SOLUTIONS TO EXAMINATION IN SF2980 RISK MANAGEMENT, 2018-04-03, 8:00–13:00.

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Allowed technical aids: Everything except computers and communication devices. All books, notes, old exams and similar are allowed. A calculator may be necessary.

Any notation introduced must be explained and defined. Assumptions must be clearly stated. Arguments and computations must be detailed so that they are easy to follow.

GOOD LUCK!

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Problem 1

An empirical estimate of $\mathrm{ES}_{0.05}(X)$ is given by

$$\frac{1}{0.05}\sum_{k=1}^{5}\frac{L_{k,100}}{100},$$

where $L_{1,100} \geq L_{2,100} \cdots \geq L_{100,100}$ are the ordered empirical losses: $L_k = -100(e^{Z_{-k+1}} - 1)$, $k = 1, \ldots, 100$. The largest losses occur for negative outcomes of the log-return, Z, and the five largest losses correspond to the log-returns approximately

$$-0.11,\ -0.08,\ -0.077,\ -0.043,\ -0.037.$$

The corresponding losses are

$$L_{1,100} = 10.5,\ L_{2,100} = 7.7,\ L_{3,100} = 7.4,\ L_{4,100} = 4.2,\ L_{5,100} = 3.6,$$

and the estimated $\mathrm{ES}_{0.05}(X)$ is 6.7.

Figure 1: This figure relates to Problem 1. It gives the qq-plot of the historlcal logreturns (y-axis) against the standard normal quantiles (x-axis).

Problem 2

We will use the property $\mathbf{w}^T\mathbf{Z} =^d \sqrt{\mathbf{w}^T\Sigma\mathbf{w}}Y_1$, of an elliptically distributed $\mathbf{Z}$ with representation $\mathbf{Z} = A\mathbf{Y}$, for $AA^T = \Sigma$ and $\mathbf{Y}$ spherically distributed. Then

$$\begin{aligned}
\mathrm{VaR}_{0.01}(X_2) &= \mathrm{VaR}_{0.01}(\mathbf{w}_2^T\mathbf{Z}) \\
&= \mathrm{VaR}_{0.01}(\sqrt{\mathbf{w}_2^T\Sigma\mathbf{w}_2}Y_1) \\
&= \sqrt{\mathbf{w}_2^T\Sigma\mathbf{w}_2}\mathrm{VaR}_{0.01}(Y_1) \\
&= \sqrt{\mathbf{w}_2^T\Sigma\mathbf{w}_2}\frac{\sqrt{\mathbf{w}_1^T\Sigma\mathbf{w}_1}\mathrm{VaR}_{0.01}(Y_1)}{\sqrt{\mathbf{w}_1^T\Sigma\mathbf{w}_1}} \\
&= \sqrt{\mathbf{w}_2^T\Sigma\mathbf{w}_2}\mathrm{VaR}_{0.05}(Y_1)\frac{\sqrt{\mathbf{w}_1^T\Sigma\mathbf{w}_1}\mathrm{VaR}_{0.01}(Y_1)}{\sqrt{\mathbf{w}_1^T\Sigma\mathbf{w}_1}\mathrm{VaR}_{0.05}(Y_1)} \\
&= \mathrm{VaR}_{0.05}(X_2)\frac{\mathrm{VaR}_{0.01}(X_1)}{\mathrm{VaR}_{0.05}(X_1)} \\
&= 110.6 \cdot \frac{45.4}{23.5} \\
&= 213.4.
\end{aligned}$$

Problem 3

The pairwise tail dependence for a Clayton copula is $\lambda_U(U,V) = 0$ and $\lambda_L(U,V) = 2^{-1/\theta}$. With $\theta = 0.5$ this gives $\lambda_L(U,V) = 0.25$. Kendall's tau for a Clayton copula is $\tau(U,V) = \theta/(2+\theta)$, so with $\theta = 0.5$ this gives $\tau(U,V) = 0.2$.

Model A: This model is consistent with all the numerical values. As just observed, since (X_1, X_2) has a Clayton copula with parameter $\theta = 0.5$ the given values for $\tau(X_1, X_2)$, $\lambda_U(X_1, X_2)$ and $\lambda_L(X_1, X_2)$ are satisfied. Moreover, since X_3 is independent of (X_1, X_2) it follows that

$$\tau(X_1, X_3) = \tau(X_2, X_3) = \lambda_U(X_1, X_3) = \lambda_U(X_2, X_3) = \lambda_L(X_1, X_3) = \lambda_L(X_2, X_3) = 0.$$

Model B: If we consider the pair (X_1, X_3) it has the copula

$$C_{13}(u_1, u_3) = C(u_1, 1, u_3) = (u_1^{-\theta} + 1 + u_3^{-\theta} - 3 + 1)^{-1/\theta} = (u_1^{-\theta} + u_3^{-\theta} - 2 + 1)^{-1/\theta},$$

which is a two-dimensional Clayton copula with parameter $\theta = 0.5$. Then $\tau(X_1, X_3) = 0.2 \neq 0$ so it is not consistent with the given numerical information.

Model C: If we consider the pair (X_1, X_2) it has a copula given by

$$C_{12}(u_1, u_2) = C(u_1, u_2, 1) = t^3_{\nu,R}(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \infty) = t^2_{\nu,R'}(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)),$$

which is a two-dimensional t_ν-copula with correlation matrix $R' = [R_{11}\ R_{12}; R_{21}\ R_{22}]$. In particular, the t-copula is symmetric, which implies that $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2)$ and this is not consistent with the given numerical information.
Answer: Model A

Problem 4

Since the distribution of X_i is regularly varying: $P(X_i > x) = (1 + x)^{-4}$, it is subexponential and the subexponential property can be used to approximate the tail of the distribution of W_j as

$$P(W_j > w) = P(X_1 + \cdots + X_7 > w) \approx 7P(X_1 > w) = 7(1 + w)^{-4},$$

for large w. Note that $P(W_j > 10) \approx 7 \cdot 11^{-4} = 4.8 \cdot 10^{-4}$ is small so $w > 10$ can be considered large. Then

$$\begin{aligned}
E[S] &= \sum_{j=1}^{52} E[\max(W_j - 10, 0)] \\
&= 52E[\max(W_1 - 10, 0)] \\
&= 52 \int_0^\infty P(\max(W_1 - 10, 0) > t)dt \\
&= 52 \int_0^\infty P(W_1 > 10 + t)dt \\
&\approx 52 \int_0^\infty 7(11 + t)^{-4}dt \\
&= 52 \cdot 7 \cdot \frac{1}{3} \cdot 11^{-3} \\
&= 0.09
\end{aligned}$$

Problem 5

In the Beta mixture model

$$P(\sum_{i=1}^{k} X_i = m \mid Z = z) = \binom{k}{m} z^m (1-z)^{k-m}.$$

By Bayes rule

$$f_{Z \mid \sum_{i=1}^{k} X_i}(Z \mid m) \propto P(\sum_{i=1}^{k} X_i = m \mid Z = z) f_Z(z) \propto z^m (1-z)^{k-m} z^{\alpha-1} (1-z)^{\beta-1},$$

so we conclude that $Z \mid \sum_{i=1}^{k} X_i = m$ has a Beta$(m+\alpha, k-m+\beta)$ distribution, and consequently

$$E[Z \mid \sum_{i=1}^{k} X_i = m] = \frac{m+\alpha}{k+\alpha+\beta}.$$

Then

$$\begin{aligned}
E\left[\sum_{i=1}^{n} X_i \mid \sum_{i=1}^{k} X_i = m\right] &= E\left[E\left[\sum_{i=1}^{n} X_i \mid Z, \sum_{i=1}^{k} X_i = m\right]\right] \\
&= E\left[E\left[m + \sum_{i=k+1}^{n} X_i \mid Z\right] \mid \sum_{i=1}^{k} X_i = m\ \right] \\
&= E[m + (n-k)Z \mid \sum_{i=1}^{k} X_i = m] \\
&= m + (n-k)E[Z \mid \sum_{i=1}^{k} X_i = m] \\
&= m + (n-k)\frac{m+\alpha}{k+\alpha+\beta}.
\end{aligned}$$