Optimal Control and Filtering

KTH 2015

The solutions to exam should be sent to **tomas.bjork@hhs.se** by Tuesday March 17. The only allowed format is Latex, in pdf form. Handwritten exams will not be considered. All notation should be clearly defined. Arguments should be complete and careful.

1. Consider a scalar state process dynamics of the form

$$dX_t = \mu\left(t, X_t, u_t\right) dt + \sigma\left(t, X_t, u_t\right) dW_t$$

where the scalar control u is without restriction (apart form being adapted). Our entire derivation of the HJB equation has so far been based on the fact that the objective function is of the form

$$\int_0^T F(t, X_t, u_t)dt + \Phi(X_T).$$

Sometimes it is natural to consider other criteria, like the exponential criterion

$$E\left[e^{\int_0^T F(t,X_t,u_t)dt+\Phi(X_T)}\right]$$

For this case we define the optimal value function $V(t,x)$ as the supremum of

$$E_{t,x}\left[e^{\int_t^T F(s,X_s,u_s)dt+\Phi(X_T)}\right]$$

Use the ideas of Dynamic Programming in order to show that the HJB equation for the expected exponential utility criterion is given by

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup\limits_{u\in R}\left\{V(t,x)F(t,x,u) + \mathcal{A}^u V(t,x)\right\} = 0, \\ V(T,x) = e^{\Phi(x)}. \end{cases}$$

2. Consider a market model with stock price S and bank account B with dynamics given by

$$\begin{aligned} dS_t &= S_t\mu(Y_t)dt + S_t\sigma(Y_t)dW_t^1, \\ dB_t &= rB_t dt \end{aligned}$$

Here, the volatility σ and local rate of return μ depend on a "Hidden Markov process" Y with dynamics

$$dY_t = a(Y_t)dt + b(Y_t)dW_t^2.$$

The Wiener processes W^1 and W^2 are assumed to be independent. Analyze, as far as you can, the problem of maximizing logarithmic utility of final wealth

$$E\left[\ln(X_T)\right],$$

where X denotes portfolio value. μ, σ, a and b are given deterministic functions. You are allowed to observe both S and Y so this is not a filtering problem.

3. Consider the filtering model

$$\begin{aligned} dX_t &= a_t dt + dV_t \\ dZ_t &= b_t dt + \sigma_t dW_t \end{aligned}$$

where

- The process σ is $\mathcal{F}_t^Z$ adapted and positive.
- W and V are, possibly correlated, Wiener processes.

Prove, along the lines in the lecture notes, that the filtering equations are given by

$$\begin{aligned} d\widehat{X}_t &= \widehat{a}_t dt + \left[\widehat{D}_t + \frac{1}{\sigma_t}\left\{\widehat{X_t b_t} - \widehat{X}_t \widehat{b}_t\right\}\right] d\nu_t \\ d\nu_t &= \frac{1}{\sigma_t}\left\{dZ_t - \widehat{b}_t dt\right\} \\ D_t &= \frac{d\langle V, W\rangle_t}{dt} \end{aligned}$$

Note: There is a typo in the overhead slides for this formula.

4. Consider a filtered probability space $\left(\Omega, \mathcal{F}, P^0, \mathbf{F}\right)$ carrying an $\mathbf{F}$-Wiener process Z. For any $\mathbf{F}$-adapted process g we can define the process L_t by

$$\begin{aligned} dL_t &= L_t g_t dZ_t \\ L_0 &= 1 \end{aligned}$$

and to emphasize the dependence of g we write $L_t = L_t(g)$. Assume now that we have a process h which is $\mathbf{F}$ adapted and independent of Z. We can then define a new measure P^1 by

$$L_t(h) = \frac{dP^1}{dP^0},$$

on $\mathcal{F}_t$. From Girsanov we then have

$$dZ_t = h_t dt + dW_t,$$

where W is P^1-Wiener. In connection with statistical testing of P^0 against P^1, based on observations of Z, it is natural to use the estimate

$$\widehat{L}_t(h) = E^0\left[L_t(h) \middle| \mathcal{F}_t^Z \right]$$

as a test statistic, where E^0 denotes integration w.r.t. P^0.

Now define the process $\Pi_t[h]$ by

$$\Pi_t[h] = E^1\left[h_t \middle| \mathcal{F}_t^Z \right]$$

where the integration is w.r.t. P^1.

Your task is to prove the formula

$$\widehat{L}_t(h) = L_t(\Pi_t[h]).$$

In other words, prove that

$$d\widehat{L}_t(h) = \widehat{L}_t(h)\Pi_t[h]\, dZ_t.$$

Note: This is a famous formula, referred to as "the separation formula for filtering and detection".

5. Consider an endowment equilibrium model with log utility

$$U(t, c) = e^{-\delta t} \ln(c)$$

and scalar endowment following GBM

$$de_t = ae_t dt + be_t W_t,$$

where a and b are real numbers and W is P-Wiener.

(a) Compute the P and Q dynamics of the price process S.

(b) Use the result of (a) and let $T \to \infty$. What do you get?

Good Luck!