EXAMINATION IN SF2980 RISK MANAGEMENT, 2017-04-10, 08:00–13:00.

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Allowed technical aids: Everything except computers and communication devices. All books, notes, old exams and similar are allowed. A calculator is necessary.

Any notation introduced must be explained and defined. Assumptions must be clearly stated. Approximations must be well motivated. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

Use the stochastic representation $X = \mu + RAU$ where $R \geq 0$, $AA^T = \Sigma$ and $U = (U_1, \dots, U_d)^T$ has uniform distribution on the unit sphere. Then

$$\begin{aligned}(X-\mu)^T\Sigma^{-1}(X-\mu) &= R^2(AU)^T\Sigma^{-1}(AU)\\ &= R^2U^TA^T(AA^T)^{-1}AU\\ &= R^2U^TA^T(A^T)^{-1}A^{-1}AU\\ &= R^2U^TU\\ &= R^2.\end{aligned}$$

The algorithm can then be written as follows:

1. Draw $Z_1, \dots, Z_d$ independently from a $\mathrm{N}(0,1)$ distribution and put $Z = (Z_1, \dots, Z_d)$.
2. Put $U = Z/|Z|$. Then U has uniform distribution on the unit sphere in $\mathbb{R}^d$.
3. Compute the Cholesky decomposition A such that $AA^T = \Sigma$.
4. Sample S from a Gamma(a, b) distribution, independently of Z, and put $R = \sqrt{S}$.
5. Put $X = \mu + RAU$.

Problem 2

Let X_i be the future net worth of the ith period and $L_i = -X_i$ be the (discounted) loss. Then

$$\mathrm{VaR}_{0.05}(X_i) = F_{L_i}^{-1}(0.95)$$

and $N = \#\{i : L_i > F_{L_i}^{-1}(0.95)\}$ has a $\mathrm{Bin}(20, 0.05)$-distribution. From the binomial table we find that $P(N \le 3) \approx 0.98$. Consequently, we reject the company's model at the 98%-level if we observe $N \ge 4$ exceedances. From Table 1 we find 4 exceedances in periods 2, 10, 17, 18. Thus, the company's model is rejected.

Problem 3

Let X_i denote the ith claim amount and suppose $X_1, X_2, \dots$ are independent. The QQplot is close to linear with slope approximately $1/4$. Consequently, $\log X_i$ has approximately the same distribution as $(1/4)Y$ where Y has standard exponential distribution. That is

$$P(\log X_i > x) = P(Y > 4x) = e^{-4x}, \quad x > 0,$$

and

$$P(X_i > x) = P(\log X_i > \log x) = e^{-4 \log x} = x^{-4}, \quad x > 1.$$

That is, X_i has approximately a Pareto distribution with parameter $\alpha = 4$. In particular, the distribution of X_i is regularly varying and the subexponential property holds. Using the subexponential approximation for $S_{30} = X_1 + \cdots + X_{30}$ we find

$$P(S_{30} > x) \approx 30P(X_1 > x) = 30x^{-4},$$

for large x. With $x = 60$ we find $P(S_{30} > 60) \approx 2.3 \cdot 10^{-6}$.

Problem 4

Let $X = (X_1, \dots, X_d)^T$ have elliptical distribution with $\mu_i = 0$ and dispersion matrix Σ with $\Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}} = \mathrm{Corr}_{i,j}$. We can write $X = AY$ where $AA^T = \Sigma$ and $Y^T = (Y_1, \dots, Y_d)$ has a spherical distribution. Then, with $\mathbf{1}^T = (1, \dots, 1) \in \mathbb{R}^d$,

$$\begin{aligned}
\text{Basic SCR} &= \mathrm{VaR}_{0.005}(X_1 + \cdots + X_d) \\
&= \mathrm{VaR}_{0.005}(\mathbf{1}^T X) \\
&= \mathrm{VaR}_{0.005}(\sqrt{\mathbf{1}^T \Sigma \mathbf{1}} Y_1) \\
&= \sqrt{\mathbf{1}^T \Sigma \mathbf{1}} \mathrm{VaR}_{0.005}(Y_1) \\
&= \sqrt{\sum_{ij} \Sigma_{ij}} \mathrm{VaR}_{0.005}(Y_1) \\
&= \sqrt{\sum_{ij} \mathrm{Corr}_{i,j} \sqrt{\Sigma_{ii}\Sigma_{jj}}} \mathrm{VaR}_{0.005}(Y_1) \\
&= \sqrt{\sum_{ij} \mathrm{Corr}_{i,j} \sqrt{\Sigma_{ii}} \mathrm{VaR}_{0.005}(Y_1) \sqrt{\Sigma_{jj}} \mathrm{VaR}_{0.005}(Y_1)} \\
&= \sqrt{\sum_{ij} \mathrm{Corr}_{i,j} \mathrm{VaR}_{0.005}(\sqrt{\Sigma_{ii}} Y_i) \mathrm{VaR}_{0.005}(\sqrt{\Sigma_{ii}} Y_j)} \\
&= \sqrt{\sum_{ij} \mathrm{Corr}_{i,j} \mathrm{VaR}_{0.005}(X_i) \mathrm{VaR}_{0.005}(X_j)} \\
&= \sqrt{\sum_{ij} \mathrm{Corr}_{i,j} \mathrm{SCR}_i \mathrm{SCR}_j}.
\end{aligned}$$

Here we have used that $\mathbf{1}^T X =^d \sqrt{\mathbf{1}^T \Sigma \mathbf{1}} Y_1$ and that $Y_1, \dots, Y_d$ have the same distribution which implies that $\mathrm{VaR}_{0.005}(Y_1) = \cdots = \mathrm{VaR}_{0.005}(Y_d)$.

Problem 5

It is impossible! To see this suppose (U_1, U_2) are counter-monotonic and (U_2, U_3) are counter-monotonic. Then $U_1 = 1 - U_2$ and $U_3 = 1 - U_2$, which implies that it is necessary that $U_1 = U_3$ so they are comonotonic. That is, (U_1, U_3) cannot be counter-monotonic. .