Avd. Matematisk statistik

KTH Matematik

EXAM FOR SF2943 TIME SERIES ANALYSIS/TIDSSERIEANALYS
WEDNESDAY 22nd MAY 2013, 14.00–19.00 HRS.

Examiner: Timo Koski, tel. 790 71 34

Allowed aids: Formulas and survey, Time series analysis (without added notes!). Pocket calculator.

Notation introduced should be defined and explained. Solutions, arguments and calculations must be clear and motivated well enough to make them easy to follow.

Each correct solution counts for 10 points. Pass (grade E) requires 25 points. Students who obtain 23 or 24 points will be offered the option to do an additional small exam to possibly raise their grade to E. Students wanting to take this option must contact the examiner within a week after the results from the exam have been published.

Solutions in Swedish are of course welcome!

The exam will be marked no later than within three weeks, and the results will be available through *Mina sidor*.

Problem 1

Let $\{X_t\}_{t=-\infty}^{\infty}$ be a stationary $AR(1)$ process given by

$$X_t + 0.25X_{t-1} = Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, 1),$$

We observe the time series at times $t = 1$ and $t = 3$, but miss the value at $t = 2$. We wish to find the linear combination of X_1 and X_3 that estimates X_2 with minimal mean square error. We denote the estimate by

$$\widehat{X}_2 = a_1X_1 + a_2X_3.$$

(a) Find a_1 and a_2 so that $E\left[\left(X_2 - \widehat{X}_2\right)^2\right]$ is minimized. (5 p)

(b) Find the value of the optimal mean squared error. (5 p)

Problem 2

Let $\{X_t, t \in \mathbb{Z}\}$ be an MA(1) process

$$X_t = Z_t + \theta Z_{t-1},$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) Find the value θ^* for θ such that the coefficient of correlation between X_t and X_{t-1} is maximized. (7 p)

(b) Find the spectral density of $\{X_t, t \in \mathbb{Z}\}$ when $\theta = \theta^*$. Are high or low frequencies dominating in the spectrum? Why is your conclusion about the frequency contents intuitively reasonable? (3 p)

Problem 3

Consider the stochastic difference equation

$$Y_t - \frac{5}{6} Y_{t-1} = Z_t - \frac{9}{20} Z_{t-1},$$

where $Z_t \sim \mathrm{WN}(0, 1)$.

(a) Show that there exists $\{Y_t \mid t = 0, \pm 1, \pm 2, \ldots\}$, which is an ARMA(1,1) -process. (3 p)

(b) Find the state space representation of $\{Y_t \mid t = 1, 2, \ldots\}$. Justify your solution careful-ly. (7 p)

Problem 4

Let $\{X_t\}_{t=0}^{\infty}$ be a time series that satisfies

$$X_t = b_t - (\theta_1 + \theta_2) Z_t - \theta_2 Z_{t-1} \tag{1}$$

where $\{b_t\}_{t=0}^{\infty}$ satisfies

$$b_t = b_{t-1} + (1 + \theta_1 + \theta_2) Z_t \tag{2}$$

and where $\{Z_t\} \sim \mathrm{WN}(0, 1)$. Show that $\{X_t\}_{t=0}^{\infty}$ is an ARIMA(0,1,2) -process. What could be the interpretation of this way of representing the process $\{X_t\}_{t=0}^{\infty}$? (10 p)

Problem 5

$\{X_t\}_{t=0}^{\infty}$ is a stationary time series. Let

$$\widehat{X}_t = \begin{cases} 0 & t = 1 \\ P_{t-1} X_t & t = 2, 3, \ldots \end{cases}$$

where $P_{t-1} X_t$ is the optimal projection of X_t onto the linear span of $X_{t-1}, \ldots, X_1$. In other words we have that

$$\widehat{X}_t \stackrel{\text{def}}{=} P_{t-1} X_t = a_{t-1,1} X_{t-1} + a_{t-1,2} X_{t-2} + \ldots a_{t-1,t-1} X_1.$$

The innovations are the one-step prediction errors defined as

$$\varepsilon_t \stackrel{\text{def}}{=} X_t - \widehat{X}_t. \tag{3}$$

We know that the innovations constitute a white noise (you need not prove this) and we write

$$v_{t-1} = E\left[\left(X_t - \widehat{X}_t\right)^2\right].$$

We introduce

$$\underline{\varepsilon}_n = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_{n-1} \\ \varepsilon_n \end{pmatrix} \quad \underline{X}_n = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} \quad \underline{\widehat{X}}_n = \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \\ \vdots \\ \widehat{X}_{n-1} \\ \widehat{X}_n \end{pmatrix}$$

and

$$\mathbf{A}_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -a_{1,1} & 1 & 0 & \dots & 0 \\ -a_{2,2} & -a_{2,1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -a_{n-1,n-1} & -a_{n-1,n-2} & -a_{n-1,n-3} & \dots & 1 \end{pmatrix}.$$

(a) Let $\mathbf{C}_n = \mathbf{A}_n^{-1}$. The matrix $\mathbf{C}_n$ is lower triangular with ones on the main diagonal, too. Show that the covariance matrix of $\underline{X}_n$ is given by

$$\Gamma_n = E\left[\underline{X}_n \underline{X}_n^T\right] = \mathbf{C}_n \mathbf{D}_n \mathbf{C}_n^T,$$

where

$$\mathbf{D}_n = \begin{pmatrix} v_0 & 0 & 0 & \dots & 0 \\ 0 & v_1 & 0 & \dots & 0 \\ 0 & 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & v_{n-1} \end{pmatrix}.$$

Justify your solution. *Aid:* Apply first (3) to $\underline{\varepsilon}_n$. (2 p)

(b) Express $\underline{\widehat{X}}_n$ in terms of $\underline{\varepsilon}_n$ via a matrix operation. (2 p)

(c) The MA(1) process $\{X_t\}_{t=-\infty}^{\infty}$ has zero mean function and satisfies

$$X_t = Z_t - 1.2 Z_{t-1},$$

where $Z_t \sim \mathrm{WN}(0, 1)$. Express $\underline{\widehat{X}}_4$ in terms of $\underline{\varepsilon}_4$.

Aid for computation:

$$C_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.4918 & 1 & 0 & 0 \\ 0 & -0.6487 & 1 & 0 \\ 0 & 0 & -0.7222 & 1 \end{pmatrix}.$$

(2 p)

(d) $\{X_t\}_{t=0}^{\infty}$ is as in (c). Find the values of v_1 and v_2. Show your computations ! (4 p)

Aid for computation:

$$\Gamma_4 = \begin{pmatrix} 2.44 & -1.2 & 0 & 0 \\ -1.2 & 2.44 & -1.2 & 0 \\ 0 & -1.2 & 2.44 & -1.2 \\ 0 & 0 & -1.2 & 2.44 \end{pmatrix}.$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.4918 & 1 & 0 & 0 \\ 0.3190 & 0.6487 & 1 & 0 \\ 0.2304 & 0.4685 & 0.7222 & 1 \end{pmatrix}.$$

You may want to check your answer w.r.t.

$$D_4 = \begin{pmatrix} 2.4400 & 0 & 0 & 0 \\ 0 & 1.8498 & 0 & 0 \\ 0 & 0 & 1.6616 & 0 \\ 0 & 0 & 0 & 1.5733 \end{pmatrix}.$$

Good luck!

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SOLUTIONS TO EXAM FOR SF2943 WEDNESDAY 22nd MAY 2013, 14.00–19.00 HRS.

Formulas and survey will be referred to as FS.

Problem 1

(a) According to FS we determine a_1 and a_2 by solving the system of (Wiener-Hopf (WH)) equations

$$\sum_{k=1}^{2} a_k Cov(Y_m, Y_k) = Cov(Y_m, X); m = 1, 2,$$

where $Y_1 = X_1$, $Y_2 = X_3$ and $X = X_2$. Since $\{X_t\}_{t=-\infty}^{\infty}$ is a stationary $AR(1)$ process with zero mean and $\{Z_t\} \sim \mathrm{WN}(0, 1)$ we get

$$Cov(Y_1, Y_1) = Cov(X_1, X_1) = \frac{1}{1 - (-0.25)^2}$$

$$Cov(Y_2, Y_2) = Cov(X_3, X_3) = \frac{1}{1 - (-0.25)^2}$$

$$Cov(Y_1, Y_2) = Cov(X_1, X_3) = \frac{1}{1 - (-0.25)^2}(-0.25),$$

and

$$Cov(Y_1, X) = Cov(X_1, X_2) = \frac{1}{1 - (-0.25)^2}(-0.25)$$

and

$$Cov(Y_2, X) = Cov(X_3, X_2) = \frac{1}{1 - (-0.25)^2}(-0.25).$$

Thus we can multiply everywhere with $1 - 0.25^2$ to get the WH-equations as

$$\begin{cases} a_1 + (-0.25)^2 a_2 = -0.25 \\ (-0.25)^2 a_1 + a_2 = -0.25 \end{cases}$$

The solution is found as

$$a_1 = a_2 = \frac{1}{1 + (-0.25)^2}(-0.25) = -0.2353.$$

ANSWER (a): $\underline{a_1 = a_2 = -0.2353.}$

(b) The value of the optimal mean squared error is by FS

$$\sigma^2_{min} = E\left[\left(X - \widehat{X}\right)^2\right] = E\left[X^2\right] - \sum_{k=1}^{2} a_k Cov(Y_k, X) =$$

$$= \frac{1}{1-0.25^2} - 2\frac{0.2353}{1-(-0.25)^2}0.25 = 0.9412.$$

ANSWER (a): $\underline{\sigma^2_{min} = 0.9412.}$

Problem 2

(a) An MA(1) process defined by

$$X_t = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

has ACVF, see the Collection of Formulas (CF),

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & \text{if } h = 0, \\ \theta\sigma^2 & \text{if } |h| = 1, \\ 0 & \text{if } |h| > 1. \end{cases}$$

The coefficient of correlation between X_t and X_{t-1} is due to stationarity

$$\rho_{X_t, X_{t-1}} = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta}{(1+\theta^2)}.$$

Let us set $\phi(\theta) = \frac{\theta}{(1+\theta^2)}$. Then we have

$$\frac{d}{d\theta}\phi(\theta) = \frac{(1+\theta^2) - 2\theta^2}{(1+\theta^2)} = \frac{1-\theta^2}{(1+\theta^2)}.$$

Thus

$$\frac{d}{d\theta}\phi(\theta) = 0 \Leftrightarrow \theta = \pm 1.$$

We see that $\phi(\theta) < 0$ for $\theta < 0$ and $\phi(\theta) > 0$ for $\theta > 0$. Hence $\theta^* = +1$ gives the maximum for $\rho_{X_t, X_{t-1}}$.

ANSWER (a): $\underline{\theta^* = +1.}$

(b) By CF MA(1) process defined by

$$X_t = Z_t + \theta^* Z_{t-1},$$

has the spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + (\theta^*)^2 + 2\theta^* \cos(\lambda)\right), \quad -\pi \le \lambda \le \pi.$$

When we insert from part (a) of this problem, we get

$$f(\lambda) = \frac{\sigma^2}{2\pi} 2(1 + \cos(\lambda)), \quad -\pi \le \lambda \le \pi.$$

This is clearly a spectrum where the low frequencies (= frequencies around $\lambda = 0$) dominate (draw a graph). Since we are maximizing the coeffient of correlation between X_t and X_{t-1}, we should also expect the process to change slowly.

ANSWER (b): $\underline{f(\lambda) = \frac{\sigma^2}{2\pi} 2(1 + \cos(\lambda)), \quad -\pi \le \lambda \le \pi.}$

Problem 3

(a) We write the linear stochastic difference equation with constant coefficients

$$Y_t - \frac{5}{6}Y_{t-1} = Z_t - \frac{9}{20}Z_{t-1},$$

as

$$\phi(B)Y_t = \theta(B)Z_t,$$

where

$$\phi(B) = 1 - \frac{5}{6}B, \quad \theta(B) = 1 - \frac{9}{20}B.$$

In order to answer the question about ARMA(1,1) we consider zeros of the polynomial $\phi(z)$,

$$\phi(z) = 1 - \frac{5}{6}z,$$

We get that

$$\phi(\xi) = 0 \Leftrightarrow \xi = \frac{6}{5} > 1,$$

Hence $\{Y_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is **stationary** and **causal**.

We see that $\phi(z)$ and $\theta(z)$ have **no common factors/ common zeros** (c.f., definition 3.1.1. in Brockwell and Davis). Hence $\{Y_t \mid t = 0, \pm 1, \pm 2, \ldots\}$ is an ARMA(1,1)-process by definition 3.1.1. in Brockwell and Davis).

Note about grading: The definition of ARMA(p,q) in Brockwell and Davis requires the condition of no common factors in $\phi(z)$ and $\theta(z)$. First the definition of causality and invertibility in the FS pp- 8−9 requires the condition of no common zeros in $\phi(z)$ and $\theta(z)$. A solution of the problem that recognizes the condition of no common zeros at some point gives credit points.

(b) On page 113 Example 11.2 of Lecture Notes we set

$$Y_t = \theta(B)X_t = \begin{pmatrix} -\frac{9}{20} & 1 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_t \end{pmatrix}$$

and observe that X_t satisfies

$$\phi(B)X_t = Z_t.$$

In matrix notation we have, see example 11.1 page 112 of Lecture Notes, for the AR(1) process X_t

$$\begin{pmatrix} X_t \\ X_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & \frac{5}{6} \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_t \end{pmatrix} + \begin{pmatrix} 0 \\ Z_{t+1} \end{pmatrix}$$

Since $\phi(B)$ defines a stationary and causal process, we get that $X_t = \sum_{j=0}^{\infty} \frac{5}{6}^j Z_{t-j}$. Then

$$\begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{5}{6}^j Z_{-j} \\ \sum_{j=0}^{\infty} \frac{5}{6}^j Z_{1-j} \end{pmatrix}$$

and this initial condition of the state process is uncorrelated with the state noise Z_t for $t = 2, \ldots$.

SUMMARY: Thus we have a state space representation with the state vector

$$\begin{pmatrix} X_t \\ X_{t+1} \end{pmatrix}$$

and the state equation

$$\begin{pmatrix} X_t \\ X_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & \frac{5}{6} \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_t \end{pmatrix} + \begin{pmatrix} 0 \\ Z_{t+1} \end{pmatrix}$$

and the observation equation

$$Y_t = \begin{pmatrix} -\frac{9}{20} & 1 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_t \end{pmatrix}$$

with no observation noise. The initial condition is

$$\begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{5}{6}^j Z_{-j} \\ \sum_{j=0}^{\infty} \frac{5}{6}^j Z_{1-j} \end{pmatrix}.$$

Problem 4

We consider by definition of ARIMA(0,1,2) the first difference

$$X_t - X_{t-1}$$

and find whether this is an MA(2) -process. From (1) we get

$$X_t - X_{t-1} = b_t - (\theta_1 + \theta_2) Z_t - \theta_2 Z_{t-1} - b_{t-1} + (\theta_1 + \theta_2) Z_{t-1} + \theta_2 Z_{t-2}$$

$$= (b_t - b_{t-1}) - (\theta_1 + \theta_2) Z_t - \theta_2 Z_{t-1} - + (\theta_1 + \theta_2) Z_{t-1} + \theta_2 Z_{t-2}.$$

Now we use (2) in $(b_t - b_{t-1})$ and get

$$= (1 + \theta_1 + \theta_2) Z_t - (\theta_1 + \theta_2) Z_t - \theta_2 Z_{t-1} + (\theta_1 + \theta_2) Z_{t-1} + \theta_2 Z_{t-2}.$$

$$= Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}.$$

Since $\{Z_t\} \sim \mathrm{WN}(0,1)$, we have now shown that $X_t - X_{t-1}$ is an MA(2) process. Thus $\{X_t\}_{t=0}^{\infty}$ is ARIMA(0,1,2).
The equations (1) and (2) are a special case of the Nelson-Beveridge decomposition of ARIMA(p,1,q). In this decomposition the process $\{X_t\}_{t=0}^{\infty}$ is a sum of a stochastic cyclic trend b_t, which is in (2) seen as a random walk, with random fluctuations around this trend.

Problem 5

(a) In view of (3) we get that

$$\underline{\varepsilon}_n = \underline{X}_n - \underline{\widehat{X}}_n.$$

But from

$$\widehat{X}_t = P_{t-1} X_t = a_{t-1,1} X_{t-1} + a_{t-1,2} X_{t-2} + \dots a_{t-1,t-1} X_1.$$

and the expression for $\mathbf{A}_n$ we get that

$$\underline{\varepsilon}_n = \mathbf{A}_n \underline{X}_n.$$

Hence

$$\underline{X}_n = \mathbf{A}_n^{-1} \underline{\varepsilon}_n = \mathbf{C}_n \underline{\varepsilon}_n.$$

Thus

$$\begin{aligned}\Gamma_n = E\left[\underline{X}_n \underline{X}_n^T\right] &= E\left[\mathbf{C}_n \underline{\varepsilon}_n \underline{\varepsilon}_n^T \mathbf{C}_n^T\right] \\ &= \mathbf{C}_n E\left[\underline{\varepsilon}_n \underline{\varepsilon}_n^T\right] \mathbf{C}_n^T \\ &= \mathbf{C}_n \mathbf{D}_n \mathbf{C}_n^T.\end{aligned}$$

Here we used the fact that the innovations constitute a white noise and that $v_{t-1} = E\left[\varepsilon_t^2\right]$ so that

$$\mathbf{D}_n = \begin{pmatrix} v_0 & 0 & 0 & \dots & 0 \\ 0 & v_1 & 0 & \dots & 0 \\ 0 & 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & v_{n-1} \end{pmatrix}.$$

(b) By

$$\underline{\varepsilon}_n = \underline{X}_n - \underline{\widehat{X}}_n.$$

we get that

$$\underline{\widehat{X}}_n = \underline{X}_n - \underline{\varepsilon}_n.$$

From (a) we have

$$= \mathbf{C}_n \underline{\varepsilon}_n - \underline{\varepsilon}_n = (\mathbf{C}_n - \mathbf{I}_n)\, \underline{\varepsilon}_n,$$

where $\mathbf{I}_n$ is the $n \times n$ unit matrix. In other words

$$\underline{\widehat{X}}_n = (\mathbf{C}_n - \mathbf{I}_n)\, \underline{\varepsilon}_n.$$

$$\text{ANSWER (b): } \underline{\underline{\widehat{X}}_n = (\mathbf{C}_n - \mathbf{I}_n)\, \underline{\varepsilon}_n.}$$

(c) From (b) we get

$$\underline{\widehat{X}}_4 = (\mathbf{C}_4 - \mathbf{I}_4)\, \underline{\varepsilon}_4.$$

With

$$C_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.4918 & 1 & 0 & 0 \\ 0 & -0.6487 & 1 & 0 \\ 0 & 0 & -0.7222 & 1 \end{pmatrix}.$$

we have

$$\underline{\widehat{X}}_4 = \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \\ \widehat{X}_3 \\ \widehat{X}_4 \end{pmatrix} = (C_4 - \mathbf{I}_4) \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4918 & 0 & 0 & 0 \\ 0 & -0.6487 & 0 & 0 \\ 0 & 0 & -0.7222 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$

or

$$\text{ANSWER (c): } \underline{\widehat{X}_1 = 0, \widehat{X}_2 = -0.4918\varepsilon_1, \widehat{X}_3 = -0.6487\varepsilon_2, \widehat{X}_4 = -0.7222\varepsilon_3.}$$

(d) We have found above that

$$\underline{\varepsilon}_n = \underline{X}_n - \underline{\widehat{X}}_n$$

and

$$\underline{\varepsilon}_n = \mathbf{A}_n \underline{X}_n.$$

Thus

$$\mathbf{A}_n \underline{X}_n = \underline{X}_n - \underline{\widehat{X}}_n$$

or

$$\underline{\widehat{X}}_n = \underline{X}_n - \mathbf{A}_n \underline{X}_n$$

$\Leftrightarrow$

$$\underline{\widehat{X}}_n = (\mathbf{I}_n - \mathbf{A}_n)\, \underline{X}_n.$$

Then

$$\underline{\widehat{X}}_4 = \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \\ \widehat{X}_3 \\ \widehat{X}_4 \end{pmatrix} = (\mathbf{I}_4 - \mathbf{A}_4)\, \underline{X}_4$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4918 & 0 & 0 & 0 \\ -0.3190 & -0.6487 & 0 & 0 \\ -0.2304 & -0.4685 & -0.7222 & 0 \end{pmatrix} \underline{X}_4$$

Thus

$$\widehat{X}_1 = 0, \widehat{X}_2 = -0.4918X_1, \widehat{X}_3 = -0.3190X_1 - 0.6487X_2.$$

Now

$$v_1 = E\left[\left(X_2 - \widehat{X}_2\right)^2\right]$$
$$= E\left[(X_2 + 0.4918X_1)^2\right].$$
$$= E\left[X_2^2\right] + 0.4918^2 E\left[X_1^2\right] + 2 \cdot 0.4918 \cdot E\left[X_2 X_1\right].$$

We recall that $\{X_t\}_{t=0}^{\infty}$ is as in (c). Then (see FS)

$$\Gamma_4 = \begin{pmatrix} 2.44 & -1.2 & 0 & 0 \\ -1.2 & 2.44 & -1.2 & 0 \\ 0 & -1.2 & 2.44 & -1.2 \\ 0 & 0 & -1.2 & 2.44 \end{pmatrix}$$

and $E\left[X_2^2\right] = E\left[X_1^2\right] = 2.44$, $E\left[X_2 X_1\right] = -1.2$. Then

$$v_1 = 2.44 + 0.4918^2 - 2 \cdot 0.4918 \cdot 1.2 = 1.8498,$$

as it should,, c.f., D_4.

$$v_2 = E\left[\left(X_3 - \widehat{X}_3\right)^2\right]$$

$$= E\left[(X_3 + 0.3190X_1 + 0.6487X_2)^2\right].$$

$$= E\left[X_3^2\right] + 0.3190^2 E\left[X_1^2\right] + 0.6487^2 E\left[X_2^2\right] + 2 \cdot 0.3190 \cdot E\left[X_3X_1\right]$$

$$+2 \cdot 0.6487 \cdot E\left[X_3X_2\right] + 2 \cdot 0.3190 \cdot 0.6487 \cdot E\left[X_1X_2\right]$$

From Γ_4 we get $E\left[X_3X_1\right] = 0$ and

$$E\left[X_3^2\right] = E\left[X_2^2\right] = E\left[X_1^2\right] = 2.44$$

and

$$E\left[X_3X_2\right] = E\left[X_1X_2\right] = -1.2.$$

Thus

$$v_2 = 2.44 + 0.3190^2 \cdot 2.44 + 0.6487^2 \cdot 2.44 - 2 \cdot 0.6487 \cdot 1.2 - 2 \cdot 0.3190 \cdot 0.6487 \cdot 1.2 = 1.6616,$$

as it should, c.f., D_4.

ANSWER (d): $\underline{v_1 = 1.8498, v_2 = 1.6616.}$