Avd. Matematisk statistik

KTH Matematik

EXAM FOR SF2943 TIME SERIES ANALYSIS/TIDSSERIEANALYS
MONDAY 26th AUGUST 2013, 08.00–13.00 HRS.

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Allowed aids: Formulas and survey, Jan Grandell: Time series analysis (without added notes!). Pocket calculator.

Notation introduced should be defined and explained. Solutions, arguments and calculations must be clear and motivated well enough to make them easy to follow.

Each correct solution counts for 10 points. Pass (grade E) requires 25 points. Students who obtain 23 or 24 points will be offered the option to do an additional small exam to possibly raise their grade to E. Students wanting to take this option must contact the examiner within a week after the results from the exam have been published.

Solutions in Swedish are of course welcome!

The exam will be marked no later than within three weeks, and the results will be available through *Mina sidor*.

Problem 1

Let $\{X_t\}_{t=-\infty}^{\infty}$ is the $MA(1)$ process given by

$$X_t = Z_t + \frac{1}{2} Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0,1).$$

We observe the time series at times $t = 1$ and $t = 3$, but the value at $t = 2$ is missed. We wish to find the linear combination of X_1 and X_3 that estimates X_2 with minimal mean square error. We denote the estimate by

$$\widehat{X}_2 = a_1 X_1 + a_2 X_3.$$

(a) Find a_1 and a_2 so that $E\left[\left(X_2 - \widehat{X}_2\right)^2\right]$ is minimized. (5 p)

(b) Find the value of the optimal mean squared error. (5 p)

Problem 2

The process $\{X_t, t \in \mathbb{Z}\}$ is a stationary ARMA(1,1) -process

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1},$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) Show that

$$Z_t = X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}.$$

Which assumptions are needed so that this representation is possible? Show next that

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} (\phi)^{j-1} Z_{t-j}.$$

Which assumptions are needed so that this representation is possible? (6 p)

(b) The conditional expectation

$$\widehat{X}_{t+1} \stackrel{\text{def}}{=} E\left[X_{t+1} \mid (X_i)_{i=-\infty}^{t}\right]$$

is the optimal mean square predictor of X_{t+1} based on the infinite past of the process $\{X_t, t \in \mathbb{Z}\}$. Verify that

$$X_{t+1} = \widehat{X}_{t+1} + Z_{t+1}.$$

Justify your computations carefully. (4 p)

Problem 3

The process $\{X_t, t \in \mathbb{Z}\}$ is a stationary and causal AR(1) -process

$$X_t - \phi X_{t-1} = Z_t,$$

where $Z_t \sim \mathrm{WN}(0, \sigma_z^2)$. We observe $\{X_t, t \in \mathbb{Z}\}$ with additive white noise

$$Y_t = X_t + W_t,$$

where $W_t \sim \mathrm{WN}(0, \sigma_w^2)$, and $E\left[W_s Z_l\right] = 0$ for each s and l.

(a) Find the autocovariance function of $\{Y_t, t \in \mathbb{Z}\}$. (3 p)

(b) Define

$$U_t \stackrel{\text{def}}{=} Y_t - \phi Y_{t-1}.$$

Show that $\{U_t, t \in \mathbb{Z}\}$ is an MA(1)-process. *Hint*: It may be helpful to study the autocovariance function of $\{U_t, t \in \mathbb{Z}\}$. Justify your solution carefully. (5 p)

(c) Represent now the process $\{Y_t, t \in \mathbb{Z}\}$ as an ARMA(1,1) -process and find its parameters. (2 p)

Problem 4

The process $\{X_t, t \in \mathbb{Z}\}$ is a stationary AR(1) -process

$$X_t - \phi X_{t-1} = Z_t,$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$. We observe $\{X_t, t \in \mathbb{Z}\}$ with additive white noise

$$Y_t = X_t + W_t,$$

where $W_t \sim \mathrm{WN}(0, \sigma^2)$. $\{Z_t, t \in \mathbb{Z}\}$ and $\{W_t, t \in \mathbb{Z}\}$ are independent white noises.

(a) Show that the spectral density of $\{Y_t, t \in \mathbb{Z}\}$ is

$$f_Y(\lambda) = f_X(\lambda) + f_W(\lambda),$$

where $f_X(\lambda)$ and $f_W(\lambda)$ are the spectral densities of $\{X_t, t \in \mathbb{Z}\}$ and $\{W_t, t \in \mathbb{Z}\}$, respectively. (5 p)

(b) Show that the spectral density of $\{Y_t, t \in \mathbb{Z}\}$ is

$$f_Y(\lambda) = \frac{\sigma^2}{2\pi}\frac{2 + \phi^2 - 2\phi\cos(\lambda)}{1 + \phi^2 - 2\phi\cos(\lambda)}, \quad -\pi \leq \lambda \leq \pi.$$

(5 p)

Problem 5

$\{X_t\}_{t=0}^{\infty}$ is a stationary time series. Let

$$\widehat{X}_t = \begin{cases} 0 & t = 1 \\ P_{t-1}X_t & t = 2, 3, \ldots \end{cases}$$

where $P_{t-1}X_t$ is the optimal projection of X_t onto the linear span of $X_{t-1}, \ldots, X_1$. In other words we have that

$$\widehat{X}_t \stackrel{\text{def}}{=} P_{t-1}X_t = a_{t-1,1}X_{t-1} + a_{t-1,2}X_{t-2} + \ldots a_{t-1,t-1}X_1.$$

The innovations are the one-step prediction errors defined as

$$\varepsilon_t \stackrel{\text{def}}{=} X_t - \widehat{X}_t. \tag{1}$$

We know that the innovations constitute a white noise (you need not prove this) and we write

$$v_{t-1} = E\left[\left(X_t - \widehat{X}_t\right)^2\right].$$

We introduce

$$\underline{\varepsilon}_n = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_{n-1} \\ \varepsilon_n \end{pmatrix} \quad \underline{X}_n = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} \quad \underline{\widehat{X}}_n = \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \\ \vdots \\ \widehat{X}_{n-1} \\ \widehat{X}_n \end{pmatrix}$$

and

$$\mathbf{A}_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -a_{1,1} & 1 & 0 & \dots & 0 \\ -a_{2,2} & -a_{2,1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -a_{n-1,n-1} & -a_{n-1,n-2} & -a_{n-1,n-3} & \dots & 1 \end{pmatrix}.$$

(a) Let $\mathbf{C}_n = \mathbf{A}_n^{-1}$. The matrix $\mathbf{C}_n$ is lower triangular with ones on the main diagonal, too. Show that the covariance matrix of $\underline{X}_n$ is given by

$$\Gamma_n = E\left[\underline{X}_n \underline{X}_n^T\right] = \mathbf{C}_n \mathbf{D}_n \mathbf{C}_n^T,$$

where

$$\mathbf{D}_n = \begin{pmatrix} v_0 & 0 & 0 & \dots & 0 \\ 0 & v_1 & 0 & \dots & 0 \\ 0 & 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & v_{n-1} \end{pmatrix}.$$

Justify your solution. *Aid:* Apply first (1) to $\underline{\varepsilon}_n$. (2 p)

(b) Express $\widehat{\underline{X}}_n$ in terms of $\underline{\varepsilon}_n$ via a matrix operation. (1 p)

(c) The stationary process $\{X_t\}_{t=-\infty}^{\infty}$ has zero mean function and

$$\Gamma_4 = \begin{pmatrix} 0.1648 & 0.0495 & 0.0148 & 0.0045 \\ 0.0495 & 0.1648 & 0.0495 & 0.0148 \\ 0.0148 & 0.0495 & 0.1648 & 0.0495 \\ 0.0045 & 0.0148 & 0.0495 & 0.1648 \end{pmatrix}. \qquad (2)$$

Check that this matrix has the necessary properties of a covariance matrix for a segment $\underline{X}_4$ of a stationary time series. You need not consider non-negative definiteness. (2 p)

(d) For the matrix Γ_4 in (2) we have

$$C_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.3000 & 1 & 0 & 0 \\ 0.09 & 0.3000 & 1 & 0 \\ 0.0270 & 0.0900 & 0.3000 & 1 \end{pmatrix}.$$

Find $\widehat{\underline{X}}_4$ in terms of $\underline{X}_4$. (5 p)

Good luck!

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SOLUTIONS TO EXAM FOR SF2943 MONDAY 26th AUGUST 2013, 08.00–13.00 HRS.

Formulas and survey will be referred to as FS.

Problem 1

(a) According to FS we determine a_1 and a_2 by solving the system of (Wiener-Hopf (WH)) equations

$$\sum_{k=1}^{2} a_k Cov(Y_m, Y_k) = Cov(Y_m, X); m = 1, 2,$$

where $Y_1 = X_1$, $Y_2 = X_3$ and $X = X_2$. Since $\{X_t\}_{t=-\infty}^{\infty}$ is an MA(1) process and $\{Z_t\} \sim \text{WN}(0,1)$ its ACVF, see the Collection of Formulas (CF), is

$$\gamma(h) = \begin{cases} \left(1 + \frac{1}{4}\right) & \text{if } h = 0, \\ \frac{1}{2} & \text{if } |h| = 1, \\ 0 & \text{if } |h| > 1. \end{cases}$$

Then we obtain

$$Cov(Y_1, Y_1) = Cov(X_1, X_1) = \frac{5}{4},$$

$$Cov(Y_2, Y_2) = Cov(X_3, X_3) = \frac{5}{4},$$

$$Cov(Y_1, Y_2) = Cov(X_1, X_3) = 0,$$

and

$$Cov(Y_1, X) = Cov(X_1, X_2) = \frac{1}{2},$$

and

$$Cov(Y_2, X) = Cov(X_3, X_2) = \frac{1}{2}.$$

Thus we get the WH-equations as

$$\begin{cases} \frac{5}{4} \cdot a_1 + 0 \cdot a_2 = \frac{1}{2} \\ 0 \cdot a_1 + \frac{5}{4} \cdot a_2 = \frac{1}{2} \end{cases}$$

The solution is found as

$$a_1 = a_2 = \frac{2}{5}.$$

ANSWER (a): $\underline{a_1 = a_2 = \frac{2}{5}.}$

(b) The value of the optimal mean squared error is by FS

$$\sigma^2_{min} = E\left[\left(X - \widehat{X}\right)^2\right] = E\left[X^2\right] - \sum_{k=1}^{2} a_k Cov(Y_k, X) =$$

$$= \frac{5}{4} - \frac{2}{5}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{5}{4} - \frac{2}{5} = \frac{17}{20}.$$

ANSWER (a): $\underline{\sigma^2_{min} = \frac{17}{20}}$.

Problem 2

(a) We must assume **invertibility**, i.e., $\mid \theta \mid < 1$. Then we can write

$$Z_t = \frac{\phi(B)}{\theta(B)} X_t$$

where $\phi(B) = 1 - \phi B$ and $\theta(B) = 1 + \theta B$. Then

$$\frac{1}{\theta(B)} = \frac{1}{1 + \theta B} = \sum_{j=0}^{\infty} (-\theta)^j B^j$$

Then

$$\frac{\phi(B)}{\theta(B)} = (1 - \phi B) \sum_{j=0}^{\infty} (-\theta)^j B^j =$$

$$= 1 + \sum_{j=1}^{\infty} (-\theta)^j B^j - \phi \sum_{j=0}^{\infty} (-\theta)^j B^{j+1}$$

We change the variable of summation in the last sum, $l = j + 1$, and obtain

$$= 1 + \sum_{j=1}^{\infty} (-\theta)^j B^j - \phi \sum_{l=1}^{\infty} (-\theta)^{l-1} B^l$$

$$= 1 + (-\theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} B^j - \phi \sum_{j=1}^{\infty} (-\theta)^{j-1} B^j$$

$$= 1 - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} B^j.$$

Thus

$$Z_t = \frac{\phi(B)}{\theta(B)} X_t = \left(1 - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} B^j\right) X_t$$

$$= X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}.$$

We need also that $(\phi + \theta) \neq 0$, **no common factors**.

Next we assume **causality** , i.e., $\mid \phi \mid < 1$. We can then write

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t =$$

where $\theta(B) = 1 + \theta B$ and $\frac{1}{\phi(B)} = \sum_{j=0}^{\infty} \phi^j B^j$. Then

$$\frac{\theta(B)}{\phi(B)} = \sum_{j=0}^{\infty} \phi^j B^j \cdot (1 + \theta B) =$$

$$= 1 + \sum_{j=1}^{\infty} \phi^j B^j + \theta B \sum_{j=0}^{\infty} \phi^j B^j$$

$$= 1 + \sum_{j=1}^{\infty} \phi^j B^j + \theta \sum_{j=0}^{\infty} \phi^j B^{j+1}$$

$$= 1 + \sum_{j=1}^{\infty} \phi^j B^j + \theta \sum_{j=1}^{\infty} \phi^{j-1} B^j$$

where we made a change of variable of summation,

$$= 1 + \phi \sum_{j=1}^{\infty} \phi^{j-1} B^j + \theta \sum_{j=1}^{\infty} \phi^{j-1} B^j$$

Thus

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t = Z_t + \left(\phi \sum_{j=1}^{\infty} \phi^{j-1} B^j + \theta \sum_{j=1}^{\infty} \phi^{j-1} B^j \right) Z_t$$

$$= Z_t + (\phi + \theta) \sum_{j=1}^{\infty} (\phi)^{j-1} Z_{t-j},$$

as was to be shown. We need obviously in addition to assume that $(\phi + \theta) \neq 0$, **no common factors**.

(b) We observe from the part (a) that

$$X_t + 1 = Z_{t+1} + (\phi + \theta) \sum_{j=1}^{\infty} (\phi)^{j-1} Z_{t+1-j},$$

The conditional expectation is then by linearity of conditional expectation/projection

$$\widehat{X}_{t+1} = E\left[X_{t+1} \mid (X_i)_{i=-\infty}^t\right]$$

$$= E\left[Z_{t+1} \mid (X_i)_{i=-\infty}^t\right] + E\left[(\phi + \theta) \sum_{j=1}^{\infty} (\phi)^{j-1} Z_{t+1-j} \mid (X_i)_{i=-\infty}^t\right].$$

By definition of white noise and causality

$$E\left[Z_{t+1} \mid (X_i)_{i=-\infty}^t\right] = E\left[Z_{t+1}\right] = 0.$$

Furhtermore, as we can write any Z_t as

$$Z_t = X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}.$$

we get

$$E\left[(\phi + \theta) \sum_{j=1}^{\infty} (\phi)^{j-1} Z_{t+1-j} \mid (X_i)_{i=-\infty}^{t}\right] = (\phi + \theta) \sum_{j=1}^{\infty} (\phi)^{j-1} Z_{t+1-j}.$$

i.e.,

$$\widehat{X}_{t+1} = (\phi + \theta) \sum_{j=1}^{\infty} (\phi)^{j-1} Z_{t+1-j}.$$

In the same manner we get by the part (a) that

$$X_{t+1} = Z_{t+1} + (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}.$$

so that by the same properties of the conditional expctation as before

$$\widehat{X}_{t+1} = (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}.$$

Hence we see that, e.g,

$$Z_{t+1} = X_{t+1} - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t+1-j}.$$

$$\Leftrightarrow$$

$$X_{t+1} = \widehat{X}_{t+1} + Z_{t+1}$$

as was to be proved.

Problem 3

(a) We try to compute

$$\begin{aligned} E[Y_t Y_{t+h}] &= E[(X_t + W_t)(X_{t+h} + W_{t+h})] \\ &= E[X_t X_{t+h}] + E[W_t X_{t+h}] + E[X_t W_{t+h}] + E[W_t W_{t+h}] \\ &= \gamma_X(h) + E[W_t X_{t+h}] + E[X_t W_{t+h}] + \gamma_W(h), \end{aligned}$$

where $\gamma_X(h)$ and $\gamma_W(h)$ are the ACVFs of $\{X_t, t \in \mathbb{Z}\}$ and $\{W_t, t \in \mathbb{Z}\}$, respectively. By causality X_t it holds that

$$X_t = \frac{1}{\phi(B)} Z_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}.$$

and thus

$$E[W_t X_{t+h}] = \sum_{j=0}^{\infty} \phi^j E[W_t Z_{t+h-j}] = 0,$$

by assumption that $E[W_s Z_l] = 0$ for each s and l. Therefore we also have

$$E[W_{t+h} X_t] = \sum_{j=0}^{\infty} \phi^j E[W_{t+h} Z_{t-j}] = 0.$$

Thus $E[Y_t Y_{t+h}] = \gamma_X(h) + \gamma_W(h)$, and we have in addition shown that $\{Y_t, t \in \mathbb{Z}\}$ is a stationary process. Let us set $\gamma_Y(h) = E[Y_t Y_{t+h}]$.

ANSWER (a): $\underline{\gamma_Y(h) = \gamma_X(h) + \gamma_W(h).}$

(b) We have defined

$$U_t = Y_t - \phi Y_{t-1}.$$

Let us insert from the above

$$Y_t - \phi Y_{t-1} = X_t + W_t - \phi(X_{t-1} + W_{t-1})$$

$$= X_t - \phi X_{t-1} + W_t - \phi W_{t-1} = Z_t + W_t - \phi W_{t-1} = Z_t + S_t,$$

where we have set $S_t \stackrel{\text{def}}{=} W_t - \phi W_{t-1}$. We study now the ACVF of the U_ts. Since $\{U_t, t \in \mathbb{Z}\}$ is obtained by linear, time invariant filtering of the stationary process $\{Y_t, t \in \mathbb{Z}\}$, it is a stationary time series. We compute thus

$$\gamma_U(h) = E[U_t U_{t+h}] = E[Z_t Z_{t+h}] + E[Z_t S_{t+h}] + E[S_t Z_{t+h}] + E[S_t S_{t+h}].$$

Here $E[Z_t Z_{t+h}] = \sigma_z^2 \delta(h)$, where $\delta(h)$ is Kronecker's delta,

$$E[Z_t S_{t+h}] = E[S_t Z_{t+h}] = 0.$$

since $E[W_s Z_l] = 0$ for each s and l. We see that since $\{W_t, t \in \mathbb{Z}\}$ is a white noise,

$$S_t = W_t - \phi W_{t-1}$$

defines an MA(1)- process. Hence by FS

$$E[S_t S_{t+h}] = \gamma_S(h) = \begin{cases} (1+\phi^2)\sigma_w^2 & \text{if } h = 0, \\ -\phi\sigma_w^2 & \text{if } |h| = 1, \\ 0 & \text{if } |h| > 1. \end{cases}$$

Hence

$$\gamma_U(h) == E[Z_t Z_{t+h}] + E[S_t S_{t+h}] = \sigma_z^2 \delta(h) + \gamma_S(h)$$

$$= \begin{cases} \sigma_z^2 + (1+\phi^2)\sigma_w^2 & \text{if } h = 0, \\ -\phi\sigma_w^2 & \text{if } |h| = 1, \\ 0 & \text{if } |h| > 1. \end{cases}$$

This ACVF has the 1-dependence of an MA(1)-process, which verifies the claim.

(c) We know now that the process $\{Y_t, t \in \mathbb{Z}\}$ is an ARMA(1,1) -process,

$$Y_t - \phi Y_{t-1} = U_t$$

We need to find θ and σ_v^2 so that

$$Y_t - \phi Y_{t-1} = V_t + \theta V_{t-1}.$$

where $V_t \sim \mathrm{WN}(0, \sigma_v^2)$. Then it must hold that

$$\begin{cases} (1+\theta^2)\sigma_v^2 & \text{if } h = 0, \\ \theta\sigma_v^2 & \text{if } |h| = 1, \\ 0 & \text{if } |h| > 1. \end{cases} = \begin{cases} \sigma_z^2 + (1+\phi^2)\sigma_w^2 & \text{if } h = 0, \\ -\phi\sigma_w^2 & \text{if } |h| = 1, \\ 0 & \text{if } |h| > 1. \end{cases}$$

This yields

$$\sigma_v^2 = -\phi\sigma_w^2/\theta,$$

and (by some algebra) that θ must satisfy the equation

$$-\phi\sigma_w^2\theta^2 - \left(\sigma_z^2 + (1+\phi^2)\sigma_w^2\right)\theta - \phi\sigma_w^2 = 0$$

Problem 4

(a) By definition of spectral density we have

$$f_Y(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_Y(h).$$

where $\gamma_Y(h)$ is the ACVF of the time series $\{Y_t, t \in \mathbb{Z}\}$. By construction we take $\{Y_t, t \in \mathbb{Z}\}$ as a stationary time series and compute

$$\begin{aligned}\gamma_Y(h) = E\left[Y_t Y_{t+h}\right] = \gamma_Y(h) = E\left[\left(X_t + W_t\right)\left(X_{t+h} + W_{t+h}\right)\right] \\ = E\left[X_t X_{t+h}\right] + E\left[W_t X_{t+h}\right] + E\left[\left(X_t W_{t+h}\right] + E\left[W_t W_{t+h}\right] \\ = \gamma_X(h) + E\left[W_t X_{t+h}\right] + E\left[\left(X_t W_{t+h}\right] + \gamma_W(h),\end{aligned}$$

where $\gamma_X(h)$ and $\gamma_W(h)$ are the ACVFs of $\{X_t, t \in \mathbb{Z}\}$ and $\{W_t, t \in \mathbb{Z}\}$, respectively.

Now, since $\{Z_t, t \in \mathbb{Z}\}$ and $\{W_t, t \in \mathbb{Z}\}$ are independent white noises and X_t is a function of $\{Z_t, t \in \mathbb{Z}\}$, we get that

$$E\left[W_t X_{t+h}\right] = E\left[\left(X_t W_{t+h}\right] = 0.$$

Thus

$$\gamma_Y(h) = \gamma_X(h) + \gamma_W(h)$$

and

$$\begin{aligned} f_Y(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_Y(h) \\ = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_X(h) + \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_W(h) \\ = f_X(\lambda) + f_W(\lambda), \end{aligned}$$

as claimed.

(b) From FS we have for stationary AR(1) with $\mid \phi \mid < 1$ and WN$(0, \sigma^2)$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{1 + \phi^2 - 2\phi\cos(\lambda)},$$

and

$$f_W(\lambda) = \frac{\sigma^2}{2\pi},$$

for white noise WN$(0, \sigma^2)$. Then

$$f_Y(\lambda) = f_X(\lambda) + f_W(\lambda) =$$

$$= \frac{\sigma^2}{2\pi} \frac{2 + \phi^2 - 2\phi\cos(\lambda)}{1 + \phi^2 - 2\phi\cos(\lambda)}.$$

Problem 5

(a) By (1) we see that

$$\underline{\varepsilon}_n = \underline{X}_n - \underline{\widehat{X}}_n.$$

But in view of

$$\widehat{X}_t = P_{t-1}X_t = a_{t-1,1}X_{t-1} + a_{t-1,2}X_{t-2} + \ldots a_{t-1,t-1}X_1.$$

and the expression for $\mathbf{A}_n$ we get that

$$\underline{\varepsilon}_n = \mathbf{A}_n \underline{X}_n.$$

Hence

$$\underline{X}_n = \mathbf{A}_n^{-1} \underline{\varepsilon}_n = \mathbf{C}_n \underline{\varepsilon}_n.$$

Thus

$$\begin{aligned}\Gamma_n = E\left[\underline{X}_n \underline{X}_n^T\right] &= E\left[\mathbf{C}_n \underline{\varepsilon}_n \underline{\varepsilon}_n^T \mathbf{C}_n^T\right] \\ &= \mathbf{C}_n E\left[\underline{\varepsilon}_n \underline{\varepsilon}_n^T\right] \mathbf{C}_n^T \\ &= \mathbf{C}_n \mathbf{D}_n \mathbf{C}_n^T.\end{aligned}$$

Here we used the fact that the innovations constitute a white noise and that $v_{t-1} = E\left[\varepsilon_t^2\right]$ so that

$$\mathbf{D}_n = \begin{pmatrix} v_0 & 0 & 0 & \ldots & 0 \\ 0 & v_1 & 0 & \ldots & 0 \\ 0 & 0 & v_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & v_{n-1} \end{pmatrix}.$$

(b) By
$$\underline{\varepsilon}_n = \underline{X}_n - \underline{\widehat{X}}_n.$$
we get that
$$\underline{\widehat{X}}_n = \underline{X}_n - \underline{\varepsilon}_n.$$
From (a) we have
$$= \mathbf{C}_n \underline{\varepsilon}_n - \underline{\varepsilon}_n = (\mathbf{C}_n - \mathbf{I}_n)\,\underline{\varepsilon}_n,$$
where $\mathbf{I}_n$ is the $n \times n$ unit matrix. In other words
$$\underline{\widehat{X}}_n = (\mathbf{C}_n - \mathbf{I}_n)\,\underline{\varepsilon}_n.$$
$$\text{ANSWER (b): } \underline{\underline{\widehat{X}}_n = (\mathbf{C}_n - \mathbf{I}_n)\,\underline{\varepsilon}_n.}$$

(c) The matrix Γ_4 is symmetric, the elements on all the left-to-right diagonals are equal to each other, as required by stationarity. The elements ($= 0.1648$) on the main diagonal are non-negative and 0.1648 is bigger than all the other values in the matrix.

(d) We have found above that
$$\underline{\varepsilon}_n = \underline{X}_n - \underline{\widehat{X}}_n$$
and
$$\underline{\varepsilon}_n = \mathbf{A}_n \underline{X}_n.$$
Thus
$$\mathbf{A}_n \underline{X}_n = \underline{X}_n - \underline{\widehat{X}}_n$$
or
$$\underline{\widehat{X}}_n = \underline{X}_n - \mathbf{A}_n \underline{X}_n$$
$\Leftrightarrow$
$$\underline{\widehat{X}}_n = (\mathbf{I}_n - \mathbf{A}_n)\,\underline{X}_n.$$
Then
$$\underline{\widehat{X}}_4 = \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \\ \widehat{X}_3 \\ \widehat{X}_4 \end{pmatrix} = (\mathbf{I}_4 - \mathbf{A}_4)\,\underline{X}_4$$
For the matrix Γ_4 in (2) we are given
$$C_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.3000 & 1 & 0 & 0 \\ 0.09 & 0.3000 & 1 & 0 \\ 0.0270 & 0.0900 & 0.3000 & 1 \end{pmatrix}$$
and we must consequently invert this matrix to get $\mathbf{A}_4$. Some electronic computing power gives now
$$A_4 = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ -0.3000 & 1.0000 & 0 & 0 \\ 0.0000 & -0.3000 & 1.0000 & 0 \\ 0.0000 & 0 & -0.3000 & 1.0000 \end{pmatrix}.$$

Hence we get

$$\mathbf{I}_4 - \mathbf{A}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{pmatrix}.$$

Hence

$$\text{ANSWER (d): } \underline{\widehat{X}_1 = 0, \widehat{X}_i = 0.3X_{i-1}, \quad i = 1, 2, 3.}$$