Avd. Matematisk statistik

KTH Matematik

EXAM FOR SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS
MONDAY 13 DECEMBER 2010, 14.00–19.00 HRS.

Examiner: Tobias Rydén, tel. 790 8469

Allowed aids: Formulas and survey, Time series analysis (without notes!). Handheld calculator.

Notation introduced should be defined and explained. Arguments and calculations must be clear and motivated well enough to make them easy to follow.

Each correct solution counts for 10 points. Pass (grade E) requires 25 points. Students who obtain 23 or 24 points will be offered the option to an additional small exam to possibly raise their grade to E. Students wanting to take this option must contact the examiner within a week after the results from the exam have been made public.

Solutions in Swedish are of course welcome!

The exam will be marked no later than Wednesday 5 January, and the results will be available through "Mina sidor".

Problem 1

Let $\{X_t\}$ be a stationary causal AR(2) process, i.e. $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$ where $\{Z_t\}$ is zero-mean white noise with some variance σ^2.

(a) Compute the best, in the MSE sense, linear predictor $\widehat{X}_{t+2}$ of X_{t+2}, given X_s for $s \leq t$. (7 p)

(b) Compute the mean square error of the predictor $\widehat{X}_{t+2}$. (3 p)

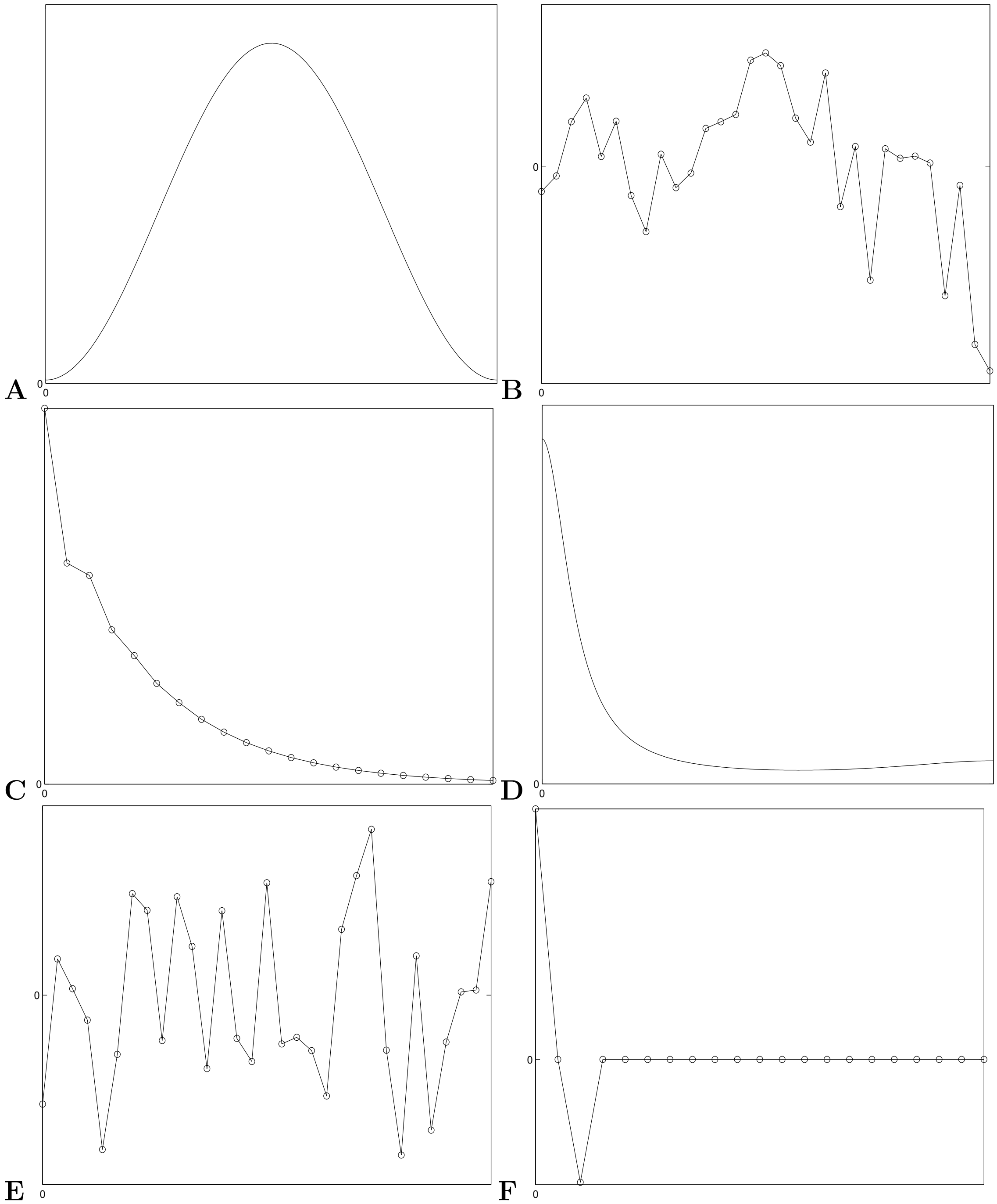
Problem 2

Let $\{X_t\}$ be a stationary MA(1) process, i.e. $X_t = Z_t + \theta Z_{t-1}$ where $\{Z_t\}$ is zero-mean white noise with some variance σ^2. In this time series, the observation at $t = 54$ happened to be missing, and we want to 'reconstruct' this observation by estimating it some way. Let X_t^* be the best, in the MSE sense, linear predictor of X_t given X_{t-1} and X_{t+1} (but no other X-variables).

(a) Compute X_t^*. (5 p)

(b) Compute the mean squared error of the predictor X_t^*. (5 p)

Problem 3



The six plots A–F above are a relisation, the autocorrelation function (ACF), and the power spectral density (PSD) for an MA(2) process, and the corresponding three plots for an AR(2) process.

Which plots are realisations, ACFs, and PSD, respectively? Group the plots into two triplets, each containing a realisation, an ACF and a PSD, such that the plots in each triplet correspond to the same time series. Which triplet corresponds to the MA(2) process, and which to the AR(2) process? (10 p)

Problem 4

Let $\{X_t\}$ be a stationary AR(2) process with autocorrelation function $\rho(\cdot)$. It is known that $\rho(1) = 1/2$ and $\rho(2) = 1/6$. Compute $\rho(3)$. (10 p)

Problem 5

In financial applications it is not unusual to consider a combination of ARCH/GARCH processes and ARMA processes, since this enables modelling the mean *and* the volatility. As an example of this, let the stationary time series $\{X_t\}$ satisfy

$$X_t - \phi X_{t-1} = Z_t,$$

where $|\phi| < 1$ and $\{Z_t\}$ is a stationary ARCH(1) process specified by

$$Z_t = h_t^{1/2} e_t, \qquad h_t = \alpha_0 + \alpha_1 Z_{t-1}^2, \qquad \{e_t\} \sim \text{IID } N(0,1),$$

where $\alpha_0 > 0$, $\alpha_1 > 0$, and e_t and $Z_{t-1}, Z_{t-2}, \ldots$ are independent for all t. The parameters are assumed to have been chosen so that necessary moments are finite.

a) Show that $\{Z_t\}$ is white noise. (5 p)

b) Compute the autocovariance function of $\{X_t\}$ in terms of ϕ, α_0, and α_1. All results given in *Formulas and survey* may be used without proofs. Other results must derived. (5 p)

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SOLUTIONS TO EXAM FOR SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS MONDAY 13 DECEMBER 2010, 14.00–19.00 HRS.

Problem 1

The problem can be solved in different ways, e.g. using the Durbin-Levinson recursion. The following is a direct approach.

(a) Expanding the recursion for X yields

$$\begin{aligned} X_{t+2} &= \phi_1 X_{t+1} + \phi_2 X_t + Z_{t+2} \\ &= \phi_1(\phi_1 X_t + \phi_2 X_{t-1} + Z_{t+1}) + \phi_2 X_t + Z_{t+2} \\ &= (\phi_1^2 + \phi_2) X_t + \phi_1\phi_2 X_{t-1} + \phi_1 Z_{t+1} + Z_{t+2}. \end{aligned}$$

We now claim that

$$\widehat{X}_{t+2} = (\phi_1^2 + \phi_2) X_t + \phi_1\phi_2 X_{t-1}.$$

This variable is a linear function of X_t and X_{t-1}, i.e. X-variables with indices at most t.

To conclude that it is the optimal linear predictor we must show that the prediction error $X_{t+2} - \widehat{X}_{t+2} = \phi_1 Z_{t+1} + Z_{t+2}$ is uncorrelated with X_s for $s \le t$. This is indeed true, because $\{Z_t\}$ is white noise. (Here we can also appeal to causality, which means that X_s can be expressed as a linear function of Z's with indices no larger than s.)

(b) Part (a) shows that the prediction error is $\phi_1 Z_{t+1} + Z_{t+2}$, and since Z_{t+1} and Z_{t+2} are uncorrelated, its variance is $\phi_1^2\sigma^2 + \sigma^2 = (1+\phi_1^2)\sigma^2$.

Problem 2

(a) All variables have zero mean, so we need not bother about means. We compute the following variance and covariances:

$$\begin{aligned} \operatorname{Var}(X_t) &= \operatorname{Var}(Z_t + \theta Z_{t-1}) = \operatorname{Var}(Z_t) + \theta^2 \operatorname{Var}(Z_{t-1}) = (1+\theta^2)\sigma^2, \\ \operatorname{Cov}(X_t, X_{t-1}) &= \operatorname{Cov}(Z_t + \theta Z_{t-1}, Z_{t-1} + \theta Z_{t-2}) = \theta \operatorname{Cov}(Z_{t-1}, Z_{t-1}) = \theta\sigma^2, \\ \operatorname{Cov}(X_{t-1}, X_{t+1}) &= 0. \end{aligned}$$

In the first computation we used that the Z's are uncorrelated, in the second computation only one out of four covariances is non-zero, and the third correlation is zero because the lag is two and the order of the MA-process is $q = 1$. By stationarity we also have $\operatorname{Var}(X_{t-1}) = \operatorname{Var}(X_{t+1}) = \operatorname{Var}(X_t)$ and $\operatorname{Cov}(X_t, X_{t+1}) = \operatorname{Cov}(X_t, X_{t+1})$.

This gives us the covariance matrix of the predictors,

$$\Gamma = \begin{pmatrix} \operatorname{Cov}(X_{t-1}, X_{t-1}) & \operatorname{Cov}(X_{t-1}, X_{t+1}) \\ \operatorname{Cov}(X_{t+1}, X_{t-1}) & \operatorname{Cov}(X_{t+1}, X_{t+1}) \end{pmatrix} = \begin{pmatrix} (1+\theta^2)\sigma^2 & 0 \\ 0 & (1+\theta^2)\sigma^2 \end{pmatrix}$$

(remembering that the variance of a random variable is its covariance with itself), and the vector of covariances between the variable to predict and the predictors,

$$\gamma = \begin{pmatrix} \mathrm{Cov}(X_t, X_{t-1}) \\ \mathrm{Cov}(X_t, X_{t+1}) \end{pmatrix} = \begin{pmatrix} \theta\sigma^2 \\ \theta\sigma^2 \end{pmatrix}.$$

Now, the requested predictor is has the form $X_t^* = a_1 X_{t-1} + a_2 X_{t+1}$, where the vector $a = (a_1,\, a_2)'$ is given by

$$a = \Gamma^{-1}\gamma = \begin{pmatrix} 1/[(1+\theta^2)\sigma^2] & 0 \\ 0 & 1/[(1+\theta^2)\sigma^2] \end{pmatrix} \begin{pmatrix} \theta\sigma^2 \\ \theta\sigma^2 \end{pmatrix} = \begin{pmatrix} \theta/(1+\theta^2) \\ \theta/(1+\theta^2) \end{pmatrix}$$

(cf. *Formulas and survey*, p.12).

Thus we find

$$X_t^* = \frac{\theta}{1+\theta^2} X_{t-1} + \frac{\theta}{1+\theta^2} X_{t+1}.$$

(b) The variance of the prediction error is (cf. *Formulas and survey*, p.13),

$$\mathrm{Var}(X_t) - \gamma'\Gamma^{-1}\gamma = \mathrm{Var}(X_t) - \gamma' a = (1+\theta^2)\sigma^2 - \frac{2\theta^2\sigma^2}{1+\theta^2} = \frac{1+\theta^4}{1+\theta^2}\sigma^2.$$

Problem 3

Plots A and D have a continuous index on the x-axis, while the other plots have a discrete index. Since a PSD is a function of a continuous variable (angular frequency or frequency), plots A and D must be PSDs. Moreover, an ACF is maximal (and equal to one) at lag zero. Plots B and E are negative at time-index 0, and can hence not be ACFs. Thus plots B and E are realisations, and plots C and F are AFCs.

Summing up: B and E are realisations, C and F are ACFs, and A and D are PSDs.

E, F and A belong together (note e.g. the negative correlation at lag 2, causing an oscillating realisation and a peak in the PSD at about half maximum frequency), and B, C and D belong together (note e.g. the slowly decaying correlation, causing a slowly varying realisation and a PSD with large contents of low frequencies).

E, F and A are for the MA(2) process (e.g., the ACF is zero for lags above 2), and B, C and D are for the AR(2) process (e.g., the ACF is non-zero for all lags).

Problem 4

Let the AR(2) process be defined by $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$, where $\{Z_t\}$ is white noise.

The Yule-Walker equations for $k = 1$ and $k = 2$ read

$$\begin{aligned} \gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(-1) &= 0 \\ \gamma(2) - \phi_1\gamma(1) - \phi_2\gamma(0) &= 0, \end{aligned}$$

where $\gamma(\cdot)$ is the ACVF. Using that $\gamma(-h) = \gamma(h)$ and dividing through by $\gamma(0)$ yields

$$\begin{aligned} \rho(1) - \phi_1 - \phi_2\rho(1) &= 0 \\ \rho(2) - \phi_1\rho(1) - \phi_2 &= 0, \end{aligned}$$

Rearranging and inserting the given values for $\rho(1)$ and $\rho(2)$, we find the system of equations

$$\begin{aligned} \phi_1 + \frac{1}{2}\phi_2 &= \frac{1}{2} \\ \frac{1}{2}\phi_1 + \phi_2 &= \frac{1}{6} \end{aligned}$$

for (ϕ_1, ϕ_2). The solution of this system is $\phi_1 = 5/9$, $\phi_2 = -1/9$.

The Yule-Walker equation for $k = 3$ is

$$\gamma(3) - \phi_1\gamma(2) - \phi_2\gamma(1) = 0,$$

so that

$$\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) = \frac{5}{9} \times \frac{1}{6} - \frac{1}{9} \times \frac{1}{2} = \frac{1}{27}.$$

Problem 5

(a) We have

$$E(Z_t) = E(h_t^{1/2})E(e_t) = 0,$$

$$\mathrm{Var}(Z_t) = E(Z_t^2) = E(h_t e_t^2) = E(h_t)E(e_t^2) = E(h_t)$$

and, for $t > s$,

$$\mathrm{Cov}(Z_s, Z_t) = E(Z_s Z_t) = E(h_s^{1/2} e_s h_t^{1/2} e_t) = E(h_s^{1/2} e_s h_t^{1/2})E(e_t) = 0.$$

Thus $\{Z_t\}$ is white noise.

(b) Since $\{Z_t\}$ is white noise it follows that $\{X_t\}$ is an AR(1) process. Using $E(Z_t^2) = E(h_t)$ and stationarity we find

$$E(Z_t^2) = \alpha_0 + \alpha_1 E(Z_t^2),$$

so that $E(Z_t^2) = \alpha_0/(1 - \alpha_1)$. Using *Formulas and survey*, p.10, we conclude that

$$\gamma(h) = \frac{\alpha_0}{1 - \alpha_1} \times \frac{\phi^{|h|}}{1 - \phi^2}.$$