EXAM FOR SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS
TUESDAY 7 JUNE 2011, 08.00–13.00 HRS.

Examiner: Tobias Rydén, tel. 790 8469

Allowed aids: Formulas and survey, Time series analysis (without notes!). Pocket calculator.

Notation introduced should be defined and explained. Solutions, arguments and calculations must be clear and motivated well enough to make them easy to follow.

Each correct solution counts for 10 points. Pass (grade E) requires 25 points. Students who obtain 23 or 24 points will be offered the option to do an additional small exam to possibly raise their grade to E. Students wanting to take this option must contact the examiner within a week after the results from the exam have been made public.

Solutions in Swedish are of course welcome!

The exam will be marked no later than 22 June, and the results will be available through "Mina sidor".

Problem 1

Let $\{X_t\}_{t=-\infty}^{\infty}$ be an AR(1) process satisfying the difference equation $X_t = \phi X_{t-1} + Z_t$, where $|\phi| < 1$ and $\{Z_t\}$ is zero-mean white Gaussian noise with variance σ^2.

Assume that due to sampling constraints we can only observe X_t at every second time-point, and define the process $Y_t = X_{2t}$.

Show that $\{Y_t\}_{t=-\infty}^{\infty}$ is also a Gaussian AR(1) process, i.e. that it satisfies a difference equation of the form $Y_t = \phi' Y_{t-1} + Z'_t$ where $\{Z'_t\}$ is zero-mean white Gaussian noise, and find its auto-regression parameter ϕ' and innovation variance $\mathrm{Var}(Z'_t)$. (10 p)

Problem 2

Let $\{X_t\}_{t=-\infty}^{\infty}$ be a stationary time series with zero mean and spectral density

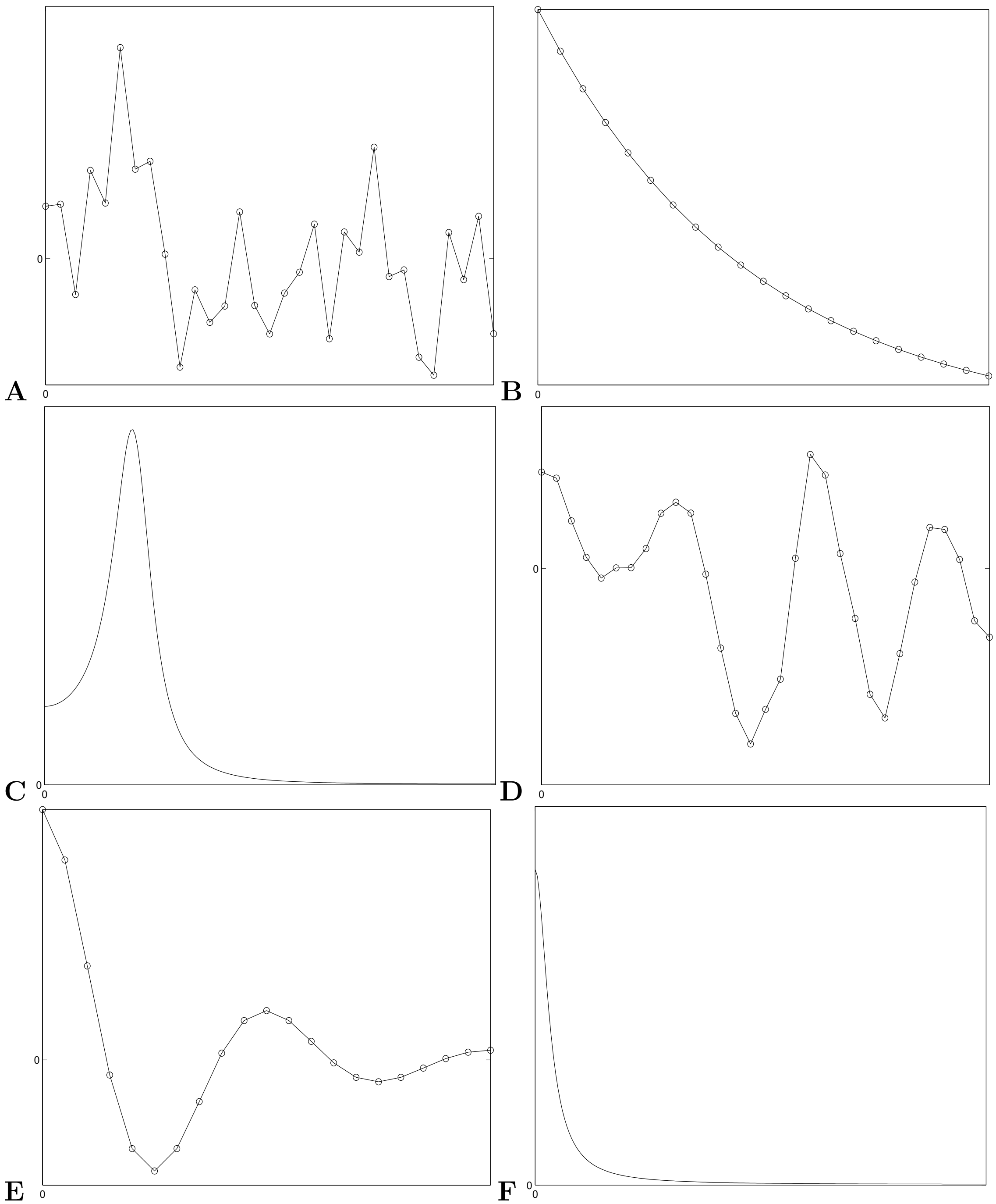
$$f_X(\lambda) = \frac{20 + 16\cos\lambda}{5 - 2\cos\lambda}, \qquad -\pi < \lambda \le \pi.$$

Consider the time series $\{Y_t\}_{t=-\infty}^{\infty}$ given by

$$Y_t = X_t - X_{t-1}, \qquad t = \ldots, -2, -1, 0, 1, 2, \ldots$$

Show that $\{Y_t\}$ is stationary and compute its spectral density. (10 p)

Problem 3



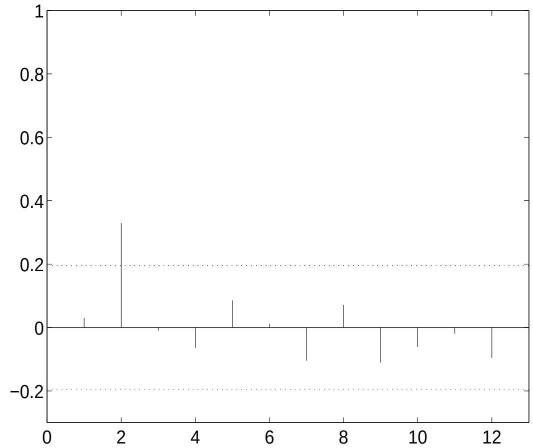
The six plots A–F above are a relisation, the autocorrelation function (ACF), and the spectral density (SD) for an AR(1) process, and the corresponding three plots for an AR(2) process. Which plots are realisations, ACFs, and SDs, respectively? Group the plots into two triplets, each containing a realisation, an ACF and an SD, such that the plots in each triplet correspond to the same time series. Which triplet corresponds to the AR(1) process, and which one to the AR(2) process? (10 p)

Problem 4

From a sequence of 100 observations from a time series, the following values of the sample autocovariance function were computed:

h	0	1	2	3	4	5
$\widehat{\gamma}(h)$	2.400	0.072	0.792	0.019	0.124	0.181

The sample partial autocorrelation function is plotted below:



(a) Model the data as an AR(p) time series with an appropriate order p and motivate your choice of order. (2 p)

(b) Use the information above to estimate the autoregression coefficients $\phi_1, \phi_2, \ldots, \phi_p$, as well as the white noise variance σ^2. (8 p)

Problem 5

Let $\{X_t\}_{t=-\infty}^{\infty}$ be an AR(1) process given by

$$X_t = 0.75X_{t-1} + Z_t, \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2). \tag{1}$$

The true relation (1) is however unknown, and it is believed (after some parameter estimation and model selection) that $\{X_t\}$ is the AR(2) process

$$X_t = 0.7X_{t-1} + 0.1X_{t-2} + Z_t, \qquad \{Z_t\} \sim \mathrm{WN}(0, 2.364^2). \tag{2}$$

Let $\widehat{X}_{t+1} = P_{\overline{\mathrm{sp}}\{X_t, X_{t-1}, \ldots\}} X_{t+1}$ be the best linear predictor of X_{t+1} in terms of X_t, $X_{t-1}, \ldots$ *when* $\{X_t\}$ *is as believed, i.e. under (2).*

Evaluate the true, i.e. under (1), mean squared error of $\widehat{X}_{t+1}$. The answer will contain σ^2. (10 p)

SOLUTIONS TO EXAM FOR SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS TUESDAY 7 JUNE 2011, 08.00–13.00 HRS.

Problem 1

We have

$$Y_t = X_{2t} = \phi X_{2t-1} + Z_{2t} = \phi(\phi X_{2t-2} + Z_{2t-1}) + Z_{2t} = \phi^2 Y_{t-1} + \phi Z_{2t-1} + Z_{2t}.$$

We write this as

$$Y_t = \phi' Y_{t-1} + Z'_t,$$

where $\phi' = \phi^2$ and $Z'_t = \phi Z_{2t-1} + Z_{2t}$.

Since $\{Z_t\}$ is WGN we find that Z_t has zero mean, $\mathrm{Var}(Z'_t) = \phi^2 \mathrm{Var}(Z_{2t-1}) + \mathrm{Var}(Z_{2t}) = (\phi^2+1)\sigma^2$ and that Z'_t has a Gaussian distribution. Moreover, since Z'_t and Z'_s depend on different Z_t's for $t \neq s$ it follows that $\{Z'_t\}$ is an independent sequence. Thus $\{Z_t\}$ is WGN.

Problem 2

The process time series $\{Y_t\}$ results from feeding $\{X_t\}$ through a causal time-invariant linear filter (TLF) with transfer function

$$h(e^{-i\lambda}) = 1 \cdot 1 + (-1)e^{-i\lambda} = 1 - e^{-i\lambda} = 1 - \cos\lambda + i\sin\lambda$$

and power transfer function

$$|h(e^{-i\lambda})|^2 = (1-\cos\lambda)^2 + \sin^2\lambda = 1 - 2\cos\lambda + \cos^2\lambda + \sin^2\lambda = 2 - 2\cos\lambda.$$

Using Theorem 9.1 of Formulas and Survey we conclude that $\{Y_t\}$ is stationary, and that its spectal density is

$$f_Y(\lambda) = |h(e^{-i\lambda})|^2 f_X(\lambda) = \frac{(2-2\cos\lambda)(20+16\cos\lambda)}{5-2\cos\lambda}, \qquad -\pi < \lambda \leq \pi.$$

Problem 3

Plots C and F have a continuous index on the x-axis, while the other plots have a discrete index. Since an SD is a function of a continuous variable (angular frequency or frequency), plots C and F must be SDs. Moreover, an ACF is maximal (and equal to one) at lag zero. Plots A and D are not maximal at time-index 0, and can hence not be ACFs. Thus plots A and D are realisations, and plots B and E are AFCs.

Summing up: A and D are realisations, B and E are ACFs, and C and F are PSDs.

D, E and C belong together (note e.g. the negative correlation at lag 5 in E, causing a realisation D with a marked dominant frequency and a peak in the spectral density C), and

A, B and F belong together (note e.g. the decaying correlation B without peaks, causing a realisation A with contents of low frequencies but without any particularly marked frequency, and a spectral density with large contents of low frequencies).

D, E and C are for the AR(2) process (an AR(1) process cannot have a pair of complex conjugate poles that cause a peaked spectrum like C), and A, B and F are for the AR(1) process (e.g., note the geometrically decaying ACF).

Problem 4

(a) As we see that the PACF is small for lags larger than 2, we set $p = 2$.

(b) We can estimate the parameters with the Yule-Walker estimates. These estimates solve the sample Yule-Walker equations

$$\begin{aligned}\widehat{\gamma}(0)\phi_1 + \widehat{\gamma}(1)\phi_2 &= \widehat{\gamma}(1)\\ \widehat{\gamma}(1)\phi_1 + \widehat{\gamma}(0)\phi_2 &= \widehat{\gamma}(2),\end{aligned}$$

that is

$$\begin{aligned}2.4\phi_1 + 0.072\phi_2 &= 0.072\\ 0.072\phi_1 + 2.4\phi_2 &= 0.792\end{aligned}$$

The solution is $\widehat{\phi}_1 = 0.020, \widehat{\phi}_2 = 0.329$. The estimate of the white noise variance is $\widehat{\sigma}^2 = \widehat{\gamma}(0) - \widehat{\phi}_1\widehat{\gamma}(1) - \widehat{\phi}_2\widehat{\gamma}(2) = 2.137$.

Problem 5

Since the innovation Z_t in (2) in uncorrelated with X_{t-1} and X_t, the best linear predictor under this model is

$$\widehat{X}_{t+1} = 0.7X_t + 0.1X_{t-1}.$$

The true autocovariance function, i.e. the ACF under (1), is

$$\gamma(h) = \frac{\sigma^2}{1 - 0.75^2}0.75^{|h|}$$

(see page 10 of Formulas and Survey).

Thus we obtain the mean squared prediction error

$$\begin{aligned}E[(X_{t+1} - \widehat{X}_{t+1})^2] &= E[(X_{t+1} - 0.7X_t - 0.1X_{t-1})^2]\\ &= \gamma(0) + (-0.7)^2\gamma(0) + (-0.1)^2\gamma(0)\\ &\quad -2 \cdot 1 \cdot 0.7\,\gamma(1) - 2 \cdot 1 \cdot 0.1\,\gamma(2) + 2 \cdot (-0.7) \cdot (-0.1)\,\gamma(1)\\ &= \frac{\sigma^2}{1 - 0.75^2}(1.5 - 1.4 \cdot 0.75 - 0.2 \cdot 0.75^2 + 0.14 \cdot 0.75)\\ &= \frac{\sigma^2 \cdot 0.4425}{0.4375} = 1.0114\,\sigma^2.\end{aligned}$$