EXAMINATION IN SF2980 RISK MANAGEMENT, 2017-01-10, 14:00–19:00.

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Allowed technical aids: Everything except computers and communication devices. All books, notes, old exams and similar are allowed. A calculator is necessary.

Any notation introduced must be explained and defined. Assumptions must be clearly stated. Approximations must be well motivated. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

- Draw $U, V_1, \dots, V_d \sim \mathrm{Unif}(0,1)$ independently.
- Put $X_i = F_i^{-1}(U)$, $i = 1\dots, d$.
- Put $U_i = \Psi_i(-\frac{\log V_i}{X_i})$, $i = 1, \dots, d$.
- Put $Y_i = G_i^{-1}(U_i)$, $i = 1, \dots, d$.

Problem 2

Let us write $r_{1j} = r_{0j} + \sum_{k=1}^{p} \sqrt{\lambda_k} o_{kj} Z_k$, where $p = 3$. We have

$$\begin{aligned} V_1 - V_0 e^{r_{01}} &= c_1 + \sum_{j=1}^{n-1} c_{j+1} e^{-r_{1j} j} - V_0 e^{r_{01}} \\ &= c_1 + \sum_{j=1}^{n-1} c_{j+1} e^{-r_{0j} j} e^{-j \sum_{k=0}^{p} \sqrt{\lambda_k} o_{kj} Z_k} - V_0 e^{r_{01}}. \end{aligned}$$

Linearizing the (second) exponential function the last expression we arrive at the approximation

$$\begin{aligned} V_1 - V_0 e^{r_{01}} &\approx c_1 + \sum_{j=1}^{n-1} c_{j+1} e^{-r_{0j} j} (1 - j \sum_{k=0}^{p} \sqrt{\lambda_k} o_{kj} Z_k) - V_0 e^{r_{01}} \\ &= b - \mathbf{a}^T \mathbf{Z}, \end{aligned}$$

where $\mathbf{a}^T = (a_1, \dots, a_p)$ and

$$b = c_1 + \sum_{j=1}^{n-1} c_{j+1} e^{-r_{0j} j} - V_0 e^{r_{01}},$$

$$a_k = \sqrt{\lambda_k} \sum_{j=1}^{n-1} c_{j+1} e^{-r_{0j} j} j o_{kj}, \quad k = 1, \dots, p.$$

Note that, since $\mathbf{Z}$ has standard t_ν-distribution (which is spherical) $b - \mathbf{a}^T\mathbf{Z}$ has the same distribution as $b - |\mathbf{a}|Z_1$, and since Z_1 has a standard t_ν-distribution it has the same distribution as $-Z_1$, which implies that $b - |\mathbf{a}|Z_1 =^d b + |\mathbf{a}|Z_1$. We conclude that

$$\begin{aligned}\mathrm{VaR}_\alpha(V_1 - V_0 e^{r_{01}}) &= \mathrm{VaR}_\alpha(b + |\mathbf{a}|Z_1) \\ &= -be^{-r_{01}} + |\mathbf{a}| t_\nu^{-1}(1-\alpha).\end{aligned}$$

Problem 3

(a) Since $\mathrm{Var}(P_t) = \mathrm{Cov}(P_t, P_t) = \sigma^2 t(1-t)$ it follows that $P_t \sim N(0, \sigma^2 t(1-t))$. Therefore $F_{P_t}(0.95) = \sigma\sqrt{t(1-t)}\Phi^{-1}(0.95)$. This function is maximized in $t^* = 1/2$ and we find that

$$\mathrm{PFE} = \frac{\sigma}{2}\Phi^{-1}(0.95).$$

(b) Let Z be a standard normal random variable, independent of S_ν, and Y have a standard t_ν-distribution. Then

$$\begin{aligned}P_t &= \frac{\sigma\sqrt{\nu-2}}{\sqrt{S_\nu}} B_t = \sigma\frac{\sqrt{\nu-2}}{\sqrt{\nu}}\sqrt{\frac{\nu}{S_\nu}} B_t =^d \sigma\sqrt{t(1-t)}\frac{\sqrt{\nu-2}}{\sqrt{\nu}}\sqrt{\frac{\nu}{S_\nu}} Z \\ &=^d \sigma\sqrt{t(1-t)}\frac{\sqrt{\nu-2}}{\sqrt{\nu}} Y.\end{aligned}$$

It follows that

$$F_{P_t}^{-1}(0.95) = \sigma\sqrt{t(1-t)}\frac{\sqrt{\nu-2}}{\sqrt{\nu}} t_\nu^{-1}(0.95),$$

which, as in (a), is maximized at $t^* = 1/2$. Consequently

$$\mathrm{PFE} = \frac{\sigma}{2}\frac{\sqrt{\nu-2}}{\sqrt{\nu}} t_\nu^{-1}(0.95),$$

Problem 4

Since $\bar{F} = 1 - F$ is regularly varying with index α it follows that the distribution of X_1 and X_2 is subexponential. The subexponential property implies that

$$\lim_{x\to\infty} \frac{p_{ind}(x)}{\bar{F}(x)} = 2.$$

(a) In the comonotonic case, $X_1 = F^{-1}(U) = X_2$ and it follows from the regular variation property that

$$\lim_{x\to\infty} \frac{p_u(x)}{p_{ind}(x)} = \lim_{x\to\infty} \frac{P(2X_1 > x)}{\bar{F}(x)} \frac{\bar{F}(x)}{p_{ind}(x)} = \lim_{x\to\infty} \frac{\bar{F}(x/2)}{\bar{F}(x)} \frac{\bar{F}(x)}{p_{ind}(x)} = 2^\alpha \cdot \frac{1}{2} = 2^{\alpha-1}.$$

(b) In the countermonotonic case, $X_1 = F^{-1}(U)$ and $X_2 = F^{-1}(1-U)$, with $U \sim \text{Unif}(0,1)$. It follows that $X_1 + X_2 = F^{-1}(U) + F^{-1}(1-U)$ is large when U is close to 1 or close to 0. When U is close to 1, $F^{-1}(U)$ is close to 1, and similarly, when U is close to 0, $F^{-1}(1-U)$ is close to 1. Therefore

$$\begin{aligned}
\lim_{x\to\infty} \frac{p_l(x)}{\bar{F}(x)} &= \lim_{x\to\infty} \frac{P(F^{-1}(U) + F^{-1}(1-U) > x)}{\bar{F}(x)} \\
&\geq \lim_{x\to\infty} \frac{P(1 + F^{-1}(1-U) > x) + P(F^{-1}(U) + 1 > x)}{\bar{F}(x)} \\
&= \lim_{x\to\infty} \frac{P(X_2 > x-1) + P(X_1 > x-1)}{\bar{F}(x)} \\
&= 2,
\end{aligned}$$

where the last equality follows from Proposition 8.1. We conclude that

$$\lim_{x\to\infty} \frac{p_l(x)}{p_{ind}(x)} \geq 1,$$

and since $p_l(x) \leq p_{ind}(x)$ the inequality must be an equality.

Problem 5

With $U_1 = F_1(X_1)$ and $U_2 = F_2(U_2)$ we can write the CQE as

$$\begin{aligned}
\text{CQE}(p) &= P(X_2 \leq F_2^{-1}(p) \mid X_1 \leq F_1^{-1}(p)) \\
&= \frac{P(X_1 \leq F_1^{-1}(p), X_2 \leq F_2^{-1}(p))}{P(X_1 > F_1^{-1}(p))} \\
&= \frac{P(U_1 \leq p, U_2 \leq p)}{p} \\
&= \frac{C(p,p)}{p},
\end{aligned}$$

where C is the copula of (X_1, X_2). Based on the sample $(F_1(x_1^{(i)}, F_2(x_2^{(i)}))$ we can estimate $C(p,p)$ empirically by

$$\frac{1}{100} \sum_{i=1}^{100} I\{F_1(x_1^{(i)}) \leq p, , F_2(x_2^{(i)}) \leq p\}.$$

The corresponding empirical estimator of CQE(p) is

$$\frac{1}{100p} \sum_{i=1}^{100} I\{F_1(x_1^{(i)}) \leq p, F_2(x_2^{(i)}) \leq p\}.$$

With $p = 0.1$ the Figure 1 (lower) gives 4 observations with $F_1(x_1^{(i)}) \leq 0.1$ and $F_2(x_2^{(i)}) \leq 0.1$ and consequently the estimate of CQE(0.1) is

$$\frac{4}{100 \cdot 0.1} = 0.4.$$