KTH Matematik

EXAMINATION IN SF2980 RISK MANAGEMENT, 2018-01-15, 14:00–19:00.

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Allowed technical aids: Everything except computers and communication devices. All books, notes, old exams and similar are allowed. A calculator may be necessary.

Any notation introduced must be explained and defined. Assumptions must be clearly stated. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

With $n = 20$, $p = 0.15$ and $np = 3$, an empirical estimate of $\mathrm{ES}_{0.15}(X)$ is given by

$$\widehat{\mathrm{ES}} = \frac{1}{0.15}\Big(\frac{L_{1,20} + L_{3,20} + L_{3,20}}{20}\Big).$$

In Figure 1 we observe that

$$\begin{aligned} L_{1,20} &= -X_{20,20} \approx 670, \\ L_{2,20} &= -X_{19,20} \approx 450, \\ L_{3,20} &= -X_{18,20} \approx 300. \end{aligned}$$

This yields: $\widehat{\mathrm{ES}} = 473$.

Problem 2

(a) The surplus at the end of the kth period is

$$\begin{aligned} V_k - G_k &= \alpha(V_{k-1} - G_k)R_k + V_{k-1} - \alpha(V_{k-1} - G_k) - G_k \\ &= (V_{k-1} - G_k)(\alpha(R_k - 1)) + V_{k-1} - G_k \\ &= (V_{k-1} - G_k)(\alpha(R_k - 1) + 1). \end{aligned}$$

Then

$$\begin{aligned} P(V_k < G_k) &= P(V_{k-1} - G_k)(\alpha(R_k - 1) + 1) < 0) \\ &= P(\alpha(R_k - 1) + 1 < 0) \\ &= P(R_k < 1 - 1/\alpha) \\ &= P(\mu + \sigma Z_k < \log(1 - 1/\alpha)) \\ &= \Phi(\frac{\log(1 - 1/\alpha) - \mu}{\sigma}) \end{aligned}$$

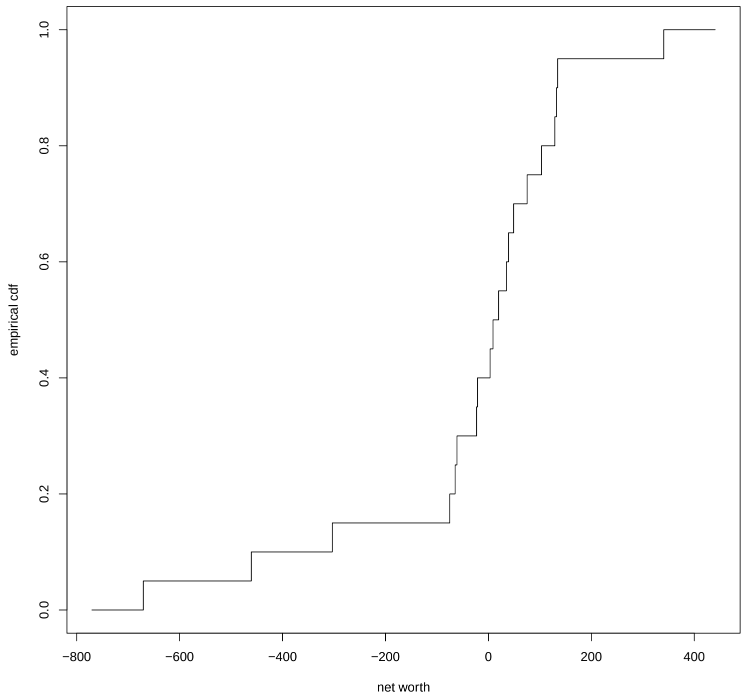

Figure 1: This figure relates to Problem 1. It gives the empirical distribution function of 20 independent observations of a net worth.

Since the probability is increasing in α we need to find α such that the above probability equals 0.01. That is

$$\Phi^{(-1)}(0.01) = \frac{\log(1 - 1/\alpha) - \mu}{\sigma}.$$

This gives

$$\alpha = \frac{1}{1 - \exp\{\mu + \sigma\Phi^{(-1)}(0.01)\}} = 2.94.$$

(b) From the calculation in (a) we have

$$\begin{aligned} \mathrm{ES}_{0.01}(V_k - G_k) &= (V_{k-1} - G_k)\mathrm{ES}_{0.01}(\alpha(R_k - 1) + 1) \\ &= (V_{k-1} - G_k)(\alpha \mathrm{ES}_{0.01}(R_k - 1) - 1). \end{aligned}$$

Since this expression is increasing, the maximum α is then the solution to

$$(V_{k-1} - G_k)(\alpha \mathrm{ES}_{0.01}(R_k - 1) - 1) = \beta(V_{k-1} - G_k).$$

Solving for α gives

$$\alpha = \frac{1 + \beta}{\mathrm{ES}_{0.01}(R_k - 1)}.$$

From Example 6.15, p. 187 we find that

$$\mathrm{ES}_{0.01}(R_k - 1) = \Big(1 - \frac{\Phi(\Phi^{-1}(0.01) - \sigma)e^{\mu + \sigma^2/2}}{0.01}\Big)$$

and consequently an explicit formula for α is

$$\alpha = \frac{1+\beta}{1 - \frac{\Phi(\Phi^{-1}(0.01)-\sigma)e^{\mu+\sigma^2/2}}{0.01}}.$$

Problem 3

From the qqplots we observe that X_1 has approximately a normal distribution whereas X_2 has a distribution with heavier tail than a normal. In particular the marginal distributions appear to be from different location-scale families. Therefore model class A is not appropriate.

The estimates of upper and lower tail dependence are very close to each other which is inconsistent with the properties of a Clayton copula. A Clayton copula has zero upper tail dependence, but positive lower tail dependence. Therefore model class C is not appropriate. Alternatively we could infer the parameter, θ, of the Clayton copula from the estimate of Kendall's tau:

$$\theta = 2\frac{\tau}{1-\tau} = 1.$$

With this value of θ the lower tail dependence must be $2^{-1/\theta} = 0.5$, which is inconsistent with the estimated value.

Only model class B remains. The qqplots indicate that the marginal distributions are consistent with model class B. Moreover, the estimates of upper and lower tail dependence are very close to each other which is consistent with the properties of the t-copula. A further confirmation would to check that the values of the tail dependence coefficients are consistent with the Kendall's tau estimates, but this involves more complicated numerical evaluation than is possible during the exam.

Problem 4

(a) Since $X = (X_1, \ldots, X_n)^T$ has an elliptical distribution we can represent it as $X = AY$ where $AA^T = \Sigma$ and $Y = (Y_1, \ldots, Y_n)^T$ has a spherical distribution. Then, from Proposition 9.3,

$$\mathrm{VaR}_{0.01}(X_1 + \cdots + X_n) = \mathrm{VaR}_{0.01}(1^T X) = \sqrt{1^T \Sigma 1}\,\mathrm{VaR}_{0.01}(Y_1) = \sqrt{1^T \Sigma 1} F_{Y_1}^{-1}(0.99).$$

Similarly

$$\mathrm{VaR}_p(X_i) = \mathrm{VaR}_{0.01}(e_i X) = \sqrt{e_i^T \Sigma e_i}\,\mathrm{VaR}_{0.01}(Y_1) = \sqrt{\Sigma_{ii}} F_{Y_1}^{-1}(1-p).$$

We must find p such that

$$\mathrm{VaR}_{0.01}(X_1 + \cdots + X_n) = \mathrm{VaR}_p(X_1) + \cdots + \mathrm{VaR}_p(X_n).$$

That is,

$$\sqrt{1^T \Sigma 1} F_{Y_1}^{-1}(0.99) = (\sqrt{\Sigma_{11}} + \cdots + \sqrt{\Sigma_{nn}}) F_{Y_1}^{-1}(1-p).$$

Solving for p gives

$$p = 1 - F_{Y_1}\Big(\frac{\sqrt{1^T \Sigma 1}}{\sqrt{\Sigma_{11}} + \cdots + \sqrt{\Sigma_{nn}}} F_{Y_1}^{-1}(0.99)\Big).$$

(b) Here Σ is the identity so $1^T \Sigma 1 = n = 60$ and $\Sigma_i i = 1$ which gives

$$p = 1 - \Phi(\Phi^{-1}(0.99)/\sqrt{60}) = 0.62$$

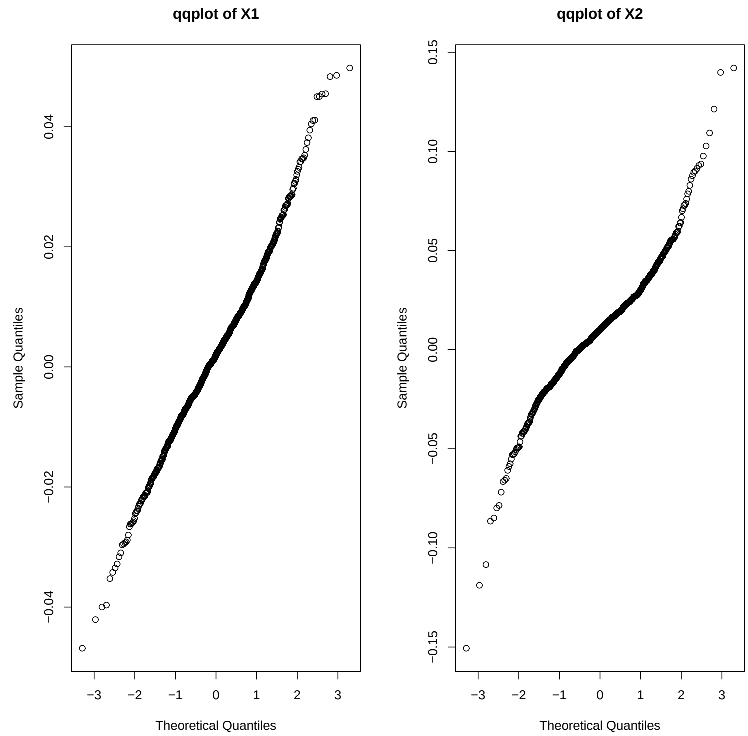

Figure 2: This figure relates to Problem 3. *Left:* quantile-quantile plot of the empirical distribution of a sample from X_1 with respect to a standard normal reference distribution. *Lower:* quantile-quantile plot of the empirical distribution of a sample from X_2 with respect to a standard normal reference distribution.

Problem 5

Let $S_1 = X_1 + \ldots X_7$, $S_2 = X_8 + \cdots + X_{14}, \ldots, S_{52} = X_{358} + \cdots + X_{364}$. Then, the sought probability is

$$\begin{aligned} P(\max\{S_1, \ldots, S_{52}\} > 100) &= 1 - P(\max\{S_1, \ldots, S_{52}\} \leq 100) \\ &= 1 - P(S_1 \leq 100)^{52} = 1 - (1 - P(S_1 > 100))^{52}. \end{aligned}$$

Note that $\bar{F}(x) = 1 - F(x) = (1 - x)^{-3}$ is regularly varying and hence the daily claim amounts has subexponential distribution. The subexponential approximation gives

$$P(S_1 > 100) = P(X_1 + \cdots + X_7 > 100) = 7P(X_1 > 100) = 7 \cdot 101^{-3}.$$

Finally, this gives

$$\begin{aligned} P(\max\{S_1, \ldots, S_{52}\} > 100) &= 1 - (1 - P(S_1 > 100))^{52} \\ &= 1 - (1 - 7 \cdot 101^{-3})^{52} \\ &\approx 52 \cdot 7 \cdot 101^{-3} \\ &= 3.5 \cdot 10^{-4}. \end{aligned}$$