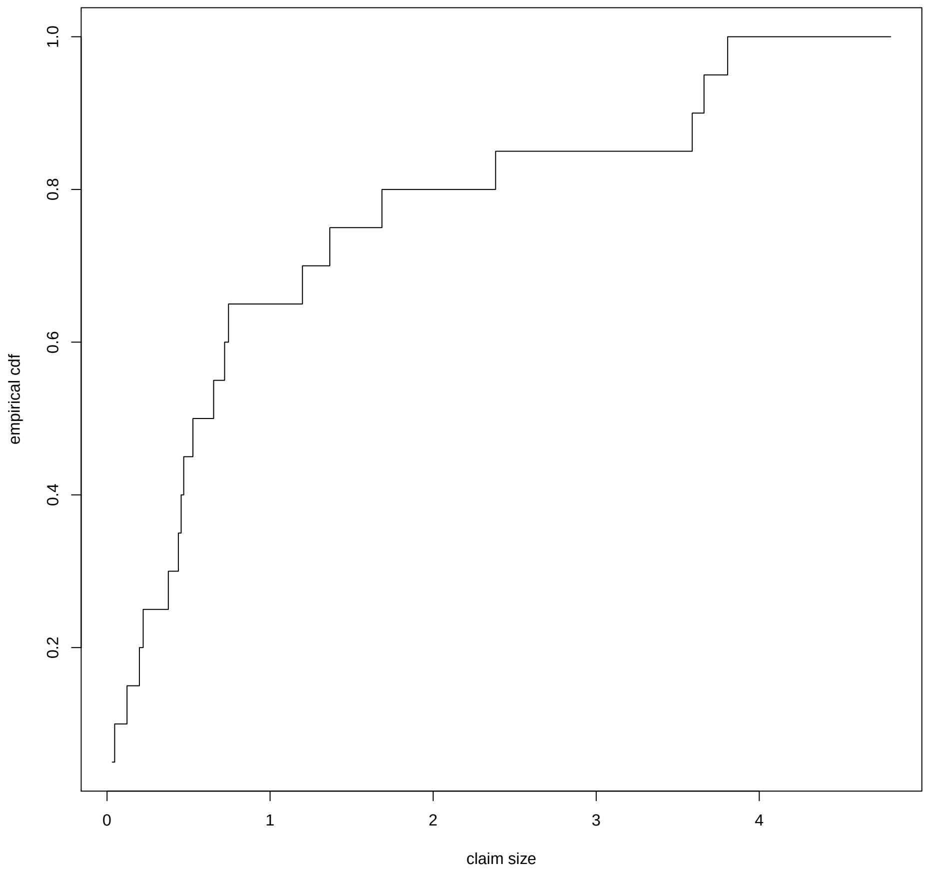
Problem 2

Let $(X_1, X_2)^{\mathrm{T}}$ have an unknown joint distribution but known marginal distributions. The marginal distributions of X_1 and X_2 are standard normal. The variables X_1 and X_2 represent the net-worth of two business lines within a company. Based on historical data the diversification effect is has been estimated. That is, we have estimated that

$$\text{Diversification effect} = \frac{\mathrm{VaR}_{0.005}(X_1 + X_2)}{\mathrm{VaR}_{0.005}(X_1) + \mathrm{VaR}_{0.005}(X_2)} = 0.75.$$

You propose to model the joint distribution of (X_1, X_2) as a bivariate normal distribution. Determine the correlation parameter of the bivariate normal distribution so that theoretical diversification effect in the proposed model equals the estimated diversification effect of 0.75.

Figure 1: Empirical distribution function of insurance claims



Problem 3

Consider the loan portfolio of an SMS loan provider consisting of $1,000$ loans, each of size $10,000$ SEK. Suppose that, for each of the loans, the probability of default within one year is 20%, and in case of default the bank makes a loss equal to 80% of the borrowed amount. Suppose further that the bank earns interest at a yearly rate of 30% on each loan that does not default. The bank decides to set aside an amount of buffer capital that equals its estimate of $\mathrm{VaR}_{0.01}(X)$, where X is the profit from interest income minus the loss from defaults over a one-year period. Estimate (approximately) the size of the buffer capital under the assumption that the default indicators follows a Beta-mixture model with default correlation $c = 0.1$. Express the estimate of the buffer capital in terms of a quantile of a Beta-distribution and state clearly the numerical values of the parameters of the Beta-distribution. Approximations must be well motivated.

Problem 4

Let (U_1, U_2) have an Archimedean copula with generator of the form

$$\Psi(t) = E[e^{-tX}] = -\frac{1}{\alpha}\log\Big(1 + (e^{-\alpha} - 1)e^{-t}\Big), \quad t > 0,$$

where $\alpha > 0$ is a parameter. Determine whether (U_1, U_2) has asymptotic dependence (tail dependence) in the lower left tail, or not. (10 p)

Problem 5

As a quantitative analyst at a bank you are investigating two different models for a vector $X = (X_1, \dots, X_d)$ of d log returns of financial asset prices. In the first model X is assumed to have a standard multivariate t-distribution with ν degrees of freedom ($\mu = 0$ and $\Sigma =$Identity). In the second model X is assumed to be a vector of independent and identically distributed random variables, with each X_i having a standard t-distribution with ν degrees of freedom (the same ν as in the first model). Determine which model that gives the highest extreme risk by determining the limit

$$\lim_{p\to 0}\frac{\mathrm{VaR}_p^{(1)}(X_1 + \cdots + X_d)}{\mathrm{VaR}_p^{(2)}(X_1 + \cdots + X_d)},$$

where $\mathrm{VaR}_p^{(1)}$ refers to the Value-at-Risk for the first model and $\mathrm{VaR}_p^{(2)}$ refers to the Value-at-Risk under the second model. Approximations must be well motivated.

HINT: You may first consider the limit $\lim_{x\to\infty}\frac{P^{(1)}(X_1+\cdots+X_d>x)}{P^{(2)}(X_1+\dots X_d>x)}$, where $P^{(1)}$ and $P^{(2)}$ refers to the probability under the first and second model, respectively. (10 p)

TABLE OF THE BINOMIAL C.D.F.

n	x	p	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
20	0		0.8179	0.6676	0.5438	0.4420	0.3585	0.2901	0.2342	0.1887	0.1516
	1		0.9831	0.9401	0.8802	0.8103	0.7358	0.6605	0.5869	0.5169	0.4516
	2		0.9990	0.9929	0.9790	0.9561	0.9245	0.8850	0.8390	0.7879	0.7334
	3		1.0000	0.9994	0.9973	0.9926	0.9841	0.9710	0.9529	0.9294	0.9007
	4		1.0000	1.0000	0.9997	0.9990	0.9974	0.9944	0.9893	0.9817	0.9710
	5		1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9981	0.9962	0.9932
	6		1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9994	0.9987
	7		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
	8		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n	x	p	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
20	0		0.1216	0.0388	0.0115	0.0032	0.0008	0.0001	0.0000	0.0000	0.0000
	1		0.3917	0.1756	0.0692	0.0243	0.0076	0.0021	0.0005	0.0001	0.0000
	2		0.6769	0.4049	0.2061	0.0913	0.0355	0.0121	0.0036	0.0009	0.0002
	3		0.8670	0.6477	0.4114	0.2252	0.1071	0.0444	0.0160	0.0049	0.0013
	4		0.9568	0.8298	0.6296	0.4148	0.2375	0.1182	0.0510	0.0189	0.0059
	5		0.9887	0.9327	0.8042	0.6172	0.4164	0.2454	0.1256	0.0553	0.0207
	6		0.9976	0.9781	0.9133	0.7858	0.6080	0.4166	0.2500	0.1299	0.0577
	7		0.9996	0.9941	0.9679	0.8982	0.7723	0.6010	0.4159	0.2520	0.1316
	8		0.9999	0.9987	0.9900	0.9591	0.8867	0.7624	0.5956	0.4143	0.2517
	9		1.0000	0.9998	0.9974	0.9861	0.9520	0.8782	0.7553	0.5914	0.4119
	10		1.0000	1.0000	0.9994	0.9961	0.9829	0.9468	0.8725	0.7507	0.5881
	11		1.0000	1.0000	0.9999	0.9991	0.9949	0.9804	0.9435	0.8692	0.7483
	12		1.0000	1.0000	1.0000	0.9998	0.9987	0.9940	0.9790	0.9420	0.8684
	13		1.0000	1.0000	1.0000	1.0000	0.9997	0.9985	0.9935	0.9786	0.9423
	14		1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9984	0.9936	0.9793
	15		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9985	0.9941
	16		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9987
	17		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	18		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Suggested solutions

Problem 1

Let F denote the true claim distribution and $Y_{F^{-1}(0.8)}$ be the number of claims exceeding the level $F^{-1}(0.8)$. Then $Y_{F^{-1}(0.8)}$ has Bin$(20, 0.2)$-distribution. We find that

$$P(X_{1,20} \leq F^{-1}(0.8)) = P(Y_{F^{-1}(0.8)} \leq 0) \approx 0.0115,$$
$$P(X_{8,20} \geq F^{-1}(0.8)) = 1 - P(Y_{F^{-1}(0.8)} \leq 7) \approx 0.0321.$$

Therefore, an exact confidence interval is $(X_{8,20}, X_{1,20}) = (0.74, 3.8)$ with confidence level $1 - (0.0115 + 0.0321) = 1 - 0.0436 = 0.9564$.

Problem 2

Let Z denote a random variable with standard normal distribution. Assuming that (X_1, X_2) has a joint normal distribution with linear correlation ρ, it follows that $X_1 + X_2 =^{\mathrm{d}} \sqrt{2+2\rho}Z$, and consequently

$$\mathrm{VaR}_{0.005}(X_1 + X_2) = F^{-1}_{-\sqrt{2+2\rho}Z/R_0}(0.995) = \frac{\sqrt{2+2\rho}}{R_0}\Phi^{-1}(0.995).$$

Since $X_1 =^{\mathrm{d}} X_2 =^{\mathrm{d}} Z$ we have $\mathrm{VaR}_{0.005}(X_1) = \mathrm{VaR}_{0.005}(X_2) = F^{-1}_{-Z/R_0}(0.995) = (1/R_0)\Phi^{-1}(0.995)$. It follows that

$$\frac{3}{4} = \frac{\mathrm{VaR}_{0.005}(X_1 + X_2)}{\mathrm{VaR}_{0.005}(X_1) + \mathrm{VaR}_{0.005}(X_2)} = \frac{\sqrt{2+2\rho}}{2}.$$

Solving for ρ gives $\rho = 1/8 = 0.125$.

Problem 3

The profit from interest minus the loss due to defaults over one year is

$$X = 10^4 \cdot (e^{0.3} - 1) \cdot (10^3 - N) - 10^4 \cdot 0.8 \cdot N = 10^7 \cdot (e^{0.3} - 1) - 10^4 \cdot (e^{0.3} - 1 + 0.8) \cdot N$$

where N is the number of defaults. In the Beta-mixture model we can do the approximation $N \approx 10^3 Z$, where $Z \sim$Beta(a, b) distributed with

$$a = \frac{1-c}{c}p = 1.8, \quad b = \frac{1-c}{c}(1-p) = 7.2.$$

Then we obtain

$$\mathrm{VaR}_{0.01}(X) \approx -10^7 \cdot (e^{0.3} - 1) + 10^7 \cdot (e^{0.3} - 1 + 0.8) \cdot F_Z^{-1}(0.99) = 2.98 \cdot 10^6.$$

On a separate note the cost for holding the buffer capital can be identified with the cost for borrowing the money for the buffer capital at a usual rate, say 0.03*, so the expected profit for the SMS loan provider is*

$$10^7 \cdot (e^{0.3} - 1) - 10^7 \cdot (e^{0.3} - 1 + 0.8) \approx 6.1 \cdot 10^5.$$

Problem 4

We have

$$\Psi^{-1}(u) = -\log\left[\frac{e^{-\alpha u}-1}{e^{-\alpha}-1}\right]$$

and consequently

$$\begin{aligned}\lambda_L &= \lim_{p\to 0}\frac{C(p,p)}{p}\\ &= \lim_{p\to 0}\frac{\Psi(2\Psi^{-1}(p))}{p}\\ &= \lim_{p\to 0} -\frac{1}{\alpha p}\log\left[1+\frac{(e^{-\alpha p}-1)^2}{e^{-\alpha}-1}\right]\end{aligned}$$

Using Taylor expansions $\log(1+x)\approx x$ and $e^{-\alpha p}\approx 1-\alpha p$ we find that the limit above is

$$\lim_{p\to 0} -\frac{1}{\alpha p}\frac{(e^{-\alpha p}-1)^2}{e^{-\alpha}-1} = \lim_{p\to 0} -\frac{1}{\alpha p}\frac{(-\alpha p)^2}{e^{-\alpha}-1} = 0.$$

Hence, there is no asymptotic dependence in the lower left tail.

Problem 5

Consider first

$$\lim_{x\to\infty}\frac{\mathrm{P}^{(1)}(X_1+\cdots+X_d>x)}{\mathrm{P}^{(2)}(X_1+\cdots+X_d>x)}.$$

Let Z have a standard t_ν distribution and denote its distribution function by F. Under model (1) we have $X_1+\cdots+X_d =^{\mathrm{d}} \sqrt{d}Z$ and it follows that

$$\mathrm{P}^{(1)}(X_1+\cdots+X_d>x) = \overline{F}(x/\sqrt{d}).$$

Under model (2), since $\overline{F}$ is regularly varying it is also subexponential and the subexponential property implies that

$$\mathrm{P}^{(1)}(X_1+\cdots+X_d>x) \sim d\overline{F}(x),$$

where $a(x)\sim b(x)$ means $\lim_{x\to\infty}\frac{a(x)}{b(x)}=1$. Consequently,

$$\lim_{x\to\infty}\frac{\mathrm{P}^{(1)}(X_1+\cdots+X_d>d)}{\mathrm{P}^{(2)}(X_1+\cdots+X_d>d)} = \lim_{x\to\infty}\frac{\overline{F}(x/\sqrt{d})}{d\overline{F}(x)} = d^{\frac{\nu}{2}-1}, \qquad (1)$$

where the last equality follows from the regular variation of $\overline{F}$. For $\nu>2$, $d^{\frac{\nu}{2}-1}>1$ which means that model (1) has higher probability of extreme values. The opposite conclusion holds for $\nu<2$.

For the ratio of Value-at-Risk, let F_1 and F_2 denote the distribution of $X_1+\cdots+X_d$ under model (1) and (2), respectively. Since the distribution of $X_1+\cdots+X_d$ is symmetric $F_i(-x)=\overline{F}_i(x)$ for $i=1,2$.

Suppose that $\nu > 2$. For x sufficiently large it follows from (1) that $\overline{F}_1(x) > \overline{F}_2(x)$. Therefore, for p sufficiently small

$$\begin{aligned}
\mathrm{VaR}_p^{(1)}(X_1 + \cdots + X_d) &= F^{-1}_{-(X_1+\cdots+X_d)}(1-p) \\
&= -F_1^{-1}(1-p) \\
&= \min\{x : F_1(x) \geq 1-p\} \\
&= \min\{x : \overline{F}_1(x) \geq p\} \\
&> \min\{x : \overline{F}_2(x) \geq p\} \\
&= \mathrm{VaR}_p^{(2)}(X_1 + \cdots + X_d).
\end{aligned}$$

This shows that for $\nu > 2$

$$\lim_{p \to 0} \frac{\mathrm{VaR}_p^{(1)}(X_1 + \cdots + X_d)}{\mathrm{VaR}_p^{(2)}(X_1 + \cdots + X_d)} > 1$$

so model (1) is more risky.
A similar argument shows that for $\nu < 2$, model (2) is more risky.