KTH Mathematics

Examination in SF2943 Time Series Analysis, August 21, 2015, 08:00–13:00.

Examiner: Filip Lindskog, tel. 790 7217, e-mail: lindskog@kth.se

Allowed aids: Pocket calculator, "Formulas and survey, Time series analysis" by Jan Grandell, without notes.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

\- -

Problem 1

Can a causal AR(2) process have autocorrelations $\rho(1) = 1/2, \rho(2) = 1/6, \rho(3) = 1/12$? (10 p)

Problem 2

Consider a causal AR(2) process $\{X_t\}$ given be $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$, where $\{Z_t\}$ is $\mathrm{WN}(0, \sigma^2)$.

(a) Determine the best linear predictor (minimizing the mean squared prediction error) of X_{t+2} based on X_s for $s \leq t$. (5 p)

(b) Determine the mean squared prediction error of the predictor in (a). (5 p)

Problem 3

Consider the MA(1) process $X_t = Z_t + \theta Z_{t-1}$, where $\{Z_t\}$ is $\mathrm{WN}(0, \sigma^2)$. To handle a missing observation X_t, it is suggested to predict X_t by the best linear predictor based on X_{t-1} and X_{t+1}. Determine this best linear predictor. (10 p)

Problem 4

Consider a sample from a time series $\{Y_t\}$ with a linear trend. The differenced series $\{\nabla Y_t\}$ is found to fit well to the model $\{W_t\}$, where $W_t = 0.1 + Z_t - 0.5 Z_{t-1} - 0.5 Z_{t-2}$, where $\{Z_t\}$ is $\mathrm{WN}(0, 4^2)$. Suggest a model for $\{Y_t\}$. (10 p)

Problem 5

Consider the time series $\{X_t\}$ given by $X_t - \alpha X_{t-1} - \alpha^2 X_{t-2} = Z_t + \beta Z_{t-1}$, where $\{Z_t\}$ is $\mathrm{WN}(0, \sigma^2)$. For which values of the parameters $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ is $\{X_t\}$ a causal ARMA process? (10 p)

Problem 1

For a causal AR(p) process

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

multiplying each side of the above expression by X_{t-k}, $k \geq 1$, and taking expectations yield

$$\begin{aligned}
\gamma(k) &= \mathrm{Cov}(X_t, X_{t-k}) \\
&= \phi_1 \mathrm{Cov}(X_{t-1}, X_{t-k}) + \cdots + \phi_p \mathrm{Cov}(X_{t-p}, X_{t-k}) + \mathrm{Cov}(Z_t, X_{t-k}) \\
&= \phi_1 \gamma(k-1) + \cdots + \phi_p \gamma(k-p).
\end{aligned}$$

Here causality is used to ensure that $\mathrm{Cov}(Z_t, X_{t-k}) = 0$. Dividing by $\gamma(0)$ now yields

$$\rho(k) = \phi_1 \rho(k-1) + \cdots + \phi_p \rho(k-p), \quad k \geq 1. \tag{1}$$

In particular (using that $\rho(h) = \rho(-h)$ and $\rho(0) = 1$), here

$$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

which gives

$$\begin{aligned}
\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \frac{1}{1-\rho(1)^2} \begin{pmatrix} \rho(1) - \rho(1)\rho(2) \\ \rho(2) - \rho(1)^2 \end{pmatrix} \\
&= \{\rho(1) = 1/2, \quad \rho(2) = 1/6\} \\
&= \begin{pmatrix} 5/9 \\ -1/9 \end{pmatrix}.
\end{aligned}$$

Now, (1) with $p = 2$ and $k = 3$ gives

$$\rho(3) = \phi_1 \rho(2) + \phi_2 \rho(1) = \frac{5}{9} \cdot \frac{1}{6} - \frac{1}{9} \cdot \frac{1}{2} = \frac{1}{27} \neq \frac{1}{12}.$$

We conclude that there is no causal AR(2) process with the given autocorrelations.

Problem 2

Notice that

$$\begin{aligned}
X_{t+2} &= \phi_1 X_{t+1} + \phi_2 X_t + Z_{t+2} = \phi_1(\phi_1 X_t + \phi_2 X_{t-1} + Z_{t+1}) + \phi_2 X_t + Z_{t+2} \\
&= \widetilde{X}_{t+2} + Z_{t+2} + \phi_1 Z_{t+1},
\end{aligned}$$

where $\widetilde{X}_{t+2} := (\phi_1^2 + \phi_2) X_t + \phi_1 \phi_2 X_{t-1}$ is a promising candidate for the best linear predictor of X_{t+2}. Since $\{X_t\}$ is causal, $\mathrm{Cov}(\widetilde{X}_{t+2} - X_{t+2}, X_s) = 0$ for $s \leq t$. Therefore, see page 12 in F&S, it is indeed the best linear predictor. The MSE is $\mathrm{E}[(\widetilde{X}_{t+2} - X_{t+2})^2] = (1 + \phi_1^2)\sigma^2$.

Problem 3

Consider linear predictors of the form $aX_{t-1} + bX_{t+1}$. The mean squared prediction errors are

$$\begin{aligned}
\mathrm{E}[(X_t - aX_{t-1} - bX_{t+1})^2] &= \mathrm{Var}(X_t - aX_{t-1} - bX_{t+1}) \\
&= (1 + a^2 + b^2)\gamma(0) - 2(a+b)\gamma(1) + 2ab\gamma(2) \\
&=: f(a,b).
\end{aligned}$$

Minimizing $f(a,b)$ corresponds to computing

$$\begin{aligned}
\frac{\partial}{\partial a} f(a,b) &= 2a\gamma(0) + 2b\gamma(2) - 2\gamma(1), \\
\frac{\partial}{\partial b} f(a,b) &= 2b\gamma(0) + 2a\gamma(2) - 2\gamma(1),
\end{aligned}$$

setting these expressions to zero and solving for (a,b), gives $a = b = \gamma(1)/(\gamma(0) + \gamma(2))$. Here, $\gamma(0) = (1+\theta^2)\sigma^2$, $\gamma(1) = \theta\sigma^2$ and $\gamma(2) = 0$, so $a = b = \theta/(1+\theta^2)$ for the best linear predictor.

Problem 4

Differencing an MA(1) process produces an MA(2) process so it makes sense to suggest an MA(1) process with linear drift: $Y_t = ct + Z_t + \theta Z_{t-1}$. Since

$$\begin{aligned}
\nabla Y_t = Y_t - Y_{t-1} &= ct - c(t-1) + Z_t + \theta Z_{t-1} - Z_{t-1} - \theta Z_{t-2} \\
&= c + Z_t + (\theta - 1)Z_{t-1} - \theta Z_{t-2}
\end{aligned}$$

the suggested model with parameters $c = 0.1$ and $\theta = 1/2$ meets the requirements.

Problem 5

We look for values of $\alpha \in (0,1)$ for which the zeros of $\phi(z) = 1 - \alpha z - \alpha^2 z^2$ are located outside the unit circle.

$$z = -\frac{1}{2\alpha} \pm \sqrt{\left(\frac{1}{2\alpha}\right)^2 + \frac{1}{\alpha^2}} = -\frac{1}{2\alpha}\left(1 \mp \sqrt{5}\right)$$

Since $-(1+\sqrt{5})/(2\alpha) < -1$ gives $\alpha \in (0, (1+\sqrt{5})/2)$ and $-(1-\sqrt{5})/(2\alpha) > 1$ gives $\alpha \in (0, (-1+\sqrt{5})/2)$, we find that causality holds for all $\alpha \in (0, (\sqrt{5}-1)/2) \approx (0, 0.62)$. Common roots of $\phi(z)$ and $\theta(z)$: $\theta(z) = 0$ gives $z = -1/\beta$. If there is a common root, then the ARMA(2,1) process reduces to a causal AR(1). Anyway, causality holds if $\alpha \in (0, (\sqrt{5}-1)/2) \approx (0, 0.62)$.