Avd. Matematisk statistik

KTH Matematik

EXAM FOR SF2943/SF2945 TIME SERIES ANALYSIS/TIDSSERIEANALYS
MONDAY 20 AUGUST 2012, 08.00–13.00 HRS.

Examiner: Tobias Rydén, tel. 790 8469

Allowed aids: Formulas and survey, Time series analysis (without notes!). Pocket calculator.

Notation introduced should be defined and explained. Solutions, arguments and calculations must be clear and motivated well enough to make them easy to follow.

Each correct solution counts for 10 points. Pass (grade E) requires 25 points. Students who obtain 23 or 24 points will be offered the option to do an additional small exam to possibly raise their grade to E. Students wanting to take this option must contact the examiner within a week after the results from the exam have been published.

Solutions in Swedish are of course welcome!

The exam will be marked no later than within three weeks, and the results will be available through *Mina sidor*.

Problem 1

Let $\{X_t\}_{t=-\infty}^{\infty}$ and $\{Y_t\}_{t=-\infty}^{\infty}$ be two independent zero mean stationary time series. Further $\{X_t\}$ has autocovariance function $\gamma_X(h) = \left(\frac{1}{2}\right)^{|h|}$ while $\{Y_t\} \sim \text{WN}(0, \sigma_Y^2)$. Define $\{Z_t\}$ by

$$Z_t = X_t(Y_t + Y_{t-1}) \quad \text{for all } t.$$

Show that $\{Z_t\}$ is an MA(1) process, and determine the parameters for the invertible version. (10 p)

Problem 2

Let $\{X_t\}_{t=-\infty}^{\infty}$ be an AR(1) process given by

$$X_t = 0.5X_{t-1} + V_t, \qquad \{V_t\} \sim \text{WN}(0, \sigma_V^2),$$

and define $\{Y_t\}_{t=-\infty}^{\infty}$ by

$$Y_t = X_t + W_t, \qquad \{W_t\} \sim \text{WN}(0, \sigma_W^2),$$

where the white noises $\{V_t\}$ and $\{W_t\}$ are independent with $\sigma_V^2 = 1$ and $\sigma_W^2 = 4$. The above model is thus a simple state-space model.

Determine the best (in the mean square error sense) linear predictor of Y_1 based on Y_0, Y_{-1}, i.e., determine $\widehat{Y}_1 = aY_0 + bY_{-1}$. (10 p)

Problem 3

Let $\{X_t\}_{t=-\infty}^{\infty}$ be an AR(2) process given by

$$X_t + 1.7X_{t-1} + 0.8X_{t-2} = Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0,1)$$

and $\{Y_t\}_{t=-\infty}^{\infty}$ be an MA(2) process, given by

$$Y_t = Z_t + 1.7Z_{t-1} + 0.8Z_{t-2}, \quad \{Z_t\} \sim \mathrm{WN}(0,1).$$

In Figures 1 and 2 random generations of the two processes are plotted. (In the generations the WN process is Gaussian.) For clarity the realizations are given as continuous functions although time is discrete.

Figure 1

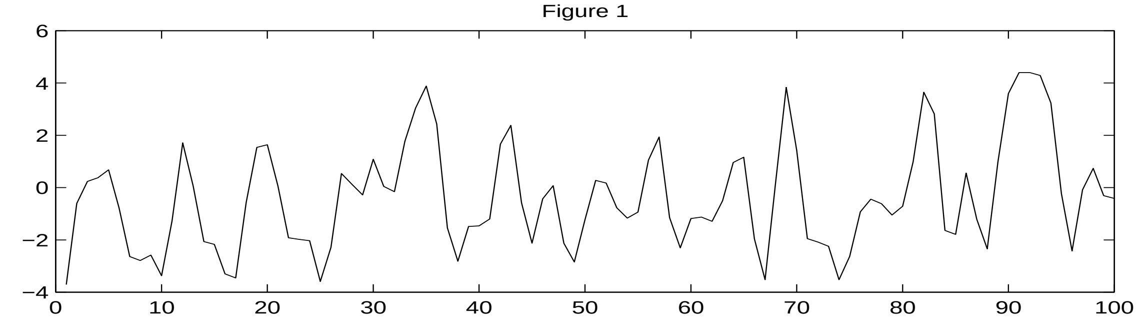
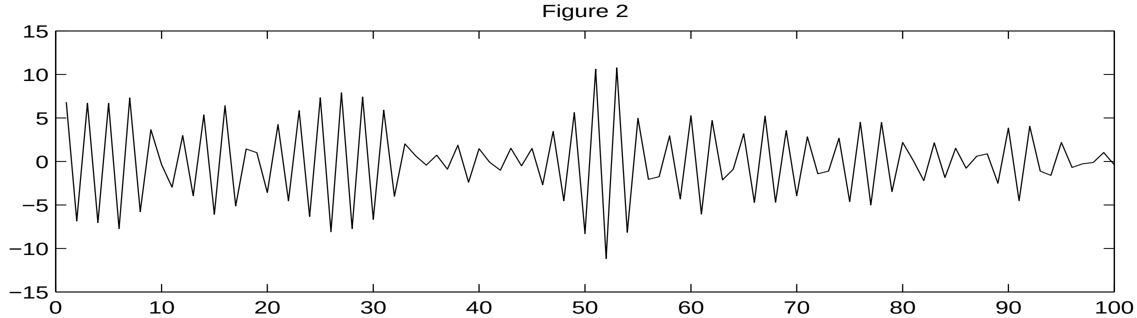


Figure 2



Determine which figure corresponds to which process. The answer must of course be motivated.

(10 p)

Problem 4

From 1,000 observations $x_1, \ldots, x_{1000}$ of an AR(2) process, one computed the sample autocovariances

$$\widehat{\gamma}(0) = 3.6840, \quad \widehat{\gamma}(1) = 2.2948 \text{ and } \widehat{\gamma}(2) = 1.8491.$$

One was interested in finding out if the first AR coefficient ϕ_1, was indeed non-zero, and wanted to address this question by using a confidence interval. Thus compute, using a method of your choice, a confidence interval for ϕ_1, with approximate confidence level 95%. (10 p)

Problem 5

Let $\{X_t\}_{t=-\infty}^{\infty}$ be a stationary time series with zero mean. Assume that the best (in the sense of mean square error) linear one-step predictor

$$\widehat{X}_{n+1|n} = P_{\overline{\text{sp}}\{X_n,\ X_{n-1},\dots\}} X_{n+1}$$

is given by

$$\widehat{X}_{n+1|n} = \sum_{i=1}^{\infty} \phi_i X_{n+1-i},$$

and that all coefficients ϕ_i are known to us. We are now interested in the best linear two-step predictor

$$\widehat{X}_{n+2|n} = P_{\overline{\text{sp}}\{X_n,\ X_{n-1},\dots\}} X_{n+2}.$$

Show that

$$\widehat{X}_{n+2|n} = \phi_1 \widehat{X}_{n+1|n} + \sum_{i=2}^{\infty} \phi_i X_{n+2-i}.$$

(10 p)

Good luck!

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SOLUTIONS TO EXAM FOR SF2943/SF2945 TIME SERIES ANALYSIS/TIDSSERIE-ANALYS, MONDAY 20 AUGUST 2012

Formulas and survey will be referred to as FS.

Problem 1

We have

$$E[Z_t] = E[X_t(Y_t + Y_{t-1})] = (\text{due to independence}) = E[X_t]E[Y_t + Y_{t-1}] = 0.$$

Thus

$$\begin{aligned}
\gamma_Z(h) &= E[Z_{t+h}Z_t] \\
&= E[X_{t+h}(Y_{t+h} + Y_{t+h-1})X_t(Y_t + Y_{t-1})] \\
&= E[X_{t+h}X_t]E[(Y_{t+h} + Y_{t+h-1})(Y_t + Y_{t-1})](\text{due to independence}) \\
&= \gamma_X(h)(\gamma_Y(h) + \gamma_Y(h+1) + \gamma_Y(h-1) + \gamma_Y(h)) \\
&= \gamma_X(h)(\gamma_Y(h-1) + 2\gamma_Y(h) + \gamma_Y(h+1)) \\
&= \left(\tfrac{1}{2}\right)^{|h|} \times \begin{cases} 2\sigma_Y^2 & \text{if } h = 0, \\ \sigma_Y^2 & \text{if } h = \pm 1, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2\sigma_Y^2 & \text{if } h = 0, \\ \frac{1}{2}\sigma_Y^2 & \text{if } h = \pm 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

If $\{Z_t\}$ is an MA(1) process, with parameters θ and σ_Z^2, we must have

$$\begin{aligned}
\sigma_Z^2(1+\theta^2) &= 2\sigma_Y^2 \\
\sigma_Z\theta &= \tfrac{1}{2}\sigma_Y^2.
\end{aligned}$$

Thus

$$\frac{1+\theta^2}{\theta} = 4 \Rightarrow \theta^2 - 4\theta + 1 = 0 \Rightarrow \theta = 2 \pm \sqrt{3}.$$

For the invertible version we must have $|\theta| < 1$, and thus

$$\theta = 2 - \sqrt{3} \approx 0.27 \quad \text{and} \quad \sigma_Z^2 = \frac{\sigma_Y^2}{2(2-\sqrt{3})} \approx 1.87\sigma_Y^2.$$

Since there exist $|\theta| < 1$ and $\sigma^2 > 0$ giving the correct autocovariance function, $\{Z_t\}$ is an MA(1) process.

Problem 2

From the projection theorem it follows that a and b are determined by

$$\begin{aligned}
E[(Y_1 - aY_0 - bY_{-1})Y_0] &= 0 \\
E[(Y_1 - aY_0 - bY_{-1})Y_{-1}] &= 0,
\end{aligned}$$

or

$$\begin{aligned}\gamma_Y(1) - a\gamma_Y(0) - b\gamma_Y(1) &= 0\\ \gamma_Y(2) - a\gamma_Y(1) - b\gamma_Y(0) &= 0.\end{aligned}$$

Simple calculations yields

$$a = \frac{\gamma_Y(1)(\gamma_Y(0) - \gamma_Y(2))}{\gamma_Y^2(0) - \gamma_Y^2(1)} \quad \text{and} \quad b = \frac{\gamma_Y(0)\gamma_Y(2) - \gamma_Y^2(1)}{\gamma_Y^2(0) - \gamma_Y^2(1)}.$$

Since $\gamma_X(h) = 0.5^{|h|}/0.75$ we have

$$\begin{aligned}\gamma_Y(0) &= \gamma_X(0) + \gamma_W(0) = \frac{1}{0.75} + \sigma_W^2 = \frac{16}{3}\\ \gamma_Y(1) &= \gamma_X(1) + \gamma_W(1) = \frac{0.5}{0.75} + 0 = \frac{2}{3}\\ \gamma_Y(2) &= \gamma_X(2) + \gamma_W(2) = \frac{0.25}{0.75} + 0 = \frac{1}{3}.\end{aligned}$$

This yields $a = 5/42$ and $b = 1/21$, so

$$\widehat{Y}_1 = \frac{5}{42}Y_0 + \frac{1}{21}Y_{-1}.$$

Problem 3

The realisation in Figure 2 is more oscillating (from one time-point to the next), which indicates a negative correlation between subsequent values. This holds for the AR process $\{X_t\}$, which is seen if its representation is written on the form

$$X_t = -1.7X_{t-1} - 0.8X_{t-2} + Z_t, \quad \{Z_t\} \sim \text{WN}(0,1).$$

The MA process $\{Y_t\}$, on the other hand, has positive correlation at lag 1.
Thus $\{Y_t\}$ corresponds to Figure 1, and $\{X_t\}$ corresponds to Figure 2.

Problem 4

We will use Yule-Walker estimation. We have

$$\widehat{\Gamma}_2 = \begin{pmatrix} 3.6840 & 2.2948\\ 2.2948 & 3.6840\end{pmatrix} \quad \text{and} \quad \widehat{\boldsymbol{\gamma}}_2 = \begin{pmatrix} 2.2948\\ 1.8491\end{pmatrix}.$$

Thus we get

$$\widehat{\boldsymbol{\phi}} = \widehat{\Gamma}_2^{-1}\widehat{\boldsymbol{\gamma}}_2 = \begin{pmatrix} 0.4435 & -0.2763\\ -0.2763 & 0.4435\end{pmatrix}\begin{pmatrix} 2.2948\\ 1.8491\end{pmatrix} = \begin{pmatrix} 0.5070\\ 0.1861\end{pmatrix} \text{ and } \widehat{\sigma}^2 = \widehat{\gamma}(0) - \widehat{\boldsymbol{\phi}}'\widehat{\boldsymbol{\gamma}}_2 = 2.1764.$$

Recall that $\widehat{\boldsymbol{\phi}} \sim \text{AN}(\boldsymbol{\phi}, \sigma^2\Gamma_2^{-1}/1000)$ where $\sigma^2\Gamma_2^{-1}/1000$ is estimated by

$$\frac{\widehat{\sigma}^2\widehat{\Gamma}_2^{-1}}{1000} = \frac{2.1764}{1000}\begin{pmatrix} 0.4435 & -0.2763\\ -0.2763 & 0.4435\end{pmatrix} = \begin{pmatrix} 0.000965 & -0.000601\\ -0.000601 & 0.000965\end{pmatrix}.$$

Thus the approximate confidence interval for ϕ_1 is given by

$$I_{\phi_1} = 0.507 \pm \lambda_{0.025}\sqrt{0.000965}$$

$$= 0.507 \pm 1.96 \cdot 0.0311 = 0.0507 \pm 0.061.$$

Note: The data came from a simulated AR(2) process with $\phi_1 = 0.5$, $\phi_2 = 0.2$ and $\sigma^2 = 2.25$.

Problem 5

From the projection theorem it follows that it is sufficient to show that

$$\mathrm{Cov}[X_{n+2} - \widehat{X}_{n+2|n}, X_{n+2-j}] = 0 \quad \text{for } j = 2, 3, \ldots,$$

where $\widehat{X}_{n+2|n}$ is as suggested in the problem formulation. We know, again from the projection theorem, that

$$\mathrm{Cov}[X_{n+1} - \widehat{X}_{n+1|n}, X_{n+1-j}] = 0 \quad \text{for } j = 1, 2, 3, \ldots \tag{1}$$

In order to use this, write

$$\begin{aligned}
\widehat{X}_{n+2|n} &= \phi_1 \widehat{X}_{n+1|n} + \sum_{i=2}^{\infty} \phi_i X_{n+2-i} \\
&= \phi_1 \widehat{X}_{n+1|n} + \sum_{i=1}^{\infty} \phi_i X_{n+2-i} - \phi_1 X_{n+1} \\
&= \widehat{X}_{n+2|n+1} - \phi_1 (X_{n+1} - \widehat{X}_{n+1|n}),
\end{aligned}$$

where we used the stationarity to justify that $\widehat{X}_{n+2|n+1}$ has the same expansion (coefficients ϕ_i) as has $\widehat{X}_{n+1|n}$.

Thus we obtain

$$\begin{aligned}
\mathrm{Cov}[X_{n+2} - \widehat{X}_{n+2|n}, X_{n+2-j}] &= \mathrm{Cov}[X_{n+2} - \widehat{X}_{n+2|n+1} + \phi_1 (X_{n+1} - \widehat{X}_{n+1|n}), X_{n+2-j}] \\
&= \mathrm{Cov}[X_{n+2} - \widehat{X}_{n+2|n+1}, X_{n+2-j}] \\
&+ \phi_1 \, \mathrm{Cov}[X_{n+1} - \widehat{X}_{n+1|n}, X_{n+2-j}].
\end{aligned}$$

Here $\mathrm{Cov}[X_{n+2} - \widehat{X}_{n+2|n+1}, X_{n+2-j}]$ equals 0 for $j = 1, 2, 3, \ldots$ by (1), but applied with $n+2$ rather than $n+1$, and $\mathrm{Cov}[X_{n+1} - \widehat{X}_{n+1|n}, X_{n+2-j}]$ equals 0 for $j = 2, 3, 4 \ldots$ by (1). Hence the proof is complete.