

Stochastic Optimal Control with Finance Applications

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- Investment theory.
- The martingale approach.
- Filtering theory.
- Optimal investment with partial information.

1. Dynamic Programming

- The basic idea.
- Deriving the HJB equation.
- The verification theorem.
- The linear quadratic regulator.

Problem Formulation

$$\max_u E \left[\int_0^T F(t, X_t, u_t) dt + \Phi(X_T) \right]$$

subject to

$$dX_t = \mu(t, X_t, u_t) dt + \sigma(t, X_t, u_t) dW_t$$

$$X_0 = x_0,$$

$$u_t \in U(t, X_t), \quad \forall t.$$

We will only consider **feedback control laws**, i.e. controls of the form

$$u_t = \mathbf{u}(t, X_t)$$

Terminology:

X = state variable

u = control variable

U = control constraint

Note: No state space constraints.

Main idea

- Embedd the problem above in a family of problems indexed by starting point in time and space.
- Tie all these problems together by a PDE—the Hamilton Jacobi Bellman equation.
- The control problem is reduced to the problem of solving the deterministic HJB equation.

NOTE:

For simplicity of notation we assume that X , W , and u are scalar.

Some notation

- For any fixed number $u \in R$, the functions μ^u and σ^u are defined by

$$\begin{aligned}\mu^u(t, x) &= \mu(t, x, u), \\ \sigma^u(t, x) &= \sigma(t, x, u),\end{aligned}$$

- For any control law \mathbf{u} , the functions $\mu^{\mathbf{u}}$, $\sigma^{\mathbf{u}}$, and $F^{\mathbf{u}}(t, x)$ are defined by

$$\begin{aligned}\mu^{\mathbf{u}}(t, x) &= \mu(t, x, \mathbf{u}(t, x)), \\ \sigma^{\mathbf{u}}(t, x) &= \sigma(t, x, \mathbf{u}(t, x)), \\ F^{\mathbf{u}}(t, x) &= F(t, x, \mathbf{u}(t, x)).\end{aligned}$$

More notation

- For any fixed number $u \in R$, the partial differential operator \mathcal{A}^u is defined by

$$\mathcal{A}^u = \mathcal{A}^u = \mu^u(t, x) \frac{\partial}{\partial x} + \frac{1}{2} [\sigma^u(t, x)]^2 \frac{\partial^2}{\partial x^2}.$$

- For any control law \mathbf{u} , the partial differential operator $\mathcal{A}^{\mathbf{u}}$ is defined by

$$\mathcal{A}^{\mathbf{u}} = \mu^{\mathbf{u}}(t, x) \frac{\partial}{\partial x} + \frac{1}{2} [\sigma^{\mathbf{u}}(t, x)]^2 \frac{\partial^2}{\partial x^2}.$$

- For any control law \mathbf{u} , the process $X^{\mathbf{u}}$ is the solution of the SDE

$$dX_t^{\mathbf{u}} = \mu(t, X_t^{\mathbf{u}}, \mathbf{u}_t) dt + \sigma(t, X_t^{\mathbf{u}}, \mathbf{u}_t) dW_t,$$

where

$$\mathbf{u}_t = \mathbf{u}(t, X_t^{\mathbf{u}})$$

Embedding the problem

For every fixed (t, x) the control problem $\mathcal{P}(t, x)$ is defined as the problem to maximize

$$E_{t,x} \left[\int_t^T F(s, X_s^{\mathbf{u}}, u_s) ds + \Phi(X_T^{\mathbf{u}}) \right],$$

given the dynamics

$$\begin{aligned} dX_s^{\mathbf{u}} &= \mu(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + \sigma(s, X_s^{\mathbf{u}}, \mathbf{u}_s) dW_s, \\ X_t &= x, \end{aligned}$$

and the constraints

$$\mathbf{u}(s, y) \in U, \quad \forall (s, y) \in [t, T] \times R.$$

The original problem was $\mathcal{P}(0, x_0)$.

The optimal value function

- The **value function**

$$\mathcal{J} : R_+ \times R \times \mathcal{U} \rightarrow R$$

is defined by

$$\mathcal{J}(t, x, \mathbf{u}) = E \left[\int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + \Phi(X_T^{\mathbf{u}}) \right]$$

given the dynamics above.

- The **optimal value function**

$$V : R_+ \times R \rightarrow R$$

is defined by

$$V(t, x) = \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}(t, x, \mathbf{u}).$$

- We want to derive a PDE for V .

Assumptions

We assume:

- There exists an optimal control law \hat{u} .
- The optimal value function V is regular in the sense that $V \in C^{1,2}$.
- A number of limiting procedures in the following arguments can be justified.

The Bellman Optimality Principle

Dynamic programming relies heavily on the following basic result.

Proposition: If \hat{u} is optimal on the time interval $[t, T]$ then it is also optimal on every subinterval $[s, T]$ with $t \leq s \leq T$.

Proof: Iterated expectations.

Basic strategy

To derive the PDE do as follows:

- Fix $(t, x) \in (0, T) \times \mathbb{R}^n$.
- Choose a real number h (interpreted as a “small” time increment).
- Choose an arbitrary control law \mathbf{u} .

Now define the control law \mathbf{u}^* by

$$\mathbf{u}^*(s, y) = \begin{cases} \mathbf{u}(s, y), & (s, y) \in [t, t + h] \times \mathbb{R} \\ \hat{\mathbf{u}}(s, y), & (s, y) \in (t + h, T] \times \mathbb{R}. \end{cases}$$

In other words, if we use \mathbf{u}^* then we use the arbitrary control \mathbf{u} during the time interval $[t, t + h]$, and then we switch to the optimal control law during the rest of the time period.

Basic idea

The whole idea of DynP boils down to the following procedure.

- Given the point (t, x) above, we consider the following two strategies over the time interval $[t, T]$:
 - I: Use the optimal law $\hat{\mathbf{u}}$.
 - II: Use the control law \mathbf{u}^* defined above.
- Compute the expected utilities obtained by the respective strategies.
- Using the obvious fact that Strategy I is least as good as Strategy II, and letting h tend to zero, we obtain our fundamental PDE.

Strategy values

Expected utility for strategy I:

$$\mathcal{J}(t, x, \hat{\mathbf{u}}) = V(t, x)$$

Expected utility for strategy II:

- The expected utility for $[t, t + h)$ is given by

$$E_{t,x} \left[\int_t^{t+h} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds \right].$$

- Conditional expected utility over $[t + h, T]$, given (t, x) :

$$E_{t,x} [V(t + h, X_{t+h}^{\mathbf{u}})].$$

- Total expected utility for Strategy II is

$$E_{t,x} \left[\int_t^{t+h} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + V(t + h, X_{t+h}^{\mathbf{u}}) \right].$$

Comparing strategies

We have trivially

$$V(t, x) \geq E_{t,x} \left[\int_t^{t+h} F(s, X_s^u, \mathbf{u}_s) ds + V(t+h, X_{t+h}^u) \right].$$

Remark

We have equality above if and only if the control law \mathbf{u} is an optimal law $\hat{\mathbf{u}}$.

Now use Itô to obtain

$$\begin{aligned} V(t+h, X_{t+h}^u) &= V(t, x) \\ &+ \int_t^{t+h} \left\{ \frac{\partial V}{\partial t}(s, X_s^u) + \mathcal{A}^u V(s, X_s^u) \right\} ds \\ &+ \int_t^{t+h} \frac{\partial V}{\partial x}(s, X_s^u) \sigma^u dW_s, \end{aligned}$$

and plug into the formula above.

We obtain

$$E_{t,x} \left[\int_t^{t+h} \left[F(s, X_s^u, \mathbf{u}_s) + \frac{\partial V}{\partial t}(s, X_s^u) + \mathcal{A}^u V(s, X_s^u) \right] ds \right] \leq 0.$$

Going to the limit:

Divide by h , move h within the expectation and let h tend to zero.

We get

$$F(t, x, u) + \frac{\partial V}{\partial t}(t, x) + \mathcal{A}^u V(t, x) \leq 0,$$

Recall

$$F(t, x, u) + \frac{\partial V}{\partial t}(t, x) + \mathcal{A}^u V(t, x) \leq 0,$$

This holds for all $u = \mathbf{u}(t, x)$, with equality if and only if $\mathbf{u} = \hat{\mathbf{u}}$.

We thus obtain the **HJB equation**

$$\frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\} = 0.$$

The HJB equation

Theorem:

Under suitable regularity assumptions the following hold:

I: V satisfies the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\} = 0,$$
$$V(T, x) = \Phi(x),$$

II: For each $(t, x) \in [0, T] \times R$ the supremum in the HJB equation above is attained by $u = \hat{u}(t, x)$.

Notation for the multi-dimensional case

- For any fixed vector $u \in R^k$, the functions μ^u , σ^u and C^u are defined by

$$\mu^u(t, x) = \mu(t, x, u),$$

$$\sigma^u(t, x) = \sigma(t, x, u),$$

$$C^u(t, x) = \sigma(t, x, u)\sigma(t, x, u)'$$

- For any control law \mathbf{u} , the functions $\mu^{\mathbf{u}}$, $\sigma^{\mathbf{u}}$, $C^{\mathbf{u}}(t, x)$ and $F^{\mathbf{u}}(t, x)$ are defined by

$$\mu^{\mathbf{u}}(t, x) = \mu(t, x, \mathbf{u}(t, x)),$$

$$\sigma^{\mathbf{u}}(t, x) = \sigma(t, x, \mathbf{u}(t, x)),$$

$$C^{\mathbf{u}}(t, x) = \sigma(t, x, \mathbf{u}(t, x))\sigma(t, x, \mathbf{u}(t, x))',$$

$$F^{\mathbf{u}}(t, x) = F(t, x, \mathbf{u}(t, x)).$$

Notation continued

- For any fixed vector $u \in R^k$, the partial differential operator \mathcal{A}^u is defined by

$$\mathcal{A}^u = \sum_{i=1}^n \mu_i^u(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_{ij}^u(t, x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

- For any control law \mathbf{u} , the partial differential operator $\mathcal{A}^{\mathbf{u}}$ is defined by

$$\mathcal{A}^{\mathbf{u}} = \sum_{i=1}^n \mu_i^{\mathbf{u}}(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_{ij}^{\mathbf{u}}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

- For any control law \mathbf{u} , the process $X^{\mathbf{u}}$ is the solution of the SDE

$$dX_t^{\mathbf{u}} = \mu(t, X_t^{\mathbf{u}}, \mathbf{u}_t) dt + \sigma(t, X_t^{\mathbf{u}}, \mathbf{u}_t) dW_t,$$

where

$$\mathbf{u}_t = \mathbf{u}(t, X_t^{\mathbf{u}})$$

Logic and problem

Note: We have shown that **if** V is the optimal value function, and **if** V is regular enough, **then** V satisfies the HJB equation. The HJB eqn is thus derived as a **necessary** condition, and requires strong *ad hoc* regularity assumptions.

Problem: Suppose we have solved the HJB equation. Have we then found the optimal value function and the optimal control law?

Answer: Yes! This follows from the **Verification theorem**.

The Verification Theorem

Suppose that we have two functions $H(t, x)$ and $g(t, x)$, such that

- H is sufficiently integrable, and solves the HJB equation

$$\begin{cases} \frac{\partial H}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u H(t, x)\} = 0, \\ H(T, x) = \Phi(x), \end{cases}$$

- For each fixed (t, x) , the supremum in the expression

$$\sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u H(t, x)\}$$

is attained by the choice $u = g(t, x)$.

Then the following hold.

1. The optimal value function V to the control problem is given by

$$V(t, x) = H(t, x).$$

2. There exists an optimal control law \hat{u} , and in fact

$$\hat{u}(t, x) = g(t, x)$$

Handling the HJB equation

1. Consider the HJB equation for V .
2. Fix $(t, x) \in [0, T] \times R^n$ and solve, the static optimization problem

$$\max_{u \in U} [F(t, x, u) + \mathcal{A}^u V(t, x)].$$

Here u is the only variable, whereas t and x are fixed parameters. The functions F , μ , σ and V are considered as given.

3. The optimal \hat{u} , will depend on t and x , and on the function V and its partial derivatives. We thus write \hat{u} as

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}(t, x; V). \quad (1)$$

4. The function $\hat{\mathbf{u}}(t, x; V)$ is our candidate for the optimal control law, but since we do not know V this description is incomplete. Therefore we substitute the expression for \hat{u} into the PDE, giving us the PDE

$$\frac{\partial V}{\partial t}(t, x) + F^{\hat{\mathbf{u}}}(t, x) + \mathcal{A}^{\hat{\mathbf{u}}}(t, x) V(t, x) = 0,$$

$$V(T, x) = \Phi(x).$$

5. Now we solve the PDE above! Then we put the solution V into expression (1). Using the verification theorem we can identify V as the optimal value function, and \hat{u} as the optimal control law.

Making an Ansatz

- The hard work of dynamic programming consists in solving the highly nonlinear HJB equation
- There are no general analytic methods available for this, so the number of known optimal control problems with an analytic solution is very small indeed.
- In an actual case one usually tries to **guess** a solution, i.e. we typically make a parameterized **Ansatz** for V then use the PDE in order to identify the parameters.
- **Hint:** V often inherits some structural properties from the boundary function Φ as well as from the instantaneous utility function F .
- Most of the known solved control problems have, to some extent, been “rigged” in order to be analytically solvable.

The Linear Quadratic Regulator

$$\min_{u \in R^k} E \left[\int_0^T \{X_t' Q X_t + u_t' R u_t\} dt + X_T' H X_T \right],$$

with dynamics

$$dX_t = \{A X_t + B u_t\} dt + C dW_t.$$

We want to control a vehicle in such a way that it stays close to the origin (the terms $x' Q x$ and $x' H x$) while at the same time keeping the “energy” $u' R u$ small.

Here $X_t \in R^n$ and $u_t \in R^k$, and we impose no control constraints on u .

The matrices Q , R , H , A , B and C are assumed to be known. We may WLOG assume that Q , R and H are symmetric, and we assume that R is positive definite (and thus invertible).

Handling the Problem

The HJB equation becomes

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial t}(t, x) + \inf_{u \in R^k} \{x'Qx + u'Ru + [\nabla_x V](t, x) [Ax + Bu]\} \\ \quad + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x) [CC']_{i,j} = 0, \\ V(T, x) = x'Hx. \end{array} \right.$$

For each fixed choice of (t, x) we now have to solve the static unconstrained optimization problem to minimize

$$u'Ru + [\nabla_x V](t, x) [Ax + Bu].$$

The problem was:

$$\min_u \quad u' R u + [\nabla_x V](t, x) [A x + B u].$$

Since $R > 0$ we set the gradient to zero and obtain

$$2u' R = -(\nabla_x V) B,$$

which gives us the optimal u as

$$\hat{u} = -\frac{1}{2} R^{-1} B' (\nabla_x V)'$$

Note: This is our candidate of optimal control law, but it depends on the unknown function V .

We now make an educated guess about the shape of V .

From the boundary function $x'Hx$ and the term $x'Qx$ in the cost function we make the Ansatz

$$V(t, x) = x'P(t)x + q(t),$$

where $P(t)$ is a symmetric matrix function, and $q(t)$ is a scalar function.

With this trial solution we have,

$$\begin{aligned} \frac{\partial V}{\partial t}(t, x) &= x'\dot{P}x + \dot{q}, \\ \nabla_x V(t, x) &= 2x'P, \\ \nabla_{xx} V(t, x) &= 2P \\ \hat{u} &= -R^{-1}B'Px. \end{aligned}$$

Inserting these expressions into the HJB equation we get

$$\begin{aligned} x' \left\{ \dot{P} + Q - PBR^{-1}B'P + A'P + PA \right\} x \\ + \dot{q} + tr[C'PC] = 0. \end{aligned}$$

We thus get the following matrix ODE for P

$$\begin{cases} \dot{P} &= PBR^{-1}B'P - A'P - PA - Q, \\ P(T) &= H. \end{cases}$$

and we can integrate directly for q :

$$\begin{cases} \dot{q} &= -tr[C'PC], \\ q(T) &= 0. \end{cases}$$

The matrix equation is a **Riccati equation**. The equation for q can then be integrated directly.

Final Result for LQ:

$$\begin{aligned} V(t, x) &= x'P(t)x + \int_t^T tr[C'P(s)C]ds, \\ \hat{u}(t, x) &= -R^{-1}B'P(t)x. \end{aligned}$$

2. Portfolio Theory

- Problem formulation.
- An extension of HJB.
- The simplest consumption-investment problem.
- The Merton fund separation results.

Recap of Basic Facts

We consider a market with n assets.

S_t^i = price of asset No i ,

h_t^i = units of asset No i in portfolio

w_t^i = portfolio weight on asset No i

X_t = portfolio value

c_t = consumption rate

We have the relations

$$X_t = \sum_{i=1}^n h_t^i S_t^i, \quad w_t^i = \frac{h_t^i S_t^i}{X_t}, \quad \sum_{i=1}^n w_t^i = 1.$$

Basic equation:

Dynamics of self financing portfolio in terms of relative weights

$$dX_t = X_t \sum_{i=1}^n w_t^i \frac{dS_t^i}{S_t^i} - c_t dt$$

Simplest model

Assume a scalar risky asset and a constant short rate.

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

$$dB_t = r B_t dt$$

We want to maximize expected utility over time

$$\max_{w^0, w^1, c} E \left[\int_0^T F(t, c_t) dt \right]$$

Dynamics

$$dX_t = X_t [w_t^0 r + w_t^1 \alpha] dt - c_t dt + w_t^1 \sigma X_t dW_t,$$

Constraints

$$\begin{aligned} c_t &\geq 0, \quad \forall t \geq 0, \\ w_t^0 + w_t^1 &= 1, \quad \forall t \geq 0. \end{aligned}$$

Nonsense!

What are the problems?

- We can obtain unlimited utility by simply consuming arbitrary large amounts.
- The wealth will go negative, but there is nothing in the problem formulations which prohibits this.
- We would like to impose a constraint of type $X_t \geq 0$ but this is a **state constraint** and DynP does not allow this.

Good News:

DynP can be generalized to handle (some) problems of this kind.

Generalized problem

Let D be a nice open subset of $[0, T] \times \mathbb{R}^n$ and consider the following problem.

$$\max_{u \in U} E \left[\int_0^\tau F(s, X_s^u, \mathbf{u}_s) ds + \Phi(\tau, X_\tau^u) \right].$$

Dynamics:

$$\begin{aligned} dX_t &= \mu(t, X_t, u_t) dt + \sigma(t, X_t, u_t) dW_t, \\ X_0 &= x_0, \end{aligned}$$

The **stopping time** τ is defined by

$$\tau = \inf \{t \geq 0 \mid (t, X_t) \in \partial D\} \wedge T.$$

Generalized HJB

Theorem: Given enough regularity the following hold.

1. The optimal value function satisfies

$$\begin{cases} \frac{\partial V}{\partial t}(t, x) + \sup_{u \in U} \{F(t, x, u) + \mathcal{A}^u V(t, x)\} = 0, & \forall (t, x) \in D \\ V(t, x) = \Phi(t, x), & \forall (t, x) \in \partial D. \end{cases}$$

2. We have an obvious verification theorem.

Reformulated problem

$$\max_{c \geq 0, w \in R} E \left[\int_0^\tau F(t, c_t) dt + \Phi(X_T) \right]$$

where

$$\tau = \inf \{t \geq 0 \mid X_t = 0\} \wedge T.$$

with notation:

$$\begin{aligned} w^1 &= w, \\ w^0 &= 1 - w \end{aligned}$$

Thus no constraint on w .

Dynamics

$$dX_t = w_t [\alpha - r] X_t dt + (rX_t - c_t) dt + w\sigma X_t dW_t,$$

HJB Equation

$$\frac{\partial V}{\partial t} + \sup_{c \geq 0, w \in R} \left\{ F(t, c) + wx(\alpha - r) \frac{\partial V}{\partial x} + (rx - c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2} \right\} = 0,$$
$$V(T, x) = 0,$$
$$V(t, 0) = 0.$$

We now specialize (why?) to

$$F(t, c) = e^{-\delta t} c^\gamma,$$

so we have to maximize

$$e^{-\delta t} c^\gamma + wx(\alpha - r) \frac{\partial V}{\partial x} + (rx - c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},$$

Analysis of the HJB Equation

In the embedded static problem we maximize, over c and w ,

$$e^{-\delta t} c^\gamma + wx(\alpha - r) \frac{\partial V}{\partial x} + (rx - c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},$$

First order conditions:

$$\begin{aligned} \gamma c^{\gamma-1} &= e^{\delta t} V_x, \\ w &= \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\alpha - r}{\sigma^2}, \end{aligned}$$

Ansatz:

$$V(t, x) = e^{-\delta t} h(t) x^\gamma,$$

Because of the boundary conditions, we must demand that

$$h(T) = 0. \quad (2)$$

Given a V of this form we have (using $\dot{\cdot}$ to denote the time derivative)

$$\begin{aligned}\frac{\partial V}{\partial t} &= e^{-\delta t} \dot{h} x^\gamma - \delta e^{-\delta t} h x^\gamma, \\ \frac{\partial V}{\partial x} &= \gamma e^{-\delta t} h x^{\gamma-1}, \\ \frac{\partial^2 V}{\partial x^2} &= \gamma(\gamma - 1) e^{-\delta t} h x^{\gamma-2}.\end{aligned}$$

giving us

$$\begin{aligned}\hat{\mathbf{w}}(t, x) &= \frac{\alpha - r}{\sigma^2(1 - \gamma)}, \\ \hat{\mathbf{c}}(t, x) &= x h(t)^{-1/(1-\gamma)}.\end{aligned}$$

Plug all this into HJB!

After rearrangements we obtain

$$x^\gamma \left\{ \dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} \right\} = 0,$$

where the constants A and B are given by

$$A = \frac{\gamma(\alpha - r)^2}{\sigma^2(1 - \gamma)} + r\gamma - \frac{1}{2} \frac{\gamma(\alpha - r)^2}{\sigma^2(1 - \gamma)} - \delta$$

$$B = 1 - \gamma.$$

If this equation is to hold for all x and all t , then we see that h must solve the ODE

$$\begin{aligned} \dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} &= 0, \\ h(T) &= 0. \end{aligned}$$

An equation of this kind is known as a **Bernoulli equation**, and it can be solved explicitly.

We are done.

Merton's Mutual Fund Theorems

1. The case with no risk free asset

We consider n risky assets with dynamics

$$dS_i = S_i\alpha_i dt + S_i\sigma_i dW, \quad i = 1, \dots, n$$

where W is Wiener in R^k . On vector form:

$$dS = D(S)\alpha dt + D(S)\sigma dW.$$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \sigma = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix}$$

$D(S)$ is the diagonal matrix

$$D(S) = \text{diag}[S_1, \dots, S_n].$$

Wealth dynamics

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW.$$

Formal problem

$$\max_{c,w} E \left[\int_0^{\tau} F(t, c_t) dt \right]$$

given the dynamics

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW.$$

and constraints

$$e'w = 1, \quad c \geq 0.$$

Assumptions:

- The vector α and the matrix σ are constant and deterministic.
- The volatility matrix σ has full rank so $\sigma\sigma'$ is positive definite and invertible.

Note: S does not turn up in the X -dynamics so V is of the form

$$V(t, x, s) = V(t, x)$$

The HJB equation is

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial t}(t, x) + \sup_{e'w=1, c \geq 0} \{F(t, c) + \mathcal{A}^{c,w}V(t, x)\} = 0, \\ V(T, x) = 0, \\ V(t, 0) = 0. \end{array} \right.$$

where

$$\mathcal{A}^{c,w}V = xw'\alpha \frac{\partial V}{\partial x} - c \frac{\partial V}{\partial x} + \frac{1}{2}x^2w'\Sigma w \frac{\partial^2 V}{\partial x^2},$$

and where the matrix Σ is given by

$$\Sigma = \sigma\sigma'.$$

The HJB equation is

$$\left\{ \begin{array}{l} V_t(t, x) + \sup_{w'e=1, c \geq 0} \left\{ F(t, c) + (xw'\alpha - c)V_x(t, x) + \frac{1}{2}x^2w'\Sigma wV_{xx}(t, x) \right\} = 0, \\ V(T, x) = 0, \\ V(t, 0) = 0. \end{array} \right.$$

where $\Sigma = \sigma\sigma'$.

If we relax the constraint $w'e = 1$, the Lagrange function for the static optimization problem is given by

$$L = F(t, c) + (xw'\alpha - c)V_x(t, x) + \frac{1}{2}x^2w'\Sigma wV_{xx}(t, x) + \lambda(1 - w'e).$$

$$\begin{aligned}
L &= F(t, c) + (xw'\alpha - c)V_x(t, x) \\
&+ \frac{1}{2}x^2w'\Sigma wV_{xx}(t, x) + \lambda(1 - w'e).
\end{aligned}$$

The first order condition for c is

$$\frac{\partial F}{\partial c}(t, c) = V_x(t, x).$$

The first order condition for w is

$$x\alpha'V_x + x^2V_{xx}w'\Sigma = \lambda e',$$

so we can solve for w in order to obtain

$$\hat{w} = \Sigma^{-1} \left[\frac{\lambda}{x^2V_{xx}}e - \frac{xV_x}{x^2V_{xx}}\alpha \right].$$

Using the relation $e'w = 1$ this gives λ as

$$\lambda = \frac{x^2V_{xx} + xV_xe'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e},$$

Inserting λ gives us, after some manipulation,

$$\hat{w} = \frac{1}{e'\Sigma^{-1}e} \Sigma^{-1}e + \frac{V_x}{xV_{xx}} \Sigma^{-1} \left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e} e - \alpha \right].$$

We can write this as

$$\hat{w}(t) = g + Y(t)h,$$

where the fixed vectors g and h are given by

$$g = \frac{1}{e'\Sigma^{-1}e} \Sigma^{-1}e,$$
$$h = \Sigma^{-1} \left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e} e - \alpha \right],$$

whereas Y is given by

$$Y(t) = \frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}.$$

We had

$$\hat{w}(t) = g + Y(t)h,$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional “optimal portfolio line”

$$g + sh,$$

in the $(n - 1)$ -dimensional “portfolio hyperplane” Δ , where

$$\Delta = \{w \in R^n \mid e'w = 1\}.$$

If we fix two points on the optimal portfolio line, say $w^a = g + ah$ and $w^b = g + bh$, then any point w on the line can be written as an affine combination of the basis points w^a and w^b . An easy calculation shows that if $w^s = g + sh$ then we can write

$$w^s = \mu w^a + (1 - \mu)w^b,$$

where

$$\mu = \frac{s - b}{a - b}.$$

Mutual Fund Theorem

There exists a family of mutual funds, given by $w^s = g + sh$, such that

1. For each fixed s the portfolio w^s stays fixed over time.
2. For fixed a, b with $a \neq b$ the optimal portfolio $\hat{w}(t)$ is, obtained by allocating all resources between the fixed funds w^a and w^b , i.e.

$$\hat{w}(t) = \mu^a(t)w^a + \mu^b(t)w^b,$$

3. The relative proportions (μ^a, μ^b) of wealth allocated to w^a and w^b are given by

$$\begin{aligned}\mu^a(t) &= \frac{Y(t) - b}{a - b}, \\ \mu^b(t) &= \frac{a - Y(t)}{a - b}.\end{aligned}$$

The case with a risk free asset

Again we consider the standard model

$$dS = D(S)\alpha dt + D(S)\sigma dW(t),$$

We also assume the risk free asset B with dynamics

$$dB = rBdt.$$

We denote $B = S_0$ and consider portfolio weights $(w_0, w_1, \dots, w_n)'$ where $\sum_0^n w_i = 1$. We then eliminate w_0 by the relation

$$w_0 = 1 - \sum_1^n w_i,$$

and use the letter w to denote the portfolio weight vector for the risky assets only. Thus we use the notation

$$w = (w_1, \dots, w_n)',$$

Note: $w \in R^n$ without constraints.

HJB

We obtain

$$dX = X \cdot w'(\alpha - re)dt + (rX - c)dt + X \cdot w'\sigma dW,$$

where $e = (1, 1, \dots, 1)'$.

The HJB equation now becomes

$$\left\{ \begin{array}{l} V_t(t, x) + \sup_{c \geq 0, w \in R^n} \{F(t, c) + \mathcal{A}^{c, w}V(t, x)\} = 0, \\ V(T, x) = 0, \\ V(t, 0) = 0, \end{array} \right.$$

where

$$\begin{aligned} \mathcal{A}^c V &= xw'(\alpha - re)V_x(t, x) + (rx - c)V_x(t, x) \\ &+ \frac{1}{2}x^2w'\Sigma wV_{xx}(t, x). \end{aligned}$$

First order conditions

We maximize

$$F(t, c) + xw'(\alpha - re)V_x + (rx - c)V_x + \frac{1}{2}x^2w'\Sigma wV_{xx}$$

with $c \geq 0$ and $w \in R^n$.

The first order conditions are

$$\begin{aligned}\frac{\partial F}{\partial c}(t, c) &= V_x(t, x), \\ \hat{w} &= -\frac{V_x}{xV_{xx}}\Sigma^{-1}(\alpha - re),\end{aligned}$$

with geometrically obvious economic interpretation.

Mutual Fund Separation Theorem

1. The optimal portfolio consists of an allocation between two fixed mutual funds w^0 and w^f .
2. The fund w^0 consists only of the risk free asset.
3. The fund w^f consists only of the risky assets, and is given by

$$w^f = \Sigma^{-1}(\alpha - re).$$

4. At each t the optimal relative allocation of wealth between the funds is given by

$$\mu^f(t) = -\frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))},$$

$$\mu^0(t) = 1 - \mu^f(t).$$

3. The Martingale Approach

- Decoupling the wealth profile from the portfolio choice.
- Lagrange relaxation.
- Solving the general wealth problem.
- Example: Log utility.
- Example: The numeraire portfolio.
- Computing the optimal portfolio.
- The Merton fund separation theorems from a martingale perspective..

Problem Formulation

Standard model with internal filtration

$$\begin{aligned}dS_t &= D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t, \\dB_t &= rB_t dt.\end{aligned}$$

Assumptions:

- Drift and diffusion terms are allowed to be arbitrary adapted processes.
- The market is **complete**.
- We have a given initial wealth x_0

Problem:

$$\max_{h \in \mathcal{H}} E^P [\Phi(X_T)]$$

where

$$\mathcal{H} = \{\text{self financing portfolios}\}$$

given the initial wealth $X_0 = x_0$.

Some observations

- In a complete market, there is a unique martingale measure Q .

- Every claim Z satisfying the budget constraint

$$e^{-rT} E^Q [Z] = x_0,$$

is attainable by an $h \in \mathcal{H}$ and vice versa.

- We can thus write our problem as

$$\max_Z E^P [\Phi(Z)]$$

subject to the constraint

$$e^{-rT} E^Q [Z] = x_0.$$

- We can forget the wealth dynamics!

Basic Ideas

Our problem was

$$\max_Z E^P [\Phi(Z)]$$

subject to

$$e^{-rT} E^Q [Z] = x_0.$$

Idea I:

We can **decouple** the optimal portfolio problem:

- Finding the optimal wealth profile \hat{Z} .
- Given \hat{Z} , find the replicating portfolio.

Idea II:

- Rewrite the constraint under the measure P .
- Use Lagrangian techniques to relax the constraint.

Lagrange formulation

Problem:

$$\max_Z E^P [\Phi(Z)]$$

subject to

$$e^{-rT} E^P [L_T Z] = x_0.$$

Here L is the likelihood process, i.e.

$$L_T = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_T$$

The Lagrangian of this is

$$\mathcal{L} = E^P [\Phi(Z)] + \lambda \{x_0 - e^{-rT} E^P [L_T Z]\}$$

i.e.

$$\mathcal{L} = E^P [\Phi(Z) - \lambda e^{-rT} L_T Z] + \lambda x_0$$

The optimal wealth profile

Given enough convexity and regularity we now expect, given the dual variable λ , to find the optimal Z by maximizing

$$\mathcal{L} = E^P [\Phi(Z) - \lambda e^{-rT} L_T Z] + \lambda x_0$$

over unconstrained Z , i.e. to maximize

$$\int_{\Omega} \{ \Phi(Z(\omega)) - \lambda e^{-rT} L_T(\omega) Z(\omega) \} dP(\omega)$$

This is a trivial problem!

We can simply maximize $Z(\omega)$ for each ω separately.

$$\max_z \{ \Phi(z) - \lambda e^{-rT} L_T z \}$$

The optimal wealth profile

Our problem:

$$\max_z \{ \Phi(z) - \lambda e^{-rT} L_T z \}$$

First order condition

$$\Phi'(z) = \lambda e^{-rT} L_T$$

The optimal Z is thus given by

$$\hat{Z} = G(\lambda e^{-rT} L_T)$$

where

$$G(y) = [\Phi']^{-1}(y).$$

The dual variable λ is determined by the constraint

$$e^{-rT} E^P [L_T \hat{Z}] = x_0.$$

Example – log utility

Assume that

$$\Phi(x) = \ln(x)$$

Then

$$g(y) = \frac{1}{y}$$

Thus

$$\hat{Z} = G(\lambda e^{-rT} L_T) = \frac{1}{\lambda} e^{rT} L_T^{-1}$$

Finally λ is determined by

$$e^{-rT} E^P [L_T \hat{Z}] = x_0.$$

i.e.

$$e^{-rT} E^P \left[L_T \frac{1}{\lambda} e^{rT} L_T^{-1} \right] = x_0.$$

so $\lambda = x_0^{-1}$ and

$$\hat{Z} = x_0 e^{rT} L_T^{-1}$$

The Numeraire Portfolio

Standard approach:

- Choose a fixed numeraire (portfolio) N .
- Find the corresponding martingale measure, i.e. find Q^N s.t.

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are Q^N -martingales.

Alternative approach:

- Choose a fixed measure Q .
- Find numeraire N such that $Q = Q^N$.

Special case:

- Set $Q = P$
- Find numeraire N such that $Q^N = P$ i.e. such that

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are Q^N -martingales under the **objective** measure P .

- This N is the **numeraire portfolio**.

Log utility and the numeraire portfolio

Definition:

The **growth optimal portfolio** (GOP) is the portfolio which is optimal for log utility (for arbitrary terminal date T).

Theorem:

Assume that X is GOP. Then X is the numeraire portfolio.

Proof:

We have to show that the process

$$Y_t = \frac{S_t}{X_t}$$

is a P martingale. From above we know that

$$X_T = x_0 e^{rT} L_T^{-1}.$$

We also have (why?)

$$X_t = e^{-r(T-t)} E^Q [X_T | \mathcal{F}_t] = e^{-r(T-t)} E^P \left[\frac{X_T L_T}{L_t} \middle| \mathcal{F}_t \right]$$

Thus

$$\begin{aligned} X_t &= e^{-r(T-t)} E^P \left[\frac{X_T L_T}{L_t} \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E^Q \left[\frac{x_0 e^{rT} L_T}{L_T L_t} \middle| \mathcal{F}_t \right] = x_0 e^{rt} L_t^{-1}. \end{aligned}$$

as expected.

Thus

$$\frac{S_t}{X_t} = x_0^{-1} e^{-rt} S_t L_t$$

which is a P martingale, since $x_0^{-1} e^{-rt} S_t$ is a Q martingale.

Back to general case: Computing L_T

We recall

$$\hat{Z} = G(\lambda e^{-rT} L_T).$$

The likelihood process L is computed by using Girsanov. We recall

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t,$$

We know from Girsanov that

$$dL_t = L_t \varphi_t^* dW_t$$

so

$$dW_t = \varphi_t dt + dW_t^Q$$

where W^Q is Q -Wiener.

Thus

$$dS_t = D(S_t) \{ \alpha_t + \sigma_t \varphi_t \} dt + D(S_t) \sigma_t dW_t^Q,$$

Computing L_T , continued

Recall

$$dS_t = D(S_t) \{ \alpha_t + \sigma_t \varphi_t \} dt + D(S_t) \sigma_t dW_t^Q,$$

The kernel φ is determined by the martingale measure condition

$$\alpha_t + \sigma_t \varphi_t = \mathbf{r}$$

where

$$\mathbf{r} = \begin{bmatrix} r \\ \vdots \\ r \end{bmatrix}$$

Market completeness implies that σ_t is invertible so

$$\varphi_t = \sigma_t^{-1} \{ \mathbf{r} - \alpha_t \}$$

and

$$L_T = \exp \left(\int_0^T \varphi_t dW_t - \frac{1}{2} \int_0^T \|\varphi_t\|^2 dt \right)$$

Finding the optimal portfolio

- We can easily compute the optimal wealth profile.
- How do we compute the optimal portfolio?

Recall:

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t,$$

wealth dynamics

$$dX_t = h_t^B dB_t + h_t^S dS_t$$

or

$$dX_t = X_t u_t^B r dt + X_t u_t^S D(S_t)^{-1} dS_t$$

where

$$h^S = (h^1, \dots, h^n), \quad u^S = (u^1, \dots, u^n)$$

Assume for simplicity that $r = 0$ or consider normalized prices.

Recall wealth dynamics

$$dX_t = h_t^S dS_t$$

alternatively

$$dX_t = h_t^S D(S_t) \sigma_t dW_t^Q$$

alternatively

$$dX_t = X_t u_t^S \sigma_t dW_t^Q$$

Obvious facts:

- X is a Q martingale.
- $X_T = \hat{Z}$

Thus the optimal wealth process is determined by

$$X_t = E^Q \left[\hat{Z} \mid \mathcal{F}_t \right]$$

Recall

$$X_t = E^Q \left[\hat{Z} \mid \mathcal{F}_t \right]$$

Martingale representation theorem gives us

$$dX_t = \xi_t dW_t^Q,$$

but also

$$dX_t = X_t u_t^S \sigma_t dW_t^Q$$

$$dX_t = h_t^S D(S_t) \sigma_t dW_t^Q$$

Thus u_t^S and h_t^S are determined by

$$u_t^S = \frac{1}{X_t} \xi_t \sigma_t^{-1}$$

$$h_t^S = \xi_t \sigma_t^{-1} D(S_t)^{-1}.$$

and

$$u_t^B = 1 - u_t^S \mathbf{e}, \quad h_t^B = X_t - h_t^S S_t$$

How do we find ξ ?

Recall

$$\begin{aligned}X_t &= E^Q \left[\hat{Z} \mid \mathcal{F}_t \right] \\dX_t &= \xi_t dW_t^Q,\end{aligned}$$

We need to compute ξ .

In a Markovian framework this follows directly from the Itô formula.

Recall

$$\hat{Z} = H(L_T) = G(\lambda L_T)$$

where

$$G = [\Phi']^{-1}$$

and

$$\begin{aligned}dL_t &= L_t \varphi_t^* dW_t, \\dW &= \varphi dt + dW_t^Q\end{aligned}$$

so

$$dL_t = L_t \|\varphi_t\|^2 dt + L_t \varphi_t^* dW_t^Q$$

Finding ξ

If the model is Markovian we have

$$\alpha_t = \alpha(S_t), \quad \sigma_t = \sigma(S_t), \quad \varphi_t = \sigma(S_t)^{-1} \{\alpha(S_t) - \mathbf{r}\}$$

so

$$\begin{aligned} X_t &= E^Q [H(L_T) | \mathcal{F}_t] \\ dS_t &= D(S_t)\sigma(S_t)dW_t^Q, \\ dL_t &= L_t\|\varphi(S_t)\|^2 dt + L_t\varphi(S_t)^* dW_t^Q \end{aligned}$$

Thus we have

$$X_t = F(t, S_t, L_t)$$

where, by the Kolmogorov backward equation

$$\begin{aligned} F_t + L\|\varphi\|^2 F_L + \frac{1}{2}L^2\|\varphi\|^2 F_{LL} + \frac{1}{2}tr \{\sigma F_{ss}\sigma^*\} &= 0, \\ F(T, s, L) &= H(L) \end{aligned}$$

Finding ξ , contd.

We had

$$X_t = F(t, S_t, L_t)$$

and Itô gives us

$$dX_t = \{F_S D(S_t) \sigma(S_t) + F_L L_t \varphi^*(S_t)\} dW_t$$

Thus

$$\xi_t = F_S D(S_t) \sigma(S_t) + F_L L_t \varphi^*(S_t).$$

and

$$\begin{aligned} u_t^S &= \frac{1}{X_t} \xi_t \sigma(S_t)^{-1} \\ h_t^S &= \xi_t \sigma(S_t)^{-1} D(S_t)^{-1}. \end{aligned}$$

Mutual Funds – Martingale Version

We now assume constant parameters

$$\alpha(s) = \alpha, \quad \sigma(s) = \sigma, \quad \varphi(s) = \varphi$$

We recall

$$\begin{aligned} X_t &= E^Q [H(L_T) | \mathcal{F}_t] \\ dL_t &= L_t \|\varphi\|^2 dt + L_t \varphi^* dW_t^Q \end{aligned}$$

Now L is Markov so we have (without any S)

$$X_t = F(t, L_t)$$

Thus

$$\xi_t = F_L L_t \varphi^*, \quad u_t^S = \frac{F_L L_t}{X_t} \varphi^* \sigma^{-1}$$

and we have fund separation with the fixed risky fund given by

$$w = \varphi^* \sigma^{-1} = \{\mathbf{r}^* - \alpha^*\} \{\sigma \sigma^*\}^{-1}.$$

3. Filtering theory

- Motivational problem.
- The Innovations process.
- The non-linear FKK filtering equations.
- The Wonham filter.
- The Kalman filter.

An investment problem with stochastic rate of return

Model:

$$dS_t = S_t \alpha(Y_t) dt + S_t \sigma dW_t$$

W is scalar and Y is some factor process. We assume that (S, Y) is Markov and adapted to the filtration \mathbf{F} .

Wealth dynamics

$$dX_t = X_t [r + u_t (\alpha - r)] dt + u_t X_t \sigma dW_t$$

Objective:

$$\max_u E^P [\Phi(X_T)]$$

Information structure:

- Complete information: We observe S and Y , so $u \in \mathbf{F}$
- Incomplete information: We only observe S , so $u \in \mathbf{F}^S$. We need **filtering theory**.

Filtering Theory – Setup

Given some filtration $\underline{\mathcal{F}}$:

$$\begin{aligned}dY_t &= a_t dt + dM_t \\dZ_T &= b_t dt + dW_t\end{aligned}$$

Here all processes are \mathbf{F} adapted and

$$\begin{aligned}Y &= \text{signal process,} \\Z &= \text{observation process,} \\M &= \text{martingale w.r.t. } \mathbf{F} \\W &= \text{Wiener w.r.t. } \mathbf{F}\end{aligned}$$

We assume (for the moment) that M and W are **independent**.

Problem:

Compute (recursively) the filter estimate

$$\hat{Y}_t = \Pi_t [Y] = E [Y_t | \mathcal{F}_t^Z]$$

The innovations process

Recall:

$$dZ_T = b_t dt + dW_t$$

Our best guess of b_t is \hat{b}_t , so the genuinely new information should be

$$dZ_t - \hat{b}_t dt$$

The **innovations process** ν is defined by

$$\nu_t = dZ_t - \hat{b}_t dt$$

Theorem: The process ν is \mathbf{F}^Z -Wiener.

Proof: By Levy it is enough to show that

- ν is an \mathbf{F}^Z martingale.
- $\nu_t^2 - t$ is an \mathbf{F}^Z martingale.

I. ν is an \mathbb{F}^Z martingale:

From definition we have

$$d\nu_t = (b_t - \hat{b}_t) dt + dW_t \quad (3)$$

so

$$\begin{aligned} E_s^Z [\nu_t - \nu_s] &= \int_s^t E_s^Z [b_u - \hat{b}_u] du + E_s^Z [W_t - W_s] \\ &= \int_s^t E_s^Z [E_u^Z [b_u - \hat{b}_u]] du + E_s^Z [E_s [W_t - W_s]] = 0 \end{aligned}$$

I. $\nu_t^2 - t$ is an \mathbb{F}^Z martingale:

From Itô we have

$$d\nu_t^2 = 2\nu_t d\nu_t + (d\nu_t)^2$$

Here $d\nu$ is a martingale increment and from (3) it follows that $(d\nu_t)^2 = dt$.

Remark 1:

The innovations process gives us a Gram-Schmidt orthogonalization of the increasing family of Hilbert spaces

$$L^2(\mathcal{F}_t^Z); \quad t \geq 0.$$

Remark 2:

The use of Itô above requires general semimartingale integration theory, since we do not know a priori that ν is Wiener.

Filter dynamics

From the Y dynamics we guess that

$$d\hat{Y}_t = \hat{a}_t dt + \text{martingale}$$

Definition: $dm_t = d\hat{Y}_t - \hat{a}_t dt$.

Proposition: m is an \mathcal{F}_t^Z martingale.

Proof:

$$\begin{aligned} E_s^Z [m_t - m_s] &= E_s^Z [\hat{Y}_t - \hat{Y}_s] - E_s^Z \left[\int_s^t \hat{a}_u du \right] \\ &= E_s^Z [Y_t - Y_s] - E_s^Z \left[\int_s^t \hat{a}_u du \right] \\ &= E_s^Z [M_t - M_s] - E_s^Z \left[\int_s^t (a_u - \hat{a}_u) du \right] \\ &= E_s^Z [E_s [M_t - M_s]] - E_s^Z \left[\int_s^t E_u^Z [a_u - \hat{a}_u] du \right] = 0 \end{aligned}$$

Filter dynamics

We now have the filter dynamics

$$d\hat{Y}_t = \hat{a}_t dt + dm_t$$

where m is an \mathcal{F}_t^Z martingale.

If the **innovations hypothesis**

$$\mathcal{F}_t^Z = \mathcal{F}_t^\nu$$

is true, then the martingale representation theorem would give us an \mathcal{F}_t^Z adapted process h such that

$$dm_t = h_t d\nu_t \tag{4}$$

The innovations hypothesis is not generally correct but FKK have proved that in fact (4) is always true.

Filter dynamics

We thus have the filter dynamics

$$d\widehat{Y}_t = \widehat{a}_t dt + h_t d\nu_t$$

and it remains to determine the gain process h .

Proposition: The process h is given by

$$h_t = \widehat{Y}_t \widehat{b}_t - \widehat{Y}_t \widehat{b}_t$$

We give a slighty heuristic proof.

Proof sketch

From Itô we have

$$d(Y_t Z_t) = Y_t b_t dt + Y_t dW_t + Z_t a_t dt + Z_t dM_t$$

using

$$d\hat{Y}_t = \hat{a}_t dt + h_t d\nu_t$$

and

$$dZ_t = \hat{b}_t dt + d\nu_t$$

we have

$$d(\hat{Y}_t Z_t) = \hat{Y}_t \hat{b}_t dt + \hat{Y}_t d\nu_t + Z_t \hat{a}_t dt + Z_t h_t d\nu_t + h_t dt$$

Formally we also should have

$$E \left[d(Y_t Z_t) - d(\hat{Y}_t Z_t) \middle| \mathcal{F}_t^Z \right] = 0$$

which gives us

$$\left(\widehat{Y_t b_t} - \hat{Y}_t \hat{b}_t - h_t \right) dt = 0.$$

The filter equations

For the model

$$\begin{aligned}dY_t &= a_t dt + dM_t \\dZ_T &= b_t dt + dW_t\end{aligned}$$

where M and W are independent, we have the FKK non-linear filter equations

$$\begin{aligned}d\hat{Y}_t &= \hat{a}_t dt + \left\{ \widehat{Y_t b_t} - \hat{Y}_t \hat{b}_t \right\} d\nu_t \\d\nu_t &= dZ_t - \hat{b}_t dt\end{aligned}$$

Remark: It is easy to see that

$$h_t = E \left[\left(Y_t - \hat{Y}_t \right) \left(b_t - \hat{b}_t \right) \middle| \mathcal{F}_t^Z \right]$$

The general filter equations

For the model

$$\begin{aligned}dY_t &= a_t dt + dM_t \\dZ_T &= b_t dt + \sigma_t dW_t\end{aligned}$$

where

- The process σ is \mathcal{F}_t^Z adapted and positive.
- There is no assumption of independence between M and W .

we have the filter

$$\begin{aligned}d\hat{Y}_t &= \hat{a}_t dt + \left[\hat{D}_t + \frac{1}{\sigma_t} \left\{ \widehat{Y}_t \widehat{b}_t - \hat{Y}_t \hat{b}_t \right\} \right] d\nu_t \\d\nu_t &= \frac{1}{\sigma_t} \left\{ dZ_t - \hat{b}_t dt \right\} \\dD_t &= \frac{d\langle M, W \rangle_t}{dt}\end{aligned}$$

Comment on $\langle M, W \rangle$

This requires semimartingale theory but there are two simple cases

- If M is Wiener then

$$d\langle M, W \rangle_t = dM_t dW_t$$

with usual multiplication rules.

- If M is a pure jump process then

$$d\langle M, W \rangle_t = 0.$$

Filtering a Markov process

Assume that Y is Markov with generator G . We want to compute $\Pi_t [f(Y_t)]$, for some nice function f . Dynkin's formula gives us

$$df(Y_t) = (Gf)(Y_t)dt + dM_t$$

Assume that the observations are

$$dZ_t = b(Y_t)dt + dW_t$$

where W is independent of Y .

The filter equations are now

$$\begin{aligned} d\Pi_t [f] &= \Pi_t [Gf] dt + \{\Pi_t [fb] - \Pi_t [f] \Pi_t [b]\} d\nu_t \\ d\nu_t &= dZ_t - \Pi_t [b] dt \end{aligned}$$

Remark: To obtain $d\Pi_t [f]$ we need $\Pi_t [fb]$ and $\Pi_t [b]$. This leads generically to an infinite dimensional system of filter equations.

On the filter dimension

$$d\Pi_t[f] = \Pi_t[Gf] dt + \{\Pi_t[fb] - \Pi_t[f] \Pi_t[b]\} d\nu_t$$

- To obtain $d\Pi_t[f]$ we need $\Pi_t[fb]$ and $\Pi_t[b]$.
- Thus we apply the FKK equations to Gf and b .
- This leads to new filter estimates to determine and generically to an **infinite dimensional** system of filter equations.
- The filter equations are really equations for the entire conditional distribution of Y .
- You can only expect the filter to be finite when the conditional distribution of Y is finitely parameterized.
- There are only very few examples of finite dimensional filters.
- The most well known finite filters are the Wonham and the Kalman filters.

The Wonham filter

Assume that Y is a continuous time Markov chain on the state space $\{1, \dots, n\}$ with (constant) generator matrix H . Define the indicator processes by

$$\delta_i(t) = I \{Y_t = i\}, \quad i = 1, \dots, n.$$

Dynkin's formula gives us

$$d\delta_t^i = \sum_j H(j, i) \delta_j dt + dM_t^i, \quad i = 1, \dots, n.$$

Observations are

$$dZ_t = b(Y_t)dt + dW_t.$$

Filter equations:

$$d\Pi_t [\delta_i] = \sum_j H(j, i) \Pi_t [\delta_j] dt + \{\Pi_t [\delta_i b] - \Pi_t [\delta_i] \Pi_t [b]\} d\nu_t$$

$$d\nu_t = dZ_t - \Pi_t [b] dt$$

We note that

$$b(Y_t) = \sum_i b(i)\delta_i(t)$$

so

$$\begin{aligned}\Pi_t [\delta_i b] &= b(i)\Pi_t [\delta_i], \\ \Pi_t [b] &= \sum_j b(j)\Pi_t [\delta_j]\end{aligned}$$

We finally have the Wonham filter

$$\begin{aligned}d\hat{\delta}_i &= \sum_j H(j, i)\hat{\delta}_j dt + \left\{ b(i)\hat{\delta}_i - \hat{\delta}_i \sum_j b(j)\hat{\delta}_j \right\} d\nu_t, \\ d\nu_t &= dZ_t - \sum_j b(j)\hat{\delta}_j dt\end{aligned}$$

The Kalman filter

$$dY_t = aY_t dt + c dV_t,$$

$$dZ_t = Y_t dt + dW_t$$

W and V are independent Wiener

FKK gives us

$$d\Pi_t [Y] = a\Pi_t [Y] dt + \left\{ \Pi_t [Y^2] - (\Pi_t [Y])^2 \right\} d\nu_t$$

$$d\nu_t = dZ_t - \Pi_t [Y] dt$$

We need $\Pi_t [Y^2]$, so use Itô to get write

$$dY_t^2 = \{2aY_t^2 + c^2\} dt + 2cY_t dV_t$$

From FKK:

$$\begin{aligned} d\Pi_t [Y^2] &= \{2a\Pi_t [Y^2] + c^2\} dt \\ &+ \{ \Pi_t [Y^3] - \Pi_t [Y^2] \Pi_t [Y] \} d\nu_t \end{aligned}$$

Now we need $\Pi_t [Y^3]$! Etc!

Define the conditional error variance by

$$H_t = \Pi_t \left[(Y_t - \Pi_t [Y])^2 \right] = \Pi_t [Y^2] - (\Pi_t [Y])^2$$

Itô gives us

$$d(\Pi_t [Y])^2 = \left[2a (\Pi_t [Y])^2 + H^2 \right] dt + 2\Pi_t [Y] H d\nu_t$$

and Itô again

$$\begin{aligned} dH_t &= \{ 2aH_t + c^2 - H_t^2 \} dt \\ &+ \left\{ \Pi_t [Y^3] - 3\Pi_t [Y^2] \Pi_t [Y] + 2(\Pi_t [Y])^3 \right\} d\nu_t \end{aligned}$$

In **this particular case** we know (why?) that the distribution of Y conditional on Z is Gaussian!

Thus we have

$$\Pi_t [Y^3] = 3\Pi_t [Y^2] \Pi_t [Y] - 2(\Pi_t [Y])^3$$

so H is deterministic (as expected).

The Kalman filter

Model:

$$dY_t = aY_t dt + c dV_t,$$

$$dZ_t = Y_t dt + dW_t$$

Filter:

$$d\Pi_t [Y] = a\Pi_t [Y] dt + H_t d\nu_t$$

$$\dot{H}_t = 2aH_t + c^2 - H_t^2$$

$$d\nu_t = dZ_t - \Pi_t [Y] dt$$

$$H_t = \Pi_t \left[(\Pi_t [Y_t] - \Pi_t [Y])^2 \right]$$

Remark: Because of the Gaussian structure, the conditional distribution evolves on a two dimensional submanifold. Hence a two dimensional filter.

Optimal investment with stochastic rate of return

- A market model with a stochastic rate of return.
- Optimal portfolios under complete information.
- Optimal portfolios under partial information.

An investment problem with stochastic rate of return

Model:

$$dS_t = S_t \alpha(Y_t) dt + S_t \sigma dW_t$$

W is scalar and Y is some factor process. We assume that (S, Y) is Markov and adapted to the filtration \mathbf{F} .

Wealth dynamics

$$dX_t = X_t \{r + u_t [\alpha(Y_t) - r]\} dt + u_t X_t \sigma dW_t$$

Objective:

$$\max_u E^P [X_T^\gamma]$$

We assume that Y is a Markov process. with generator \mathcal{A} . We will treat several cases.

A. Full information, Y and W independent.

The HJB equation for $F(t, x, y)$ becomes

$$F_t + \sup_u \left\{ u [\alpha - r] x F_x + r x F_x + \frac{1}{2} u^2 x^2 \sigma^2 F_{xx} \right\} + \mathcal{A}F = 0$$

where \mathcal{A} operates on the y -variable. Obvious boundary condition

$$F(t, x, y) = x^\gamma$$

First order condition gives us:

$$\hat{u} = \frac{r - \alpha}{x \sigma^2} \cdot \frac{F_x}{F_{xx}}$$

Plug into HJB:

$$F_t - \frac{(\alpha - r)^2}{2\sigma^2} \frac{F_x^2}{F_{xx}} + r x F_x + \mathcal{A}F = 0$$

Ansatz:

$$F(T, x, y) = x^\gamma G(t, y)$$

$$F_t = x^\gamma G_t, \quad F_x = \gamma x^{\gamma-1} G, \quad F_{xx} = \gamma(\gamma-1)x^{\gamma-2} G$$

Plug into HJB:

$$x^\gamma G_t + x^\gamma \frac{(\alpha - r)^2}{2\sigma^2} \beta G + r\gamma x^\gamma G + x^\gamma \mathcal{A}G = 0$$

where $\beta = \gamma/(\gamma - 1)$.

$$\begin{aligned} G_t(t, y) + H(y)G(t, y) + \mathcal{A}G(t, y) &= 0, \\ G(T, y) &= 1. \end{aligned}$$

here

$$H(y) = r\gamma - \frac{[\alpha(y) - r]^2}{2\sigma^2} \beta$$

Kolmogorov gives us

$$H(y) = E_{t,y} \left[e^{\int_t^T H(Y_s) ds} \right]$$

B. Full information, Y and W dependent.

Now we allow for dependence but restrict Y to dynamics of the form.

$$\begin{aligned}dS_t &= S_t \alpha(Y_t) dt + S_t \sigma dW_t \\dX_t &= X_t \{r + u_t [\alpha(Y_t) - r]\} dt + u_t X_t \sigma dW_t \\dY_t &= a(Y_t) dt + b(Y_t) dW_t\end{aligned}$$

with **the same** Wiener process W driving both S and Y . The imperfectly correlated case is a bit more messy but can also be handled.

HJB Equation for $F(t, x, y)$:

$$\begin{aligned}F_t + aF_y + \frac{1}{2}b^2 F_{yy} + rx F_x \\+ \sup_u \left\{ u [\alpha - r] x F_x + \frac{1}{2} u^2 x^2 \sigma^2 F_{xx} + u x b \sigma F_{xy} \right\} = 0.\end{aligned}$$

Ansatz.

HJB:

$$F_t + aF_y + \frac{1}{2}b^2 F_{yy} + rx F_x + \sup_u \left\{ u [\alpha - r] x F_x + \frac{1}{2}u^2 x^2 \sigma^2 F_{xx} + uxb\sigma F_{xy} \right\} = 0.$$

After a lot of thinking we make the Ansatz

$$F(t, x, y) = x^\gamma h^{1-\gamma}(t, y)$$

and then it is not hard to see that h satisfies a standard parabolic PDE. (Plug Ansatz into HJB).

See Zariphopoulou for details and more complicated cases. See Björk-Davis-Landn for a different approach.

C. Partial information.

Model:

$$dS_t = S_t \alpha(Y_t) dt + S_t \sigma dW_t$$

Assumption: Y cannot be observed directly.

Requirement: The control u must be \mathcal{F}_t^S adapted. We thus have a partially observed system.

Idea: Project the S dynamics onto the smaller \mathcal{F}_t^Z filtration and add filter equations in order to reduce the problem to the completely observable case.

Define Z by

$$dZ_t = \frac{dS_t}{\sigma S_t}$$

and note (why?) that $\mathcal{F}_t^Z = \mathcal{F}_t^S$. We have

$$dZ_t = \frac{\alpha(Y_t)}{\sigma} dt + dW_t$$

Projecting onto the \mathcal{S} -filtration

Recall that

$$dZ_t = \frac{\alpha Y_t}{\sigma} dt + \sigma dW_t$$

From filtering theory we know that

$$dZ_t = \frac{\hat{\alpha}_t}{\sigma} dt + \sigma d\nu_t$$

where ν is $\mathcal{F}_t^{\mathcal{S}}$ -Wiener and

$$\hat{\alpha}_t = E [\alpha(Y_t) | \mathcal{F}_t^{\mathcal{S}}]$$

Projecting onto the S -filtration ct'd

Recall

$$\begin{aligned}dZ_t &= \frac{dS_t}{\sigma S_t} \\dZ_t &= \frac{\hat{\alpha}_t}{\sigma} dt + \sigma d\nu_t\end{aligned}$$

From these equations we have have the following S dynamics on the S filtration

$$dS_t = S_t \hat{\alpha}_t dt + S_t \sigma d\nu_t$$

and wealth dynamics

$$dX_t = X_t \{r + u_t (\hat{\alpha}_t - r)\} dt + u_t X_t \sigma d\nu_t$$

Reformulated problem

We now have the problem

$$\max_u E [X_T^\gamma]$$

for Z -adapted controls given wealth dynamics

$$dX_t = X_t \{r + u_t (\hat{\alpha}_t - r)\} dt + u_t X_t \sigma d\nu_t$$

If we now can model Y such that the (linear!) observation dynamics for Z will produce a finite filter for $\hat{\alpha}$, then **we are back in the completely observable case** with $\alpha(Y_t)$ replaced by $\hat{\alpha}_t$.

We need a finite dimensional filter!

Two choices for Y

- Linear Y dynamics. This will give us the Kalman filter. See Brendle
- Y as a Markov chain. This will give us the Wonham filter. See Bäuerle and Rieder.

Kalman case (Brendle)

Assume that

$$dS_t = Y_t S_t dt + S_t \sigma dW_t$$

with Y dynamics

$$dY_t = aY_t dt + c dV_t$$

where W and V are independent. Observations:

$$dZ_t = \frac{Y_t}{\sigma} dt + dW_t$$

We have a standard Kalman filter.

Wealth dynamics

$$dX_t = X_t \left\{ r + u_t \left(\hat{Y}_t - r \right) \right\} dt + u_t X_t \sigma d\nu_t$$

Kalman case, solution.

$$\max_u E [X_T^\gamma]$$

$$\begin{aligned} dX_t &= X_t \left\{ r + u_t \left(\hat{Y}_t - r \right) \right\} dt + u_t X_t \sigma d\nu_t \\ d\hat{Y}_t &= a\hat{Y}_t dt + H_t \sigma d\nu_t \end{aligned}$$

where H is deterministic and given by a Riccati equation.

We are back in standard completely observable case with state variables X and \hat{Y} .

Thus the optimal value function is of the form

$$F(t, x, \hat{y}) = x^\gamma h^{1-\gamma}(t, \hat{y})$$

where h solves a parabolic PDE and can be computed explicitly.

References

We have used the following texts but see also other references therein (and /or go the author's home pages):

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Zariphopoulou, T. (2001) "A solution approach to valuation with unhedgable risk". *Finance and Stochastics*,

References Continued

For very interesting and deep works on duality, which are outside the scope of these lectures, see

Schachermayer, W. “Utility Maximisation in Incomplete Markets”. In: *Stochastic Methods in Finance* (M. Frittelli, W. Runggaldier, eds.), Springer Lecture Notes in Mathematics, Vol. 1856 (2004), pp. 225-288.

Rogers, L.C.G. “Duality in constrained optimal investment and consumption problems: a synthesis”. In *Paris-Princeton Lectures on Mathematical Finance 2002*, Springer Lecture Notes in Mathematics 1814, 2003, 95–131.