KTH Matematik

EXAMINATION IN SF2980 RISK MANAGEMENT, 2017-01-10, 14:00–19:00.

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Allowed technical aids: Everything except computers and communication devices. All books, notes, old exams and similar are allowed. A calculator is necessary.

Any notation introduced must be explained and defined. Assumptions must be clearly stated. Arguments and computations must be detailed so that they are easy to follow.

Good luck!

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Problem 1

A limitation of Archimedean copulas for a d-dimensional vector $(U_1, \dots, U_d)$ is that all pairs (U_i, U_j), $i \neq j$, has, by construction, the same Kendall's tau and the same tail dependence. This makes Archimedean copulas rather inflexible when modelling more than two assets. A possible remedy is to construct a copula in the following way. Let $U, V_1, \dots, V_d$ be independent $\mathrm{U}(0,1)$ random variables and put $X_i = F_i^{-1}(U)$ for possibly different distribution functions F_i on $(0, \infty)$, but the same U for each i. Let $\Psi_i(t) = E[e^{-tX_i}]$ and finally, let the copula be given as the joint distribution of the vector

$$(U_1, \dots, U_d) = \left(\Psi_1(-\frac{\log V_1}{X_1}), \dots, \Psi_d(-\frac{\log V_d}{X_d}) \right).$$

Let $Y = (Y_1, \dots, Y_d)$ be a random vector where Y_i has marginal distribution G_i, $i = 1, \dots, d$ and the vector Y has the copula described above. Suggest an algorithm for sampling from the joint distribution of Y. You should give a step-by-step instruction of the algorithm for sampling from the joint distribution of Y. (10 p)

Problem 2

Consider a bond portfolio that pays the amounts $c_1, c_2, \ldots, c_n$ at times $1, 2, \ldots, n$, respectively, where time is measured in years. Today is time 0. The zero rates today are given by the vector $\mathbf{r}_0 = (r_{01}, \ldots, r_{0n})$ where r_{0j} is the zero rate observed today for a payment j years into the future. Let $\mathbf{r}_1 = (r_{11}, \ldots, r_{1n})$ denote the zero rates observed at time 1, where r_{1j} is the zero rate, observed at time 1, for a payment at time $j+1$. Suppose $\mathbf{r}_1$ is modeled by

$$\mathbf{r}_1 = \mathbf{r}_0 + \sqrt{\lambda_1}\mathbf{o}_1 Z_1 + \sqrt{\lambda_2}\mathbf{o}_2 Z_2 + \sqrt{\lambda_3}\mathbf{o}_3 Z_3,$$

where $\lambda_1 > \lambda_2 > \lambda_3 > 0$, $\mathbf{o}_1$, $\mathbf{o}_2, \mathbf{o}_3$ are vectors in $\mathbb{R}^n$ (can be interpreted as eigenvectors of the first three principal components), and (Z_1, Z_2, Z_3) is a three-dimensional random vector with standard bivariate t distribution with ν degrees of freedom. Let V_0 and V_1 denote the value of the bond portfolio at time 0 and time 1, respectively. Use linearization to derive an explicit expression for $\mathrm{VaR}_p(V_1 - V_0 e^{r_{01}})$. (10 p)

Problem 3

In counterparty credit risk it is of importance to determine the potential future exposure (PFE) of a traded portfolio between the two counterparties. The PFE is the maximum of the 0.95-quantile of the distribution of the mark-to-market value of the traded portfolio over the lifetime of the portfolio. More precisely, let P_t be the mark-to-market value at time t of the traded portfolio and F_{P_t} its cdf, then

$$\mathrm{PFE} = \max_{0 \le t \le 1} F_{P_t}^{-1}(0.95).$$

(a) For a large portfolio it may be reasonable to approximate P_t by a Brownian bridge, which is a Gaussian process with fixed start and end points. To this end, suppose that $P_t = \sigma B_t$ for all $t \in [0,1]$ where B_t is a standard Brownian bridge: for any $s, t \in [0,1]$ the joint distribution of (B_s, B_t) is Gaussian with zero mean and covariance $\mathrm{Cov}(B_s, B_t) = \min(s,t) - st$. Compute the PFE. (5 p)

(b) After inspecting the model in (a) you may realize that you need a model with heavier tails than the Brownian bridge. For $\nu > 2$, let S_ν be a random variable with χ^2-distribution with ν degrees of freedom, independent of the Brownian bridge. Put

$$P_t = \frac{\sigma\sqrt{\nu - 2}}{\sqrt{S_\nu}} B_t, \quad 0 \le t \le 1.$$

Compute the PFE. (5 p)

Problem 4

In operational risk you may encounter risk factors with very heavy tails, but the dependence is unknown and difficult to estimate. In that case it is useful to get upper and lower bounds on the risk. In this problem you will derive asymptotic upper and lower bounds on loss probabilities. Let (X_1, X_2) be loss variables arising in operational risk and suppose the marginal distribution of X_1 and X_2 is a Pareto distribution with cdf $F(x) = 1 - x^{-\alpha}$, $x \geq 1$, $\alpha > 0$. In the case X_1 and X_2 are independent we write $p_{ind}(x) = P(X_1 + X_2 > x)$.

(a) For the upper bound, suppose X_1 and X_2 are comonotonic and write $p_u(x) = P(X_1 + X_2 > x)$. Determine (5 p)

$$\lim_{x\to\infty} \frac{p_u(x)}{p_{ind}(x)}.$$

(b) For the lower bound, suppose X_1 and X_2 are countermonotonic and write $p_l(x) = P(X_1 + X_2 > x)$. Determine (5 p)

$$\lim_{x\to\infty} \frac{p_l(x)}{p_{ind}(x)}.$$

Problem 5

Let (X_1, X_2) be a random vector with continuous marginal distributions F_1, F_2 and a copula C. The conditional quantile exceedance at level p is defined as the conditional probability

$$\text{CQE}(p) = P(X_1 \leq F_1^{-1}(p) \mid X_2 \leq F_2^{-1}(p)).$$

In Figure 1 (upper) you find 100 independent outcomes $(x_1^{(i)}, x_2^{(i)}), i = 1, \ldots, 100$, from the joint distribution of (X_1, X_2). In Figure 1 (lower) you find the points $(F_1(x_1^{(i)}), F_2(x_2^{(i)})), i = 1, \ldots, 100$. Suggest an empirical estimator of CQE(0.1) and use the samples pictured in Figure 1 to estimate CQE(0.1) empirically (no parametric model may be used). (10 p)

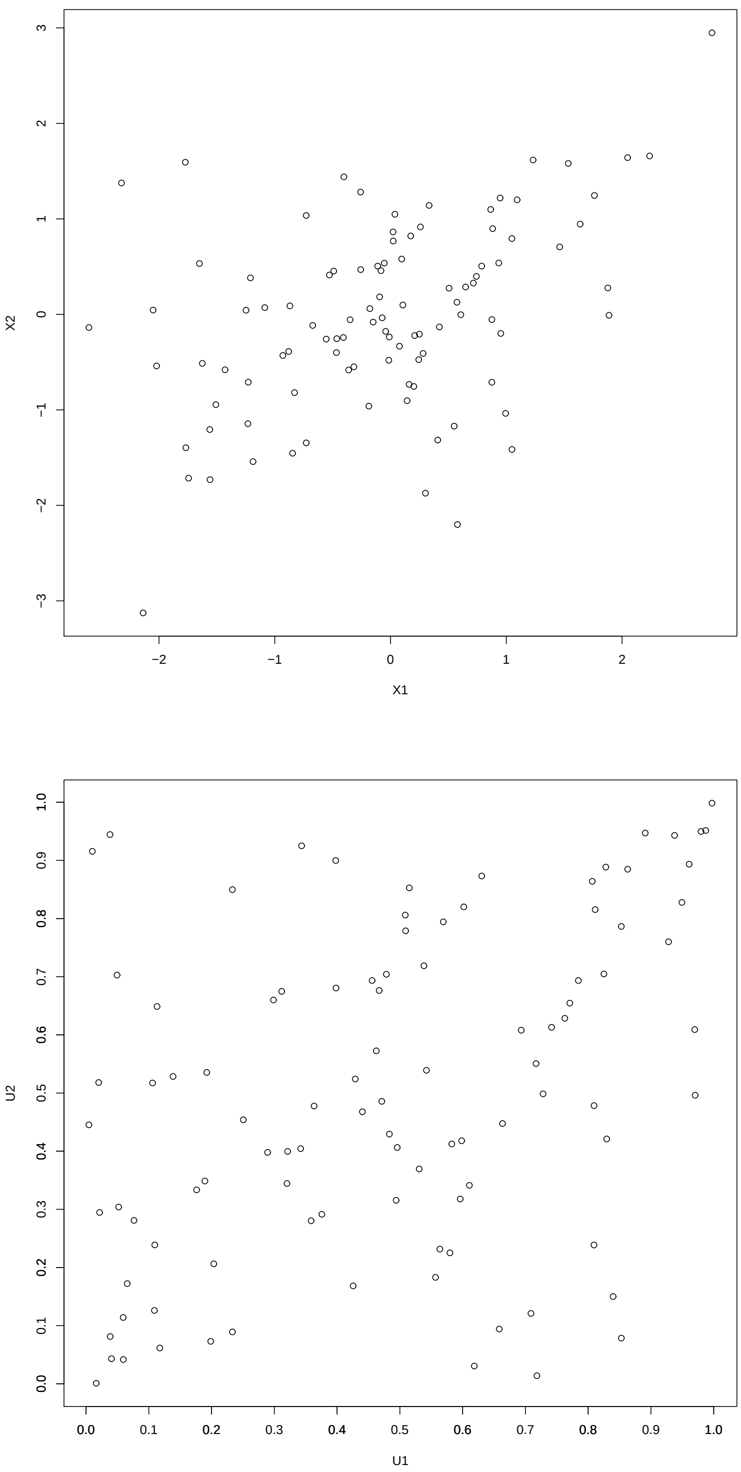

Figure 1: This figure relates to Problem 5. *Upper:* the outcomes $x^{(i)} = (x_1^{(i)}, x_2^{(i)}), i = 1, \ldots, 100$, independently sampled from the joint distribution of (X_1, X_2). *Lower:* the points $(F_1(x_1^{(i)}), F_2(x_2^{(i)}))$.