KTH Mathematics

Examination in SF2943 Time Series Analysis, June 4, 2014, 14:00–19:00.

Examiner: Filip Lindskog, tel. 790 7217, e-mail: lindskog@kth.se

Allowed aids: Pocket calculator, “Formulas and survey, Time series analysis” by Jan Grandell, without notes.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

The formulas $\sum_{k=1}^{n} k = n(n+1)/2$ and $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ may be useful.

- -

Problem 1

A person records the observations of a stationary time series $\{X_t\}$ with zero mean and ACVF $\gamma_X(h)$. At each time t, with probability p the person erroneously records the value of the process at time $t-1$ instead of time t. This happens independent for each time t and the probability p does not vary over time.

(a) Show that the actually recorded data series $\{Y_t\}$ is stationary and compute its ACVF $\gamma_Y(h)$. (5 p)

(b) Suppose that $\{X_t\}$ is a causal AR(1) process with an ACF satisfying $\rho_X(1) = 0.5$. Compute the ACF $\rho_Y(h)$ for the value of p that maximizes $\rho_Y(h)$. (5 p)

Problem 2

A company wants to do accurate one-step prediction of a causal AR(1) process $\{X_t\}$. However, due to imperfect measurement equipment the company only manages to make predictions of X_t based on noisy observations Y_s for $s < t$. It is assumed that

$$\begin{aligned} X_t &= 0.5X_{t-1} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0,1), \\ Y_t &= X_t + W_t, \quad \{W_t\} \sim \mathrm{IID}(0, 0.5^2), \end{aligned}$$

where the noise sequences $\{Z_t\}$ and $\{W_t\}$ are independent. The company predicts X_t with the best linear predictor (minimizing the mean squared prediction error) based on the most recent observation Y_{t-1}. An employee claims that buying a new measurement equipment that gives no measurement error would reduce the mean squared prediction error by more than 10%. Is this claim correct? (10 p)

Problem 3

Figure 1 shows spectral densities of the four time series:

$$X_t = -0.5X_{t-1} + Z_t, \tag{1}$$
$$X_t = 0.5X_{t-1} + 0.2X_{t-2} + Z_t, \tag{2}$$
$$X_t = 0.3X_{t-1} - 0.7X_{t-2} + Z_t, \tag{3}$$
$$X_t = Z_t + 0.7Z_{t-1}, \tag{4}$$

where $\{Z_t\} \sim \mathrm{WN}(0,1)$. For each of the time series, determine which spectral density corresponds to that time series. (10 p)

Problem 4

Based on a sample of size 500 from a stationary time series, the values of the sample autocovariance function for lags $0, 1, \ldots, 5$ and the values of the sample autocorrelation function and sample partial autocorrelation function for lags $1, 2, \ldots, 6$ were found to be

```
ACVF  1.670 -0.968  0.261  0.048 -0.016 -0.034
ACF  -0.570  0.154  0.029 -0.010 -0.020  0.009
PACF -0.570 -0.253 -0.010  0.072  0.023 -0.015
```

The sample ACF and the sample PACF are shown in Figure 2. Suggest a time series model and estimate its parameters. (10 p)

Problem 5

A person with little knowledge of time series analysis wants to understand the time series data shown in the left plot in Figure 3. However, the sample ACVF shown in the right plot in Figure 3 does not resemble any of the plots found in the textbook. Suggest a time series model for the data series and estimate its parameters. (10 p)

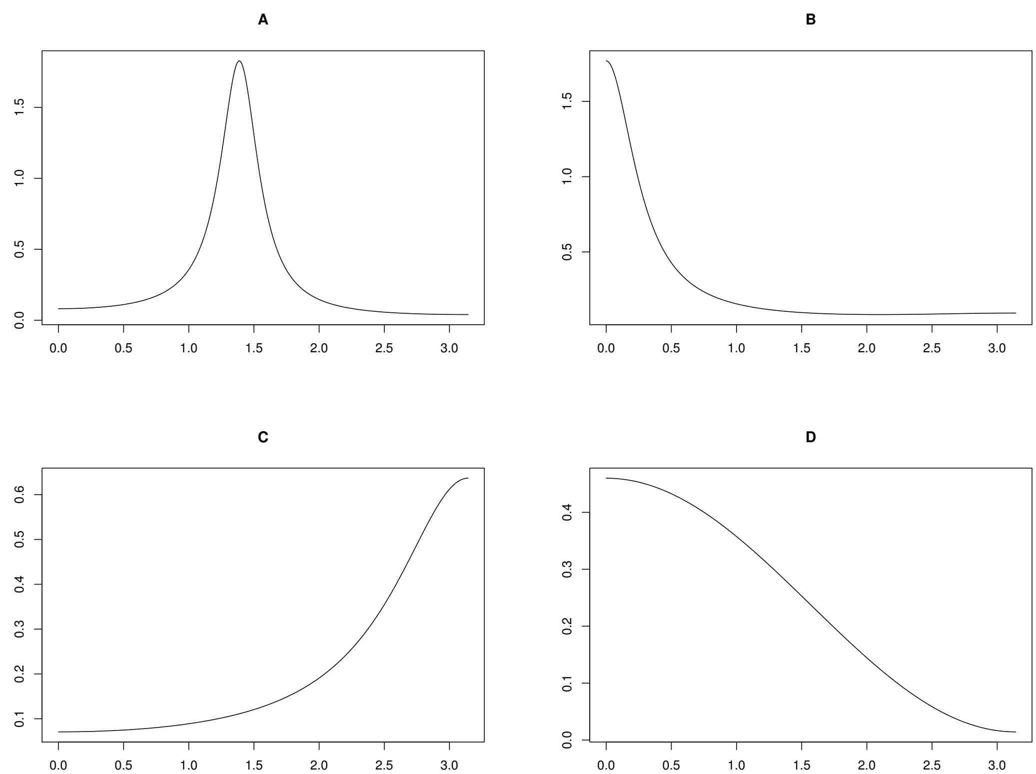

Figure 1: Four spectral densities of the models considered in Problem 3.

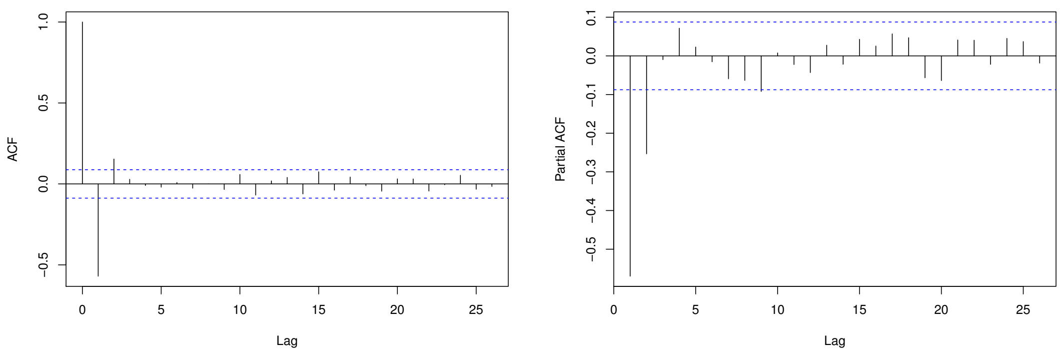

Figure 2: The sample ACF (left) and sample PACF (right) of the data analyzed in Problem 4.

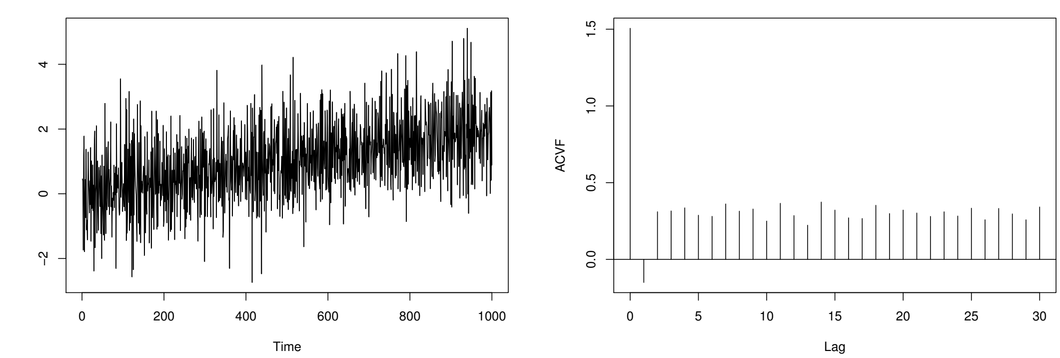

Figure 3: The trajectory of the data series (left) and the sample ACVF (right) of the data analyzed in Problem 5.

Problem 1

Write $Y_t = I_t X_t + (1 - I_t)X_{t-1}$, where $\{I_t\}$ is an iid sequence, independent of $\{X_t\}$, with $P(I_t = 1) = 1-p = 1-P(I_t = 0)$. Then $\mathrm{E}[Y_t] = \mathrm{E}[I_t]\,\mathrm{E}[X_{t-1}]+\mathrm{E}[1-I_t]\,\mathrm{E}[X_t] = 0$,

$$\mathrm{Var}(Y_t) = \mathrm{E}[Y_t^2] = (1-p)\,\mathrm{E}[X_t^2] + p\,\mathrm{E}[X_{t-1}^2] = \gamma_X(0)$$

and, for $h \geq 1$,

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t+h}) &= \mathrm{E}[Y_t Y_{t+h}] \\
&= (1-p)^2\,\mathrm{E}[X_t X_{t+h}] + p^2\,\mathrm{E}[X_{t-1}X_{t+h-1}] \\
&\quad + p(1-p)\,\mathrm{E}[X_t X_{t+h-1}] + p(1-p)\,\mathrm{E}[X_{t-1}X_{t+h}] \\
&= (p^2 + (1-p)^2)\gamma_X(h) + p(1-p)\gamma_X(h-1) + p(1-p)\gamma_X(h+1).
\end{aligned}$$

We conclude that $\{Y_t\}$ is stationary. Moreover,

$$\gamma_Y(h) = \begin{cases} \gamma_X(h), & h = 0, \\ \gamma_X(h) + p(1-p)(\gamma_X(h-1) + \gamma_X(h+1) - 2\gamma_X(h)), & h \geq 1, \\ \gamma_Y(-h), & h \leq -1. \end{cases}$$

For an AR(1) process with $\phi \in (0,1)$, for $h \geq 1$,

$$\begin{aligned}
\gamma_Y(h) - \gamma_X(h) &= p(1-p)(\gamma_X(h-1) + \gamma_X(h+1) - 2\gamma_X(h)) \\
&= p(1-p)(\phi^{-1} + \phi - 2)\phi^h(1-\phi^2)^{-1}\sigma^2 > 0.
\end{aligned}$$

Since $\rho_X(1) = \phi = 0.5$ and $p(1-p)$ is maximized for $p = 1/2$, for $h \geq 1$,

$$\rho_Y(h) = (1/2)^h + \frac{1}{4}(2 + 1/2 - 2)(1/2)^h = 1.125(1/2)^h = 1.125\rho_X(h).$$

Problem 2

The best linear predictor is

$$\begin{aligned}
P(X_t \mid Y_{t-1}) &= \frac{\mathrm{Cov}(X_t, Y_{t-1})}{\mathrm{Var}(Y_{t-1})} Y_{t-1} \\
&= \frac{\mathrm{Cov}(\phi X_{t-1} + Z_t, X_{t-1} + W_{t-1})}{\mathrm{Var}(X_{t-1} + W_{t-1})} Y_{t-1} \\
&= \frac{(1-\phi^2)^{-1}\phi\sigma_Z^2}{(1-\phi^2)^{-1}\sigma_Z^2 + \sigma_W^2} Y_{t-1}.
\end{aligned}$$

The prediction error is

$$X_t - P(X_t \mid Y_{t-1}) = \phi X_{t-1} + Z_t - \frac{(1-\phi^2)^{-1}\phi\sigma_Z^2}{(1-\phi^2)^{-1}\sigma_Z^2 + \sigma_W^2}(X_{t-1} + W_{t-1}).$$

Since X_{t-1}, Z_t, W_{t-1} are uncorrelated, the mean squared prediction error is

$$\begin{aligned}
\mathrm{E}[(X_t - P(X_t \mid Y_{t-1}))^2] &= \Big(\frac{\phi\sigma_W^2}{(1-\phi^2)^{-1}\sigma_Z^2 + \sigma_W^2}\Big)^2 \frac{\sigma_Z^2}{1-\phi^2} + \sigma_Z^2 \\
&\quad + \Big(\frac{(1-\phi^2)^{-1}\phi\sigma_Z^2}{(1-\phi^2)^{-1}\sigma_Z^2 + \sigma_W^2}\Big)^2 \sigma_W^2.
\end{aligned}$$

Moreover, $\mathrm{E}[(X_t - P(X_t \mid X_{t-1}))^2] = \sigma_Z^2$. With numerical values inserted

$$\frac{\mathrm{E}[(X_t - P(X_t \mid X_{t-1}))^2]}{\mathrm{E}[(X_t - P(X_t \mid Y_{t-1}))^2]} = 0.95,$$

i.e. having a measurement equipment that enabled perfect measurements would reduce the mean squared prediction error by 5%. The claim is therefore not correct.

Problem 3

The correct combination is $A - (3)$, $B - (2)$, $C - (1)$, $D - (4)$.

Problem 4

The sample PACF (and ACF) indicates that an AR(2) model is plausible. The Yule-Walker estimate of $(\phi_1, \phi_2, \sigma^2)$ is

$$\begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \end{pmatrix} = \frac{1}{\widehat{\gamma}(0)^2 - \widehat{\gamma}(1)^2} \begin{pmatrix} \widehat{\gamma}(0) & -\widehat{\gamma}(1) \\ -\widehat{\gamma}(1) & \widehat{\gamma}(0) \end{pmatrix} \begin{pmatrix} \widehat{\gamma}(1) \\ \widehat{\gamma}(2) \end{pmatrix}$$

$$\widehat{\sigma}^2 = \widehat{\gamma}(0) - (\widehat{\phi}_1, \widehat{\phi}_2) \begin{pmatrix} \widehat{\gamma}(1) \\ \widehat{\gamma}(2) \end{pmatrix}.$$

Inserting numerical values gives $(\widehat{\phi}_1, \widehat{\phi}_2, \widehat{\sigma}^2) \approx (-0.74, -0.27, 1.03)$.
Note that the sample ACF (and PACF) also shows that an MA(2) process is plausible. The innovations algorithm can be used to estimate the parameters.