EXAMINATION IN SF2980 RISK MANAGEMENT, 2016-03-14, 08:00–13:00.

Examiner: Henrik Hult, tel. 790 6911, e-mail: hult@kth.se

Allowed technical aids: Everything except computers and communication devices. All books, notes, old exams and similar are allowed. A calculator is necessary.

Any notation introduced must be explained and defined. Assumptions must be clearly stated. Approximations must be well motivated. Arguments and computations must be detailed so that they are easy to follow.

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Good luck!

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Problem 1
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(a) Note that $Y = \#\{i : X_i > F^{-1}(0.90)\}$ has a $\text{Bin}(20, 0.10)$-distribution. Taking $a(X_1, \ldots, X_{20}) = X_{k:20}$, the kth order statistic, we find that

$$P(a(X_1, \ldots, X_{20}) < F^{-1}(0.9)) = P(X_{k:20} < F^{-1}(0.90)) = P(Y \leq k - 1).$$

Consulting the table for the Binomial distribution we find that $P(Y \leq 4) = 0.9568$ and we conclude that $k = 5$ is the appropriate choice.

(b) From the qqplot in Figure 1 we see that $X_{5:20} \approx 11$

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Problem 2
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We are given $10^6 = \text{VaR}_{0.005}(w^{\text{T}}X) = \sqrt{w^{\text{T}}\Sigma w}\Phi^{-1}(0.995)$, with Φ the cdf of the standard normal. A $t_\nu(0, R)$-distribution with dispersion matrix R has covariance $\frac{\nu}{\nu-2}R$. With $\nu = 4$ the dispersion matrix of the revised model is given by $R = \frac{1}{2}\Sigma$. Therefore the Value-at-Risk for the revised model is

$$\text{VaR}_{0.005}(w^{\text{T}}X) = \sqrt{w^{\text{T}}Rw}\,t_4^{-1}(0.995)$$

$$= \frac{1}{\sqrt{2}}\sqrt{w^{\text{T}}\Sigma w}\Phi^{-1}(0.995)\frac{t_4^{-1}(0.995)}{\Phi^{-1}(0.995)}$$

$$= \frac{1}{\sqrt{2}}10^6\frac{4.60}{2.58} = 1.26 \cdot 10^6.$$

Problem 3

(a) The upper bound is

$$P(X_1 > x, X_2 > x) \le \min\{P(X_1 > x), P(X_2 > x)\} = P(X_1 > x) = e^{-x},$$

where $\min\{P(X_1 > x), P(X_2 > x)\}$ is attained for a comonotone pair, i.e. $X_1 = X_2 = F^{-1}(U)$, with $U \sim \mathrm{Unif}(0, 1)$.

(b) The lower bound is

$$\begin{aligned}
P(X_1 > x, X_2 > x) &= 1 - P(X_1 \le x \text{ or } X_2 \le x) \\
&\ge 1 - P(X_1 \le x) - P(X_2 \le x) \\
&= P(X_1 > x) + P(X_2 > x) - 1,
\end{aligned}$$

where the bound is attained for a countermonotone pair: $(X_1, X_2) = (F^{-1}(U), F^{-1}(1-U))$. Indeed, for this choice

$$\begin{aligned}
P(X_1 > x, X_2 > x) &= P(U > F(x), 1 - U > F(x)) \\
&= P(F(x) < U < 1 - F(x)) \\
&= [1 - F(x) - F(x)] \vee 0 \\
&= [2(1 - F(x)) - 1] \vee 0 \\
&= [2e^{-x} - 1] \vee 0.
\end{aligned}$$

(c) For $x = F^{-1}(0.99)$ we have $e^{-x} = 1 - F(F^{-1}(0.99)) = 0.01$ and $p^*(x) = 0.01$, $p_*(x) = 0$.

Problem 4

Let $g(s) = \max[s - 20, 0]$ so that $Y = g(S)$. Since g is nondecreasing we have

$$F_Y^{-1}(0.995) = F_{g(S)}^{-1}(0.995) = g(F_S^{-1}(0.995)) = \max[F_S^{-1}(0.995) - 20, 0].$$

Since $\bar{F}(x) = \int_x^\infty f(z)dz = \int_x^\infty 2(1 + z)^{-3}dz = (1 + x)^{-2}$ we see that $\bar{F}$ is regularly varying:

$$\lim_{t \to \infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = \lim_{t \to \infty} \frac{(1 + tx)^{-2}}{(1 + t)^{-2}} = \lim_{t \to \infty} \left[\frac{\frac{1}{t} + x}{\frac{1}{t} + 1}\right]^{-2} = x^{-2}.$$

Since $\bar{F}$ is regularly varying, the subexponential approximation gives

$$P(S > s) \approx 12\bar{F}(s) = 12(1 + s)^{-2}$$

and for the quantile $F_S^{-1}(0.995) \approx (0.005/12)^{-1/2} - 1 = 48$. Consequently,

$$F_Y^{-1}(0.995) = \max[F_S^{-1}(0.995) - 20, 0] \approx 28.$$

There was a misprint in the original formulation, which stated that $f(x) = \frac{1}{2}(1 + x)^{-3}$. With this misspecified density the corresponding $\bar{F}(x) = \frac{1}{4}(1+x)^{-2}$, $F_S^{-1}(0.995) \approx (0.005/3)^{-1/2} - 1 = 23.5$ and $F_Y^{-1}(0.995) \approx 3.5$.

(b) Note first that

$$E[Y] = \int_0^\infty P(Y > y)dy = \int_0^\infty P(\max[S - 20, 0] > y)dy = \int_0^\infty P(S > 20 + y)dy.$$

With the subexponential approximation (which is valid because 21 can be considered quite large in view of (a)) we get

$$E[Y] \approx \int_0^\infty 12(1 + 20 + y)^{-2}dy = \ldots = \frac{12}{21}.$$

Problem 5

Let $X_i = F_i^{-1}(U)$. Then

$$
\begin{aligned}
C(u_1, \ldots, u_d) &= P(U_1 \le u_1, \ldots, U_d \le u_d) \\
&= P\left(\Psi_1\left(-\frac{\log V_1}{X_1}\right) \le u_1, \ldots, \Psi_d\left(-\frac{\log V_d}{X_d}\right) \le u_d\right) \\
&= P\left(V_1 \le e^{-X_1\Psi_1^{-1}(u_1)}, \ldots, V_d \le e^{-X_d\Psi_d^{-1}(u_d)}\right) \\
&= P\left(V_1 \le e^{-F_1^{-1}(U)\Psi_1^{-1}(u_1)}, \ldots, V_d \le e^{-F_1^{-1}(U)\Psi_d^{-1}(u_d)}\right) \\
&= E\left[P\left(V_1 \le e^{-F_1^{-1}(U)\Psi_1^{-1}(u_1)}, \ldots, V_d \le e^{-F_1^{-1}(U)\Psi_d^{-1}(u_d)} \mid U\right)\right] \\
&= E\left[\prod_i e^{-F_i^{-1}(U)\Psi_i^{-1}(u_i)}\right] \\
&= E\left[e^{-\sum_i F_i^{-1}(U)\Psi_i^{-1}(u_i)}\right].
\end{aligned}
$$

For the exponential distribution $F_i^{-1}(u) = -\frac{1}{\lambda_i}\log(1 - u)$, $\Psi_i(t) = \lambda_i/(t + \lambda_i)$ and $\Psi_i^{-1}(u_i) = \lambda_i(1/u_i - 1)$. Thus, the above expression for the copula reduces to

$$
\begin{aligned}
C(u_1, \ldots, u_d) &= E\left[e^{\log(1-U)\sum_i(1/u_i-1)}\right] \\
&= E[(1 - U)^{\sum_i(1/u_i-1)}] \\
&= \int_0^1 (1 - u)^{\sum_i(1/u_i-1)}du \\
&= \left(\sum_i(1/u_i - 1) + 1\right)^{-1}.
\end{aligned}
$$

Note that in this exponential setting, this actually reduces to a standard Archimedian copula, in fact Clayton with $\theta = 1$.

(b) The lower tail dependence is given by (with $d = 2$)

$$\lim_{u \to 0} \frac{C(u, u)}{u} = \lim_{u \to 0} \frac{1}{u(\frac{2}{u} - 1)} = \frac{1}{2}.$$