Variance Risk Premiums in Currency Options

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Abstract

Synthetic variance swap rates, computed from currency option implied volatility quotes using the vanna-volga method, are compared to realized exchange rate variance in an effort to determine the existence of a variance risk premium. Due to conflicting results for different time periods and different currency pairs, no conclusion is reached. The variance swap rate consistent with vanna-volga prices is found to be given by a simple expression, but may significantly underestimate the true variance swap rate. Since for currency variance swaps the notional amount could be given in either of the two involved currencies there are two variance swap rates. The difference between the two rates is explored and is found to be small and determined by the slope of the implied volatility smile.

Acknowledgements

I am grateful to Boualem Djehiche, my supervisor at KTH, for insightful comments and suggestions, and to Johan Blixt for providing the data, suggesting the topic and the vanna volga method, as well as giving valuable advice. Any errors are mine.

1 Introduction

A variance swap is a contract which pays the difference between the realized variance of price movements of an underlying financial asset, and a fixed variance swap rate, times a notional amount. Thus it is a forward contract on the realized variance. Since the realized variance can be replicated using European options of the same maturity, the variance swap rate is more or less uniquely determined by option prices. This variance swap rate is the risk neutral expectation of the realized variance. It is interesting to investigate whether this risk neutral expectation is equal to the "real world" expectation. If not, a *variance risk premium* can be said to exist (assuming investors have rational expectations).

In this study synthetic variance swap rates computed from implied volatility data on currency options are compared to realized variance, in order to determine the existence of variance risk premium in currency options.

For exchange rates there are two notions of risk neutral expected variance. A variance swap could be denominated in either of the two involved currencies. Hence there are two variance swap rates, one where the notional amount is in the domestic currency and one where it is in the foreign currency. The difference between the two rates is captured by the slope of the implied volatility smile. For example, options on the USDJPY exchange rate often exhibit a steep downward sloping implied volatility smile; (out-of-the-money) USD calls have lower implied volatility than USD puts. It is showed that this implies that the variance swap rate of a dollar denominated variance swap is lower than that of a yen denominated variance swap. It is therefore interesting to investigate variance swap return from both the foreign and domestic perspective and to study the difference between the two rates.

Synthetic variance swap returns have previously been used to quantify the variance risk premium in index options and options on individual stocks. Strongly significant negative risk premiums have been found, particularly in index options, by Petersson and Šarić (2008), Carr and Wu (2007) and Bondarenko (2007) among others.

In other work evidence of a variance/volatility risk premium in currency options have been found using other methods. Jorion (1995) investigates the predictive power of Black-Scholes implied volatilities in futures options on the German mark, Japanese yen and Swiss franc and finds that implied volatility is a biased estimate of realized volatility. Guo (1998) uses a parametric approach where parameters of the Heston (1993) model are estimated from options on the USDDEM exchange rate between 1987 and 1992. Evidence of a negative variance risk premium is found. Sarwar (2001) investigates the volatility risk premium in options traded on the Philadelphia Stock Exchange, using a parametric specification based on the Heston model. No significant volatility risk premium is found in options on the British pound. Low and Zhang (2005) calculate the returns from delta hedged straddles. They find that these returns are related to the volatility risk premium. Evidence of negative risk premiums is found in options on the British pound, the euro, the Japanese yen and the Swiss franc against the U.S. dollar.

The synthetic variance swap approach has the advantage of not relying on a particular model of the underlying price process. (This also applies to the method used by Low and Zhang.) Moreover the returns on variance swaps are independent of the sign of price movements and of the underlying price level.

It is also important to consider how variance swap returns are related to returns on other investment opportunities. A possible explanation for the variance risk premium found in stock and index options is the negative correlation between variance and stock market returns. In the capital asset pricing model this means that long variance swaps have negative beta and therefore a negative excess return should be expected (since the expected return of the market portfolio is assumed to be positive). This explanation is tested by Petersson and Šarić (2008) and Carr and Wu (2007) in a CAPM framework. It is found that the CAPM beta only partially explains the variance risk premium, meaning that variance swap returns have negative alpha.

If currency volatility too is negatively correlated with stock market returns this could be an explanation of the variance risk premium, if there is one. In other words, the premium earned by sellers of currency variance is in this model compensation for bearing systematic risk.

The paper is organized as follows. Section 2 deals with the theory of the variance risk premium. In section 3 the replication of variance swaps is described and some properties of the foreign variance swap rate are derived. Section 4 outlines the methodology of the empirical analysis. Section 5 reports the results. In section 6 the accuracy of the method is investigated and section 7 concludes the paper.

2 The Variance Risk Premium

The variance risk premium of a time period $[t, T]$ is the difference between the time t expectation of the realized variance with respect to the risk neutral measure and the expectation with respect to the objective probability measure.

As shown by Carr and Wu (2007) the variance risk premium can be expressed in terms of the covariance with a pricing kernel. Let $\mathsf{E}_t^{\mathbb{Q}}$ denote the expectation with respect to the measure $\mathbb{Q}$ conditional on the filtration $\mathcal{F}_t$. Absence of arbitrage implies the existence of a martingale measure $\mathbb{Q}$ with respect to which the expected value of any time T payoff is its forward price (Björk, 2004).[1] The variance swap rate $\mathrm{KV}_{t,T}$ (which is the forward price of realized variance, determined by vanilla option prices) can therefore be written

$$\mathrm{KV}_{t,T} = \mathsf{E}_t^{\mathbb{Q}} \left[\mathrm{RV}_{t,T}\right].$$

Define L_T as the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to the objective probability

[1]This is the T-forward measure, with the price of a zero-coupon bond as numeraire.

measure $\mathbb{P}$ on $\mathcal{F}_T$,

$$L_T = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \quad \text{on } \mathcal{F}_T.$$

Using Bayes' formula this can be rewritten as

$$\mathrm{KV}_{t,T} = \frac{\mathsf{E}_t^{\mathbb{P}}\left[L_T \mathrm{RV}_{t,T}\right]}{\mathsf{E}_t^{\mathbb{P}}\left[L_T\right]}. \tag{1}$$

This can be decomposed into

$$\mathrm{KV}_{t,T} = \mathsf{E}_t^{\mathbb{P}}\left[\mathrm{RV}_{t,T}\right] + \mathrm{Cov}_t^{\mathbb{P}}\left[\frac{L_T}{\mathsf{E}_t^{\mathbb{P}}\left[L_T\right]}, \mathrm{RV}_{t,T}\right].$$

Thus the variance swap rate is the expected realized variance plus the covariance with the pricing kernel. Carr and Wu define the negative of the covariance term as the variance risk premium and note that the sample average of the difference between the realized variance and the variance swap rate is an estimate of the average risk premium.

Dividing by $\mathrm{KV}_{t,T}$ and rearranging

$$\mathsf{E}_t^{\mathbb{P}}\left[\frac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}} - 1\right] = -\,\mathrm{Cov}_t^{\mathbb{P}}\left[\frac{L_T}{\mathsf{E}_t^{\mathbb{P}}\left[L_T\right]}, \frac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}} - 1\right].$$

This captures the variance excess return since the cost of the realized variance is the discounted variance swap rate. In line with Bondarenko (2007), in this investigation the variance risk premium is measured as the sample average of variance swap excess returns defined by

$$\widehat{\mathrm{RP}}_{t,T} = \frac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}} - 1.$$

The null-hypothesis to be tested states that the market is indifferent to variance risk in which case the covariance terms would be zero and the sample average of $\widehat{\mathrm{RP}}_{t,T}$ for a large number of observations would be close to zero.

Alternatively it could be measured as the average of $\mathrm{RV}_{t,T} - \mathrm{KV}_{t,T}$ as in Carr and Wu (2007) and Petersson and Šarić (2008). But as Bondarenko points out the magnitude of this difference is positively related to the variance swap rate. And because the variance swap rate tend to vary, this quantity is much more heteroskedastic. This estimate puts greater weight on observations where the swap rate is high.

Both Carr and Wu (2007) and Petersson and Šarić (2008) also measure the *log variance risk premium*, defined as

$$\ln \frac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}}.$$

But this quantity cannot be used to test the hypothesis of indifference to variance risk since, even if $\mathrm{KV}_{t,T} = \mathsf{E}_t^{\mathbb{P}}\left[\mathrm{RV}_{t,T}\right]$, by Jensen's inequality

$$\mathsf{E}_t^{\mathbb{P}}\left[\ln \tfrac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}}\right] \leq \ln\, \mathsf{E}_t^{\mathbb{P}}\left[\tfrac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}}\right] = 0.$$

The capital asset pricing model is the special case where the pricing kernel is a linear function of the return on the market portfolio. The expected excess return of an asset is the product of its market portfolio beta and the expected excess return of the market portfolio. Hence

$$\mathsf{E}_t^{\mathbb{P}}\left[\frac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}} - 1\right] = \frac{\mathrm{Cov}_t^{\mathbb{P}}\left[\mathrm{ER}_{t,T}^m, \frac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}} - 1\right]}{\mathrm{Var}_t^{\mathbb{P}}\,\mathrm{ER}_{t,T}^m}\,\mathsf{E}_t^{\mathbb{P}}\mathrm{ER}_{t,T}^m,$$

where $\mathrm{ER}_{t,T}^m$ is the excess return of the market portfolio. If σ is the relative risk aversion, $\mathsf{E}_t^{\mathbb{P}}\mathrm{ER}_{t,T}^m = \sigma\,\mathrm{Var}_t^{\mathbb{P}}\,\mathrm{ER}_{t,T}^m$ (Smith and Wickens, 2002). Therefore the above is equivalent to equation 1 with

$$\frac{L_T}{\mathsf{E}_t^{\mathbb{P}}\left[L_T\right]} = 1 + \mathsf{E}_t^{\mathbb{P}}\left[\sigma \mathrm{ER}_{t,T}^m\right] - \sigma \mathrm{ER}_{t,T}^m.$$

3 Replicating the Realized Variance Using Options

The realized variance in a variance swap is defined as

$$\mathrm{RV} = \frac{N_y}{N}\sum_{i=1}^{N} \ln\left(\frac{S_i}{S_{i-1}}\right)^2. \tag{2}$$

where N is the number of business days between inception and expiry, N_y is the number of business days in a year and S_i is the price of the underlying asset on day i. Let F_i be the forward price of the underlying on day i for delivery at the time of maturity. Define

$$R_i = \ln\frac{S_i}{S_{i-1}}, \quad Y_i = \ln\frac{F_i}{F_{i-1}}.$$

Consider the following payoff

$$f(F_N) = -\ln\frac{F_N}{F_0} + \frac{F_N - F_0}{F_0}. \tag{3}$$

at maturity and the terminal value of a dynamic strategy consisting of on day day $i-1$ holding forwards contracts to buy $\frac{1}{F_{i-1}} - \frac{1}{F_0}$ units of the underlying. This approximately replicates the realized variance since

$$
\begin{aligned}
&-\ln\frac{F_N}{F_0}+\frac{F_N-F_0}{F_0}+\sum_{i=1}^{N}\left(\frac{1}{F_{i-i}}-\frac{1}{F_0}\right)(F_i-F_{i-i})\\
&=\sum_{i=1}^{N}\left[\left(\frac{1}{F_{i-i}}-\frac{1}{F_0}\right)(F_i-F_{i-i})-\ln\frac{F_i}{F_{i-1}}\right]+\frac{F_N-F_0}{F_0}\\
&=\sum_{i=1}^{N}\left[\frac{F_i-F_{i-1}}{F_{i-1}}-\ln\frac{F_i}{F_{i-1}}\right]\\
&=\sum_{i=1}^{N}\left[e^{Y_i}-1-Y_i\right]\\
&=\sum_{i=1}^{N}\frac{1}{2}Y_i^2+\sum_{i=1}^{N}\sum_{j=3}^{\infty}\frac{Y_i^j}{j!}\\
&=\sum_{i=1}^{N}\frac{1}{2}R_i^2+\sum_{i=1}^{N}\left(R_i(Y_i-R_i)+\frac{1}{2}(R_i-Y_i)^2+\sum_{j=3}^{\infty}\frac{Y_i^j}{j!}\right).
\end{aligned}
$$

This follows from Taylor expansion of the exponential function. The first term is the desired sum of squared relative price movements. The second term is a replication error which is related to the difference between movements of the spot price and of the forward price and to the higher moments of the price movements. The approximation is better when price movements are small (volatility is low), when the distribution of price movements is symmetric and platykurtic and when the difference between the variance of the forward price and of the spot price is small.

By differentiating the pay off $f(F)$ it can be seen that the dynamic forward strategy amounts to delta hedging of the (forward) intrinsic value of the payoff, or equivalently delta hedging at zero volatility. Differentiating again reveals that the gamma of the payoff at zero volatility is proportional to $\frac{1}{F^2}$ which explains why the leading term of the daily profit or loss is proportional to the square of relative price movements, as intended.

As is shown in Carr and Madan (1998), in continuous time, assuming diffusion dynamics for the price process, when the realized variance is the quadratic variation of the logarithm of the price and interest rates are deterministic, and the replicating strategy is carried out continuously, the replication error is exactly zero.

Ignoring the replication error, the variance swap rate should equal the forward price of the payoff $f(F_N)$. Carr and Madan (1998) also show that this payoff can be constructed by a continuum of European out of the money put and call options with strikes between zero and infinity, weighted by the square of the inverse of the strike since[2]

$$
\begin{aligned}
\int_0^{F_0} \frac{1}{K^2}(K-F_N)^+ \mathrm{d}K &+ \int_{F_0}^{\infty} \frac{1}{K^2}(F_N-K)^+ \mathrm{d}K \\
&= \begin{cases} \int_{F_N}^{F_0} \frac{K-F_N}{K^2}\mathrm{d}K & \text{if } F_N < F_0, \\ \int_{F_0}^{F_N} \frac{F_N-K}{K^2}\mathrm{d}K & \text{if } F_N > F_0 \end{cases} \\
&= \int_{F_0}^{F_N} \frac{F_N-K}{K^2}\mathrm{d}K \\
&= -\ln\frac{F_N}{F_0} + \frac{F_N-F_0}{F_0}.
\end{aligned}
\tag{4}
$$

And so, the forward price of realized variance as defined in (2) should equal

$$
\mathrm{KV} = 2\frac{N_y}{N}\left(\int_0^{F_0} \frac{1}{K^2}P(K)\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{K^2}C(K)\mathrm{d}K\right), \tag{5}
$$

where $P(K)$ and $C(K)$ are the forward prices of put and call options with strike K.

3.1 The foreign variance swap rate

If the underlying is an exchange rate the variance swap rate defined in (5) is the price of the realized variance in the domestic currency. This is the variance swap rate when the notional amount is in the domestic currency. What is the variance swap rate when the notional amount is in the foreign currency? The replicating portfolio will contain a different set of options.

To simplify things, let us consider a variance swap on the EURUSD exchange rate. Let F_0 denote the forward price of one euro in dollars. Let $P(K)$ be the (forward) price in dollars of an option to sell one euro for K dollars and let $C(K)$ be the price in dollars of an option to buy one euro for K dollars. The dollar denominated variance swap rate is then given by (suppressing the normalization factor $2N_y/N$)

$$
\mathrm{KV} = \int_0^{F_0} \frac{1}{K^2}P(K)\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{K^2}C(K)\mathrm{d}K. \tag{6}
$$

Let $\widetilde{F}_0$ denote the inverted (forward) exchange rate, the price of a dollar in euro. And define the prices in euro of options to buy or sell a dollar for L euro as $\widetilde{P}(L)$ and $\widetilde{C}(L)$, respectively. The euro denominated variance swap rate is then given by

$$
\widetilde{\mathrm{KV}} = \int_0^{\widetilde{F}_0} \frac{1}{L^2}\widetilde{P}(L)\mathrm{d}L + \int_{\widetilde{F}_0}^{\infty} \frac{1}{L^2}\widetilde{C}(L)\mathrm{d}L. \tag{7}
$$

Now, an option to sell one dollar for L euro is the same as L options to buy one euro. If $L = 1/K$ the option prices are therefore related by

$$
\widetilde{P}(L) = \frac{1}{F_0 K}C(K), \quad \widetilde{C}(L) = \frac{1}{F_0 K}P(K).
$$

[2] $(F_N - K)^+$ is shorthand for $\max(F_N - K, 0)$

Inserting into (7) and changing variables we have

$$\begin{aligned}
\widetilde{\mathrm{KV}} &= \int_{\infty}^{F_0} K^2 \frac{1}{F_0 K} C(K) \left(-\frac{1}{K^2}\mathrm{d}K\right) + \int_{F_0}^{0} K^2 \frac{1}{F_0 K} P(K) \left(-\frac{1}{K^2}\mathrm{d}K\right) \\
&= \int_{0}^{F_0} \frac{1}{F_0 K} P(K)\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{F_0 K} C(K)\mathrm{d}K.
\end{aligned} \tag{8}$$

This is the foreign variance swap rate in terms of option prices in the domestic currency. Interestingly, this turns out to be equal to the gamma swap rate.[3] A gamma swap is a variance swap where the realized variance is defined as the average of the squared daily returns weighted by the price level of the underlying (Lee, 2008):

$$\mathrm{RV}_\Gamma = \frac{N_y}{N} \sum_{i=1}^{N} \frac{S_i}{S_0} \ln\left(\frac{S_i}{S_{i-1}}\right)^2.$$

Entering into a foreign denominated variance swap with a notional amount of $1/F_0$ units of the foreign currency, and simultaneously entering into a forward contract to buy $\widetilde{\mathrm{KV}}/F_0$ units of the foreign currency, the profit or loss at expiry (in the domestic currency at the then prevailing exchange rate $F_N = S_N$) will be

$$\mathrm{RV}\frac{F_N}{F_0} - \widetilde{\mathrm{KV}},$$

which is similar to the gamma swap payoff. Since the variance swap rates can also be expressed as the time 0 risk neutral expected variance this also implies that the difference between the foreign and domestic variance swap rates is

$$\begin{aligned}
\widetilde{\mathrm{KV}} - \mathrm{KV} &= \mathsf{E}_0^{\mathbb{Q}}\left[\mathrm{RV}\frac{F_N}{F_0} - \mathrm{RV}\right] \\
&= \mathsf{E}_0^{\mathbb{Q}}\left[\mathrm{RV}\left(\frac{F_N}{F_0} - 1\right)\right] \\
&= \mathrm{Cov}_0^{\mathbb{Q}}\left[\mathrm{RV}, \frac{F_N}{F_0} - 1\right] + \mathsf{E}_0^{\mathbb{Q}}\left[\mathrm{RV}\right] \mathsf{E}_0^{\mathbb{Q}}\left[\frac{F_N}{F_0} - 1\right] \\
&= \mathrm{Cov}_0^{\mathbb{Q}}\left[\mathrm{RV}, \frac{F_N}{F_0}\right].
\end{aligned}$$

Thus the difference is proportional to the risk neutral covariance between the realized variance and the underlying exchange rate.

If the implied volatility is symmetric in *log moneyness*, defined as $\ln\frac{K}{F_0}$, it is easily shown that

$$C(K) = \tfrac{K}{F_0} P\left(\tfrac{F_0^2}{K}\right).$$

[3] To be precise, it is the gamma swap rate when the underlying is the forward price (or futures price assuming deterministic interest rates).

This, (6) and (8) imply

$$\begin{aligned}
\widetilde{\mathrm{KV}} - \mathrm{KV} &= \int_0^{F_0} \left(\frac{1}{F_0 K} - \frac{1}{K^2} \right) P(K) \mathrm{d}K + \int_{F_0}^{\infty} \left(\frac{1}{F_0 K} - \frac{1}{K^2} \right) \frac{K}{F_0} P\left(\tfrac{F_0^2}{K} \right) \mathrm{d}K \\
&= \int_0^{F_0} \left(\frac{1}{F_0 K} - \frac{1}{K^2} \right) P(K) \mathrm{d}K + \int_{F_0}^{0} \left(\frac{X}{F_0^3} - \frac{X^2}{F_0^4} \right) \frac{F_0}{X} P(X) \left(-\tfrac{F_0^2}{X^2} \mathrm{d}X \right) \\
&= 0.
\end{aligned}$$

Thus, when the smile is symmetric the foreign and domestic variance swap rates are equal. If the smile is not symmetric the difference is then

$$\widetilde{\mathrm{KV}} - \mathrm{KV} = \int_{F_0}^{\infty} \left(\frac{1}{F_0 K} - \frac{1}{K^2} \right) \left(C(K) - \frac{K}{F_0} P\left(\tfrac{F_0^2}{K} \right) \right) \mathrm{d}K.$$

Since the first factor of the integrand is strictly positive, this means that if out-of-the-money calls have lower (higher) implied volatility than the corresponding out-of-the-money puts, the foreign variance swap rate is lower (higher) than the domestic.

4 Methodology

The principal purpose of this paper is to determine whether there exists a negative variance risk premium in currency options, or more precisely, whether the average expected returns of long variance swaps on exchange rates are negative. The hypothesis that such a premium exists will be accepted if the null hypothesis, that the average expected return of variance swap is zero, is rejected and the sign of the average return is negative. Attributing the average difference between the price and the (objective probability measure) expected value entirely to a risk premium implicitly assumes that investors have rational expectations, that their expectations are not systematically biased.

Another objective is to examine whether currency volatility is negatively correlated with stock market returns, as is the case with equity volatility. If it is, the capital asset pricing model predicts that there will exist a negative risk premium (if the stock market is taken to mean “the market portfolio”). A related objective is therefore to determine to what extent the capital asset pricing model can account for currency variance swap returns.

A third objective is to quantify the difference between the foreign and domestic variance swap rates.

In section 3 it was showed that the variance swap rate can be calculated as a function of option prices of every conceivable strike between zero and infinity. In practice such data are not available, and assumptions about the prices of options with strikes far from the money have to made. However, as the strike goes to zero or infinity the value of out-of-the-money options quickly goes to zero. The price of deep out-of-the-money options therefore constitute a small part of the variance swap rate and need not be estimated with great accuracy.

In this paper three implied volatility quotes are used to infer an implied volatility smile using the vanna-volga-method. Below these quotes and the vanna-volga method are described. It is shown that assuming vanna-volga prices for options, the variance swap rate is given as a closed form expression.

4.1 Market quotes

In the foreign exchange market, options are referred to by their Black-Scholes delta and implied volatility. Moreover instead of giving the implied volatility of individual options, the prices of instruments that are combinations of options are quoted. This section tries to describe these quotes.

4.1.1 Delta conventions

Using the implied volatility and the delta of an option, the strike and the premium can be found. There are however many definitions of delta. Therefore "the entire FX quotation story generally becomes a mess" (Wystup, 2006).

Delta is the derivative of the premium with respect to the underlying exchange rate. It is the amount of the foreign currency to sell in order to make the overall position locally risk free (in the Black-Scholes sense) if one option is bought.

The Black-Scholes price of a call option C and a put option P is given by

$$\begin{aligned} C &= e^{-r_d t}\left[F\Phi(d_1) - K\Phi(d_2)\right], \\ P &= e^{-r_d t}\left[-F\Phi(-d_1) + K\Phi(-d_2)\right], \\ d_1 &= \frac{\log(F/K) + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}, \\ d_2 &= \frac{\log(F/K) - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}, \end{aligned} \tag{9}$$

where r_d and r_f are the domestic and foreign interest rates, t is the time to maturity and $F = Se^{(r_d - r_f)t}$ is the forward price, S is the spot price, and K is the strike. Φ is the standard normal cumulative distribution function.

For example, in the EURUSD and GBPUSD cases the normal Black-Scholes delta is used according to Bossens et al. (2009), that is

$$\Delta_{\text{call}} = \frac{\partial C}{\partial S} = e^{-r_f t}\Phi(d_1), \quad \Delta_{\text{put}} = \frac{\partial P}{\partial S} = -e^{-r_f t}\Phi(-d_1) \tag{10}$$

These cases are straight forward because USD is the *domestic* currency.[4]

In the USDJPY case on the other hand, where USD is the *foreign* currency, a different notion of delta is used. In this case S is the price of one dollar in yen, and C is the price in yen of an

[4] EURUSD refers to the price of one euro in dollars. Therefore, USD is referred to as the domestic currency, and EUR as the foreign.

option to buy one dollar. One long option with a notional amount of one dollar and $-\Delta$ dollars has a (locally) constant yen value. The regular Black-Scholes delta is thus the amount of dollars to sell in order to make the combined position insensitive to exchange rate movements, in yen terms. This amount is expressed as dollars per dollar of notional, a dimensionless quantity. What usually is referred to in the USDJPY case is instead, according to Bossens et al. (2009), the *premium included delta*. In order for the position to have a constant *dollar* value an amount of yen y should be bought. This yen amount is then expressed in dollars to get the premium included delta $\widetilde{\Delta}_{\text{call}} = y/S$. So for every dollar of notional $\widetilde{\Delta}$ dollars should be sold in order to make the position delta neutral in dollar terms.[5] The dollar value of the USD call option is C/S and the dollar value of the yen amount is y/S. y is therefore given by

$$0 = \frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{C}{S} + \frac{y}{S}\right) = \frac{1}{S}\frac{\partial C}{\partial S} - \frac{C}{S^2} - \frac{y}{S^2}.$$

Accordingly the premium included delta is defined as

$$\widetilde{\Delta}_{\text{call}} = \Delta_{\text{call}} - \frac{C}{S} = \frac{K}{S}e^{-r_d t}\Phi(d_2). \tag{11}$$

Similarly the premium included delta of a put is

$$\widetilde{\Delta}_{\text{put}} = \Delta_{\text{put}} - \frac{P}{S} = -\frac{K}{S}e^{-r_d t}\Phi(-d_2). \tag{12}$$

Different currency pairs use different conventions for delta. According to Wystup (2006) "in the professional interbank market there is one notion of delta per currency pair". Beneder and Elkenbracht-Huizing (2003) assert that for all currency pairs except EURUSD the premium included convention is used, whereas Bossens et al. (2009) claim the standard Black-Scholes delta is used for GBPUSD too. Fortunately for out-of-the-money options (which are used in the volatility quotes described below) the difference is "not huge" as stated by Wystup.

There is another aspect of the definition of delta. Instead of the *spot* delta defined above sometimes the *forward delta* is used. This is the derivative of the forward premium with respect to the underlying forward price which is

$$\Delta^F_{\text{call}} = \Phi(d_1), \qquad \Delta^F_{\text{put}} = -\Phi(-d_1), \tag{13}$$

$$\widetilde{\Delta}^F_{\text{call}} = \tfrac{K}{F}\Phi(d_2), \qquad \widetilde{\Delta}^F_{\text{put}} = -\tfrac{K}{F}\Phi(-d_2). \tag{14}$$

The difference between the definitions is greater when the time to maturity is longer and for short maturities it should make little difference.

[5]The USD call is also a JPY put. Let r_d as before refer to the JPY interest rate and d_1' be given by (9) with F replaced by $1/F$ and K by $1/K$. The regular Black-Scholes delta of this put is then $\Delta' = -e^{-r_d t}\Phi(-d_1') = -e^{-r_d t}\Phi(d_2)$. This is the amount of yen to *sell* per *yen* of notional to make the position delta neutral in dollar terms.

4.1.2 Volatility quotes

Three main price quotes describe the implied volatility smile on the foreign exchange options market, the delta neutral straddle, the 25 delta risk reversal and the 25 delta butterfly (Bisesti et al., 2005; Castagna and Mercurio, 2005). The quotes are related to the prices of the instruments with the same name. When trying to estimate an implied volatility smile the smile should correctly price these instruments. Below a call option with delta 0.25 will be referred to as a 25 delta call option and a 25 delta put option will mean a put option with delta -0.25.

The delta neutral straddle is a combination of a put an call option with the same strike such that the combined delta is zero. Its price is quoted as the volatility of the options. This is referred to as the at-the-money (ATM) volatility, hereafter denoted by σ_{ATM}. Using the appropriate delta and premium convention, the ATM strike can be found.[6]

A strangle is a combination of long out of the money put and call options. Its price is quoted as the difference between the *strangle volatility* and the ATM volatility. The butterfly quote refers to this difference. Let $\sigma_{25\Delta\text{BF}}$ denote the 25 delta butterfly quote. The strikes of the involved options and the premium of the strangle should be calculated so that delta of the put option is -0.25 and the delta of the call option is 0.25 using the strangle volatility for both options (Bossens et al., 2009). Thus the strikes, $K^{\text{BF}}_{\text{put}}$ and $K^{\text{BF}}_{\text{call}}$, are given by

$$\begin{aligned} \Delta_{\text{put}}(K^{\text{BF}}_{\text{put}}, \sigma_{\text{ATM}} + \sigma_{25\Delta\text{BF}}) &= -0.25, \\ \Delta_{\text{call}}(K^{\text{BF}}_{\text{call}}, \sigma_{\text{ATM}} + \sigma_{25\Delta\text{BF}}) &= 0.25, \end{aligned} \tag{15}$$

and the price of the strangle is

$$P_{25\Delta\text{STR}} = P_{\text{call}}(K^{\text{BF}}_{\text{call}}, \sigma_{\text{ATM}} + \sigma_{25\Delta\text{BF}}) + P_{\text{put}}(K^{\text{BF}}_{\text{put}}, \sigma_{\text{ATM}} + \sigma_{25\Delta\text{BF}}). \tag{16}$$

Here $P_{\text{call}}(K, \sigma)$ and $\Delta_{\text{call}}(K, \sigma)$ refers to the Black-Scholes price and delta of a call option with strike K and volatility σ. The interpolated smile, hereafter denoted by $\sigma(K)$ should price this strangle correctly so that

$$P_{\text{call}}(K^{\text{BF}}_{\text{call}}, \sigma(K^{\text{BF}}_{\text{call}})) + P_{\text{put}}(K^{\text{BF}}_{\text{put}}, \sigma(K^{\text{BF}}_{\text{put}})) = P_{25\Delta\text{STR}}. \tag{17}$$

However, it will not necessarily match the price of the individual put and call options in the strangle in (16), since it also has to take into account the risk reversal.

A risk reversal is a combination of a short out of the money put option and a long out of the money call option. Its price is quoted as the difference in implied volatility between the two options. Let $\sigma_{25\Delta\text{RR}}$ denote the 25 delta risk reversal quote. It is not entirely clear how the risk reversal quote should best be incorporated as a constraint on the smile. One method, described in Bossens et al. (2009), is to require that the difference between the volatility of the

[6] There are other slightly different definitions of “at the money”.

25 delta call and 25 delta put options be equal to the risk reversal quote, so that

$$\begin{aligned}\sigma(K_{25\Delta \mathrm{Call}}) - \sigma(K_{25\Delta \mathrm{put}}) &= \sigma_{25\Delta \mathrm{RR}},\\ \Delta_{\mathrm{put}}(K_{25\Delta \mathrm{put}}, \sigma(K_{25\Delta \mathrm{put}})) &= -0.25,\\ \Delta_{\mathrm{call}}(K_{25\Delta \mathrm{call}}, \sigma(K_{25\Delta \mathrm{call}})) &= 0.25.\end{aligned} \tag{18}$$

Another method is to calculate the strikes of the quoted risk reversal as

$$\begin{aligned}\Delta_{\mathrm{put}}(K^{\mathrm{RR}}_{\mathrm{put}}, \sigma_{\mathrm{ATM}} + \sigma_{25\Delta \mathrm{BF}} - \tfrac{1}{2}\sigma_{25\Delta \mathrm{RR}}) &= -0.25,\\ \Delta_{\mathrm{call}}(K^{\mathrm{RR}}_{\mathrm{call}}, \sigma_{\mathrm{ATM}} + \sigma_{25\Delta \mathrm{BF}} + \tfrac{1}{2}\sigma_{25\Delta \mathrm{RR}}) &= 0.25,\end{aligned}$$

and the price as

$$P_{25\Delta \mathrm{RR}} = P_{\mathrm{call}}(K^{\mathrm{BF}}_{\mathrm{call}}, \sigma_{\mathrm{ATM}} + \sigma_{25\Delta \mathrm{BF}} - \tfrac{1}{2}\sigma_{25\Delta \mathrm{RR}}) - P_{\mathrm{put}}(K^{\mathrm{RR}}_{\mathrm{put}}, \sigma_{\mathrm{ATM}} + \sigma_{25\Delta \mathrm{RR}} + \tfrac{1}{2}\sigma_{25\Delta \mathrm{RR}}),$$

and require that the interpolated smile prices this risk reversal correctly, i.e.

$$P_{\mathrm{call}}(K^{\mathrm{RR}}_{\mathrm{call}}, \sigma(K^{\mathrm{RR}}_{\mathrm{call}})) - P_{\mathrm{put}}(K^{\mathrm{RR}}_{\mathrm{put}}, \sigma(K^{\mathrm{RR}}_{\mathrm{put}})) = P_{25\Delta \mathrm{RR}}. \tag{19}$$

In practice it makes little difference which of the two methods is used.

It is sometimes assumed as a simplification that the 25 delta risk reversal quote is simply the difference between the 25 delta call volatility, $\sigma_{25\Delta \mathrm{C}}$, and the 25 delta put volatility, $\sigma_{25\Delta \mathrm{P}}$, and similarly that the 25 delta butterfly is the average of the 25 delta call and put volatilities less the ATM-volatility, so that

$$\sigma_{25\Delta \mathrm{RR}} = \sigma_{25\Delta \mathrm{C}} - \sigma_{25\Delta \mathrm{P}}, \tag{20}$$

$$\sigma_{25\Delta \mathrm{BF}} = \frac{\sigma_{25\Delta \mathrm{C}} + \sigma_{25\Delta \mathrm{P}}}{2} - \sigma_{\mathrm{ATM}}. \tag{21}$$

Using this assumption the volatilities for three strikes (including the ATM-volatility) can be found by

$$\sigma_{25\Delta \mathrm{P}} = \sigma_{\mathrm{ATM}} - \tfrac{1}{2}\sigma_{25\Delta \mathrm{RR}} + \sigma_{25\Delta \mathrm{BF}}, \tag{22}$$

$$\sigma_{25\Delta \mathrm{C}} = \sigma_{\mathrm{ATM}} + \tfrac{1}{2}\sigma_{25\Delta \mathrm{RR}} + \sigma_{25\Delta \mathrm{BF}}. \tag{23}$$

and the problem of finding the volatility of other strikes is a problem of interpolation and extrapolation. In practice using (22) and (23) is a good approximation if the smile is not very steep, but it can lead to substantial errors for currency pairs with steep smiles such as USDJPY.

4.2 The vanna-volga method

Using the known implied volatilities of three strikes—typically the at-the-money (ATM) volatility, the 25 delta put and call volatilities—the vanna-volga method infers the entire volatility smile (at least in the 5 delta put to 5 delta call range, according to Castagna and Mercurio

(2005)). It is not a consistent model of the underlying price process but a valuation method that is commonly used in the foreign exchange options market (Castagna and Mercurio, 2007).

Castagna and Mercurio (2007) gives a theoretical justification which is based on a hedging argument. The idea is to form a portfolio of the three options with known volatilities and an option with an unknown volatility that is to be calculated, which is locally risk free in a model where options are valued with a volatility that is independent of strike but stochastic.

Since we are interested in the forward price of the realized variance and therefore in the forward prices of the options in the replicating portfolio, let us consider only forward prices of options. A hedging argument similar to that found in Castagna and Mercurio (2005) is presented below using forward prices of options. The resulting price is equivalent to the standard vanna-volga price.

Let the three strikes with known volatilities σ_i be denoted by K_i for $i = 1, 2, 3$. where $K_1 < K_2 < K_3$ and K_2 is the ATM strike. Assume that call options for all three strikes are traded and let $C_t^{\mathrm{BS}}(K)$ denote the Black-Scholes forward price of a call option with strike K and a volatility σ_t which is equal to the ATM volatility. In the Black and Scholes (1973) model the forward price of an option is given by

$$\begin{aligned} C^{\mathrm{BS}}(K) &= F\Phi(d_1) - K\Phi(d_2), \\ d_1 &= \frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(F/K) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \end{aligned} \tag{24}$$

where $T - t$ is the time to maturity, σ is the implied volatility, K is and F is the underlying forward exchange rate.

Now let K denote the strike of the option with unknown implied volatility and let F_t the be the underlying forward exchange rate. Consider a strategy where, at time t, it is agreed to buy one unit of the option with unknown value, and $-\Delta_t$ units of the underlying and $-x_t^i$ units of the call option with strike K_i, $i = 1, 2, 3$. In a world where options are valued with a common but stochastic volatility, the resulting profit or loss at the time of maturity is then

$$V = \int_0^T \left(\mathrm{d}C_t^{\mathrm{BS}}(K) - \Delta_t \mathrm{d}F_t - \sum_{i=1}^{3} x_t^i \mathrm{d}C_t^{\mathrm{BS}}(K_i) \right). \tag{25}$$

By Itô's lemma

$$
\begin{aligned}
&\mathrm{d}C_t^{\mathrm{BS}}(K) - \Delta_t \mathrm{d}F_t - \sum_{i=1}^{3} x_t^i \mathrm{d}C_t^{\mathrm{BS}}(K_i) \\
&= \left(\frac{\partial}{\partial t} C_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_t^i \frac{\partial}{\partial t} C_t^{\mathrm{BS}}(K_i) \right) \mathrm{d}t + \left(\frac{\partial}{\partial F} C_t^{\mathrm{BS}}(K) - \Delta_t - \sum_{i=1}^{3} x_t^i \frac{\partial}{\partial F} C_t^{\mathrm{BS}}(K_i) \right) \mathrm{d}F_t \\
&+ \frac{1}{2} \left(\frac{\partial^2}{\partial F^2} C_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_t^i \frac{\partial^2}{\partial F^2} C_t^{\mathrm{BS}}(K_i) \right) \mathrm{d}\langle F \rangle_t + \left(\frac{\partial}{\partial \sigma} C_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_t^i \frac{\partial}{\partial \sigma} C_t^{\mathrm{BS}}(K_i) \right) \mathrm{d}\sigma_t \\
&+ \frac{1}{2} \left(\frac{\partial^2}{\partial \sigma^2} C_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_t^i \frac{\partial^2}{\partial \sigma^2} C_t^{\mathrm{BS}}(K_i) \right) \mathrm{d}\langle \sigma \rangle_t + \left(\frac{\partial^2}{\partial F \partial \sigma} C_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_t^i \frac{\partial^2}{\partial F \partial \sigma} C_t^{\mathrm{BS}}(K_i) \right) \mathrm{d}\langle \sigma_t, F_t \rangle.
\end{aligned}
\tag{26}
$$

By letting x^i be the solution to the following system of equations

$$
\begin{aligned}
\frac{\partial}{\partial \sigma} C_t^{\mathrm{BS}}(K) &= \sum_{i=1}^{3} x_t^i \frac{\partial}{\partial \sigma} C_t^{\mathrm{BS}}(K_i), \\
\frac{\partial^2}{\partial F \partial \sigma} C_t^{\mathrm{BS}}(K) &= \sum_{i=1}^{3} x_t^i \frac{\partial^2}{\partial F \partial \sigma} C_t^{\mathrm{BS}}(K_i), \\
\frac{\partial^2}{\partial \sigma^2} C_t^{\mathrm{BS}}(K) &= \sum_{i=1}^{3} x_t^i \frac{\partial^2}{\partial \sigma^2} C_t^{\mathrm{BS}}(K_i),
\end{aligned}
\tag{27}
$$

the last three terms in (26) become equal to zero. The three partial derivatives in (27) are the forward counterparts of the sensitivities known respectively as *vega*, *vanna* and *volga*. Since in the Black-Scholes formula the ratio between vega and *gamma*—the second derivative with respect to the underlying price—is independent of the strike, the third term too becomes equal to zero. If Δ_t matches the combined delta[7] of the options only the first term is left. Now, the forward price of an option as defined in (24) satisfies the following partial differential equation:[8]

$$
\frac{\partial C^{\mathrm{BS}}}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 C^{\mathrm{BS}}}{\partial F^2} = 0. \tag{28}
$$

Using this and the fact that the combined gamma is equal to zero we have

$$
V = \int_0^T \left(\frac{\partial}{\partial t} C_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_t^i \frac{\partial}{\partial t} C_t^{\mathrm{BS}}(K_i) \right) \mathrm{d}t = 0. \tag{29}
$$

Rearranging (25) and using that the value of an option at maturity is its intrinsic value yields

$$
C_0^{\mathrm{BS}}(K) = (F_T - K)^+ - \int_0^T \Delta_t \mathrm{d}F_t - \int_0^T \sum_{i=1}^{3} x_t^i \mathrm{d}C_t^{\mathrm{BS}}(K_i). \tag{30}
$$

[7] The first derivative of the forward options price with respect to the underlying forward price.

[8] Compare this to $\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 C}{\partial F^2} - rC = 0$ which is the partial differential equation used to derive the "Black 76" price which is just (24) with a discount factor (Black, 1976).

Thus, in a model where the smile is flat but stochastic, an option can be perfectly replicated at a cost of $C_0^{\mathrm{BS}}(K)$. Let $C_t^{\mathrm{MKT}}(K_i)$ denote the market prices of the options, that is (24) with $\sigma = \sigma_i$. Consider what happens when the strategy is applied in reality. Using (30) we have

$$\begin{aligned}(F_T - K)^+ - \int_0^T \Delta_t \mathrm{d}F_t - \int_0^T \sum_{i=1}^3 x_t^i \mathrm{d}C_t^{\mathrm{MKT}}(K_i) \\ = C_0^{\mathrm{BS}}(K) - \int_0^T \sum_{i=1}^3 x_t^i \left[\mathrm{d}C_t^{\mathrm{MKT}}(K_i) - \mathrm{d}C_t^{\mathrm{BS}}(K_i)\right] \\ = C_0^{\mathrm{BS}}(K) + \sum_{i=1}^3 x_t^i \left[C_t^{\mathrm{MKT}}(K_i) - C_t^{\mathrm{BS}}(K_i)\right] - \int_0^T \sum_{i=1}^3 \left[C_t^{\mathrm{MKT}}(K_i) - C_t^{\mathrm{BS}}(K_i)\right] \mathrm{d}x_t^i.\end{aligned} \tag{31}$$

If the last term can be considered negligible the option can be replicated at a cost of

$$C(K) = C_0^{\mathrm{BS}}(K) + \sum_{i=1}^3 x_t^i \left[C_t^{\mathrm{MKT}}(K_i) - C_t^{\mathrm{BS}}(K_i)\right]. \tag{32}$$

This is defined as the vanna-volga price. Vega, vanna and volga (or rather their forward counterparts) can be found by differentiating (24) to be

$$\mathrm{Vega}(K) = F\sqrt{T}\varphi(d_1(K)), \tag{33}$$

$$\mathrm{Vanna}(K) = -\mathrm{Vega}(K)\frac{d_2(K)}{F\sigma\sqrt{T}}, \tag{34}$$

$$\mathrm{Volga}(K) = \frac{d_1(K)d_2(K)}{\sigma}\mathrm{Vega}(K). \tag{35}$$

Castagna and Mercurio (2005) show that the system (27) always has a unique solution.

If the implied volatility for three strikes are known, the vanna-volga method is particularly simple because the entire smile is then given as a closed-form expression of the implied volatilities. But when trying to fit the smile to the risk reversal and butterfly quotes described in section 4.1 numerical methods must be used.

Since the vanna-volga method is not a consistent model of the underlying price there is no guarantee that the method will produce arbitrage free prices. In fact, when the risk reversal is large the smile is steep the method produces negative premiums for some out-of-the-money options (Bossens et al., 2009).

4.3 The vanna-volga price of realized variance

In section 3 the variance swap rate was found to be related to the forward price of the payoff

$$f(F_T) = -\ln\frac{F_T}{F_0} + \frac{F_T - F_0}{F_0} \tag{36}$$

at maturity where F_0, F_T are the forward prices of the underlying at inception and expiry. In this section the vanna-volga price of this contract is derived. Defining

$$V = \begin{bmatrix} \text{Vega}(K_1) & \text{Vega}(K_2) & \text{Vega}(K_3) \\ \text{Vanna}(K_1) & \text{Vanna}(K_2) & \text{Vanna}(K_3) \\ \text{Volga}(K_1) & \text{Volga}(K_2) & \text{Volga}(K_3) \end{bmatrix}, \quad \boldsymbol{\partial} = \begin{bmatrix} \frac{\partial}{\partial\sigma} \\ \frac{\partial^2}{\partial F \partial\sigma} \\ \frac{\partial^2}{\partial\sigma^2} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} C^{\mathrm{MKT}}(K_1) - C^{\mathrm{BS}}(K_1) \\ C^{\mathrm{MKT}}(K_2) - C^{\mathrm{BS}}(K_2) \\ C^{\mathrm{MKT}}(K_3) - C^{\mathrm{BS}}(K_3) \end{bmatrix},$$

the vanna-volga (forward) price (32) of call, $C(K)$, and put, $P(K)$, options can be expressed as

$$C(K) = (1 + \mathbf{y}^\top V^{-1}\boldsymbol{\partial})C^{\mathrm{BS}}(K), \quad P(K) = (1 + \mathbf{y}^\top V^{-1}\boldsymbol{\partial})P^{\mathrm{BS}}(K).$$

The generalization of the vanna-volga method to put options is trivial since puts and calls with the same strike has the same vega, vanna and volga. Let V^f denote the forward price of the payoff $f(F_T)$. Using (4) we get

$$\begin{aligned} V^f &= \int_0^{F_0} \frac{1}{K^2} P(K)\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{K^2} C(K)\mathrm{d}K \\ &= \int_0^{F_0} \frac{1}{K^2}(1 + \mathbf{y}^\top V^{-1}\boldsymbol{\partial})P^{\mathrm{BS}}(K)\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{K^2}(1 + \mathbf{y}^\top V^{-1}\boldsymbol{\partial})C^{\mathrm{BS}}(K)\mathrm{d}K \\ &= (1 + \mathbf{y}^\top V^{-1}\boldsymbol{\partial}) \left(\int_0^{F_0} \frac{1}{K^2} P^{\mathrm{BS}}(K)\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{K^2} C^{\mathrm{BS}}(K)\mathrm{d}K \right). \end{aligned} \tag{37}$$

The Black-Scholes forward price defined in (24) can be expressed as a risk neutral expected value:

$$C^{\mathrm{BS}}(K) = \mathsf{E}\left[(X - K)^+\right].$$

where X is a stochastic variable in the Black-Scholes model defined by

$$X = F \exp\left(\sigma\sqrt{T}z - \tfrac{1}{2}T\sigma^2\right), \quad z \sim N(0,1).$$

Of course, the Black-Scholes price of the portfolio of options replicating the payoff $f(F_T)$ is simply the expected value of the payoff:

$$\begin{aligned} &\int_0^{F_0} \frac{1}{K^2} P^{\mathrm{BS}}(K)\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{K^2} C^{\mathrm{BS}}(K)\mathrm{d}K \\ &\quad = \int_0^{F_0} \frac{1}{K^2}\,\mathsf{E}\left[(K - X)^+\right]\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{K^2}\,\mathsf{E}\left[(X - K)^+\right]\mathrm{d}K \\ &\quad = \mathsf{E}\left[\int_0^{F_0} \frac{1}{K^2}(K - X)^+\mathrm{d}K + \int_{F_0}^{\infty} \frac{1}{K^2}(X - K)^+\mathrm{d}K\right] \\ &\quad = \mathsf{E}\left[f(X)\right]. \end{aligned} \tag{38}$$

This expected value can be calculated as

$$\begin{aligned} \mathsf{E}\left[f(X)\right] &= \mathsf{E}\left[-\ln\frac{X}{F_0} + \frac{X - F_0}{F_0}\right] \\ &= \mathsf{E}\left[-\ln\frac{X}{F} - \ln\frac{F}{F_0} + \frac{X - F_0}{F_0}\right] \\ &= \tfrac{1}{2}T\sigma^2 - \ln\frac{F}{F_0} + \frac{F - F_0}{F_0}. \end{aligned} \tag{39}$$

The Black-Scholes price of realized variance is therefore the volatility squared, which is expected because the volatility is a constant in the model. By (37), (38) and (39) the vanna-volga price of the replicating portfolio is[9]

$$\begin{aligned} V^f &= (1+\mathbf{y}^\top V^{-1}\boldsymbol{\partial})\left(\tfrac{1}{2}T\sigma^2 - \ln\frac{F}{F_0} + \frac{F-F_0}{F_0}\right) \\ &= \tfrac{1}{2}T\left(\sigma^2 + \mathbf{y}^\top V^{-1}\begin{bmatrix}2\sigma\\0\\2\end{bmatrix}\right). \end{aligned}$$

Since $\sigma = \sigma_2$ it follows that

$$C^{\text{MKT}}(K_2) - C^{\text{BS}}(K_2) = 0.$$

Let $k_i = \ln(K_i/F_0)$. By the definitions of vanna, volga and vega and by Cramers rule, the variance swap rate as defined in (5), consistent with vanna-volga option prices is[10]

$$\begin{aligned} \sigma^2+\mathbf{y}^\top V^{-1}\begin{bmatrix}2\sigma\\0\\2\end{bmatrix} &= \sigma^2 + \frac{2T\sigma^3(k_2+k_3)+4\sigma k_2k_3+4T\sigma^3+T^2\sigma^5}{-2(k_1-k_2)(k_3-k_1)\text{Vega}(K_1)}\left(C^{\text{MKT}}(K_1)-C^{\text{BS}}(K_1)\right) \\ &+ \frac{2T\sigma^3(k_1+k_2)+4\sigma k_1k_2+4T\sigma^3+T^2\sigma^5}{-2(k_3-k_1)(k_2-k_3)\text{Vega}(K_3)}\left(C^{\text{MKT}}(K_3)-C^{\text{BS}}(K_3)\right). \end{aligned} \tag{40}$$

The denominators are never zero since K_i are distinct and the vega of an option is never zero. The expression depends on K_i/F but not on K_i or F. If the driftless definition of delta is used and the strikes K_i are given in terms of delta, the vanna-volga price of realized variance can be calculated without considering interest rates.

4.4 Data

The data used are quotes on one month ATM straddles, 25 delta risk reversals and 25 delta butterflies from British Bankers Association (BBA)[11] and from Bloomberg. The Bloomberg data covers 2003–2009 and the BBA data covers 2001–2008. Combined these datasets give daily implied volatility quotes between August, 2001 and January, 2009, roughly 1800 observations per currency pair. The following currency pairs are included:

- GBPUSD, pound sterling/U.S. dollar
- EURUSD, euro/U.S. dollar

[9]Although F and F_0 are equal they represent different quantities, so in principle it is important to distinguish between them. The differential operator $\boldsymbol{\partial}$ does not operate on F_0 which is a parameter in the definition of the payoff $f(F_T)$, whereas F is a parameter of the Black-Scholes forward price formula.

[10]It is assumed here that T in the Black-Scholes price of the options and $\frac{N}{N_y}$ in the definition of the realized variance (2) are equal.

[11]Available at `http://www.bba.org.uk/`

- EURGBP, euro/pound sterling
- EURSEK, euro/Swedish krona
- USDJPY, U.S. dollar/Japanese yen
- USDCAD, U.S. dollar/Canadian dollar
- AUDUSD, Australian dollar/U.S. dollar

Data on the spot exchange rate for the currency pairs from Bloomberg are also used.

4.5 Estimating variance swap returns

At every day for which there are options data for a currency pair a variance swap rate and the corresponding realized variance are calculated. The time of maturity is set to the first "business day" that is more than 30 calendar days away. A business day is defined as a day for which there is a spot price. The realized variance is calculated as defined in section 3,

$$\mathrm{RV} = \frac{N_y}{N} \sum_{i=1}^{N} \ln \left(\frac{S_i}{S_{i-1}} \right)^2, \tag{41}$$

where N is the number of business days between inception and expiry. N_y, the number of business days in a year, is found to be approximately 260. S_i is the spot exchange rate on the i:th day.

The variance swap rate is calculated using (40), with the volatility of 25 delta put and calls and the ATM volatility as input. These inputs are obtained by fitting vanna-volga prices to the ATM volatility and the 25 delta risk reversal and butterfly quotes. The risk reversal quote is incorporated as a constraint on the smile according to (18) as advocated by Bossens et al. (2009). The butterfly quote constrains the smile according to (17). The time-to-maturity parameter t in the Black-Scholes formula is defined as the factor N/N_y in (41) for simplicity. Delta is interpreted as driftless delta, as defined in (14). As noted above, in this way interest rates do not have to be taken into account. The premium included delta definition is used for the following currency pairs.

- EURGBP
- EURSEK
- USDJPY
- USDCAD

The vanna-volga method is not guaranteed to produce arbitrage free prices, especially not far from the money. As noted earlier, for very large risk reversals the method sometimes produces negative out-of-the-money option prices, though for typical market data this does not happen. To check whether this happens in the sample, the actual vanna-volga prices that are implicitly integrated in (40), are computed for each day. It is found that in the USDJPY case the method failed in this manner frequently between the summer of 2007 and 2009, when the smile was very steep. It is concluded that the vanna-volga method is not suitable for these conditions. To get an estimate of the variance swap rate, data on 10 delta risk reversals and butterflies which are available between 2003 and 2009 are also used for this currency pair. Natural cubic splines are used to interpolate between 10 and 25 delta puts and calls and the ATM volatility and then a constant volatility beyond these strikes are assumed.[12] This is in line with Petersson and Šarić (2008). The variance swap rate is then calculated by numerical integration of (5) with the iterpolated smile. Before 2003 when 10 delta quotes are not available the vanna-volga method is used as the interpolated smile is reasonable. None of the other currency pairs are affected by this with the exception of AUDUSD which exhibited negative option prices on a few days.

The absence of option prices for deep out-of-the-money strikes, and sparsity of available prices is a problem when estimating the variance swap rate. Some assumptions always have to made about the prices of options with strikes for which there are no available quotes. As described above, the data provide three constraints on the volatility smile. As an approximation this is equivalent to prices of three options: the 25 delta put, the 25 delta call and ATM put or call. This could seem like very little information to extrapolate from. However, the three instruments are commonly traded and are said "to characterize the FX options market" (Bisesti et al., 2005). The vanna-volga method for extrapolating is constructed for use with these quotes, has a theoretical justification, is commonly used in the market and is thought to be accurate in a wide range of strikes (Castagna and Mercurio, 2007).

This is also in line with Carr and Wu (2007), who use a minimum of three prices (for individual stocks) and Petersson and Šarić (2008) who use a minimum of four prices. Both then interpolate implied volatility and use a constant implied volatility beyond available strikes.[13]

Since exchange traded index and stock options have fixed monthly expiries, Carr and Wu and Petersson and Šarić interpolate between the two closest maturities to achieve synthetic variance swap rates with a constant time to maturity. OTC foreign exchange option quotes, on the other hand, already refer to options that have a constant time to maturity so this problem is avoided. Overall therefore it can be argued that this method is of comparable accuracy to previously employed methods.

Using the realized variance and the estimated variance swap rate, the excess return is calculated for each day as defined in section 2:

$$\widehat{\mathrm{RP}}_{t,T} = \frac{\mathrm{RV}_{t,T}}{\mathrm{KV}_{t,T}} - 1.$$

[12]This shape of the smile sometimes violates another no-arbitrage condition; because of the discontinuity in the derivative of the implied volatility with respect to strike, the price is not always convex in strike. However this is deemed less serious than negative prices, and the result is anyway a more conservative estimate of the variance swap rate.

[13]Carr and Wu interpolate linearly in log-strike and Petersson and Šarić use natural cubic splines.

4.6 Hypothesis testing

The null-hypothesis of indifference to variance risk should be rejected if the observed data is sufficiently improbable under the hypothesis. The null-hypothesis states that the average (conditional) expected excess return of variance swaps during the period was zero. Let $\hat{\theta}$ denote the estimated average excess return on variance swaps during the period. Under the hypothesis $\hat{\theta}$ has zero expected value. Since the estimated variance swap returns are calculated over overlapping periods, consecutive estimates will be strongly autocorrelated. Moreover heteroskedasticity can be expected. In order to estimate the variance of $\hat{\theta}$ the heteroskedasticity and autocorrelation consistent estimator of Newey and West (1987) is used. Using this an asymptotically normally distributed t-statistic is calculated for testing the hypothesis. The number of lags is set to 25.

In order to test the hypothesis of correlation with stock market returns and to determine whether the CAPM beta can account for variance swap returns, the following regression on U.S. dollar denominated variance swaps is estimated

$$\widehat{\mathrm{RP}}_{t,T} = \alpha + \beta\, \mathrm{ER}^m_{t,T} + \epsilon_{t,T}$$

where ER^m is the excess return of the market portfolio. The MSCI World Index,[14] in U.S. dollars with dividends included, is used as a proxy for the market portfolio. Excess returns are calculated using the yield on one month treasury bills.[15] The MSCI data is monthly so the observations are from non-overlapping periods and are found not to be autocorrelated. Therefore an ordinary least squares regression can be used. Heteroskedasticity is expected, however, and there are only about 85 observations. This needs to be taken into account when testing the coefficients for significance. Long and Ervin (2000) recommend the use of a heteroskedasticity consistent estimator of the covariance matrix known as "HC3" for small samples.[16] Consequently this estimator is used for significance tests of the coefficients.

5 Results and further analysis

In table 1 the results of the hypothesis tests of zero average risk premium are reported. The average risk premium is estimated to less than zero for all currency pairs, but AUDUSD. However they are mostly insignificantly less than zero.

There are three explanations for why the null-hypothesis could not be rejected in most cases. First, the average expected return could indeed be zero or even positive; the hypothesis may be wrong.

[14] Available at `http://www.mscibarra.com/`

[15] Available at `http://www.ustreas.gov/`

[16] To achieve consistent significance tests Long and Ervin recommend *not* basing the decision to use the estimator on a test for heteroskedasticity.

	Estimate	Std. Error	t value	Pr(>|t|)
GBPUSD	-0.050	0.035	-1.410	0.159
EURUSD	-0.073	0.030	-2.435	0.015*
EURGBP	-0.053	0.033	-1.588	0.113
EURSEK	-0.163	0.063	-2.578	0.010*
USDJPY	-0.071	0.052	-1.353	0.176
USDCAD	-0.055	0.056	-0.977	0.328
AUDUSD	0.109	0.098	1.115	0.265

Table 1: The estimated average risk premium and tests of the null-hypothesis of zero average for the domestic variance swap rate. (*** $p < .001$,** $p < .01$,* $p < .05$)

	Estimate	Std. Error	t value	Pr(>|t|)
GBPUSD	-0.049	0.035	-1.391	0.164
EURUSD	-0.073	0.030	-2.445	0.015*
EURGBP	-0.054	0.033	-1.628	0.104
EURSEK	-0.164	0.063	-2.599	0.009**
USDJPY	-0.066	0.053	-1.254	0.210
USDCAD	-0.055	0.056	-0.979	0.328
AUDUSD	0.113	0.099	1.142	0.254

Table 2: The estimated average risk premium and tests of the null-hypothesis of zero average for the foreign variance swap rate. (*** $p < .001$,** $p < .01$,* $p < .05$)

Second, the sample period might not be representative, meaning that there is a negative variance risk premium in currency options, but due to some unusual influential events it could not be observed. If an extraordinary negative event happens during the sample period it can be argued that it is likely investors appear not to properly take into account the possibility of such events, because the likelihood of them occurring appear greater than it actually is.

In figures 1–7 on pages 30–36 historic variance swap returns are illustrated. As can be seen all currency pairs experienced sudden exchange rate movements and consequently sudden increases in volatility during the fall of 2008. In many cases this led to positive returns on variance swaps. For example the Australian dollar lost almost 40 percent of its value against the U.S. dollar in a matter of months, causing realized variance 10 times greater than the variance swap rate.

These recent high levels of realized variance caused by sudden large exchange rate movements can be considered to constitute such unlikely adverse events for sellers of variance. Based on the seven years of data the apparent frequency of exchange rate movements of this magnitude is once every seven years, but it is possible that this frequency is much lower and that in the long run the expected return of long variance swaps is indeed negative. The large positive returns not only affect the estimate itself but also cause the variance of the estimate to be greater thereby reducing the significance.

In order to gauge the impact of the recent turbulence on the hypothesis tests the sample is divided into two periods of equal length and the variance risk premium is estimated in each. The results are reported in table 3. For the first period between 2001 and 2005 the estimated variance risk premium is less than zero for all currency pairs, in some cases with a high significance. In the latter period, on the other hand, none of the currency pairs have significant variance risk premiums. It is difficult to tell which of the two periods is more representative without more data.

The third explanation, is that the there is a negative risk premium and the sample is representative but there were not enough observations to make it statistically significant. This means that if more observations were available the null-hypothesis might have been rejected.

5.1 Extending the dataset using an approximation

Intuitively, the most important of the three inputs determining the variance swap rate is the ATM volatility. It controls the overall level of the implied volatility smile and it directly determines the prices of near-the-money options which have most influence on the variance swap rate since they are the most expensive. As shown in section 4.3 if the smile is flat the variance swap rate is the volatility squared.

Table 4 reports the results of regressions of the variance swap rate on the ATM volatility squared. As can be seen the slope of the regressions are above one, reflecting the convexity of the smile. The intercepts are very close to zero. As is evident from the coefficients of

	Estimate	Std. Error	t value	Pr(>|t|)
2001-08-30–2005-05-14				
GBPUSD	-0.106	0.042	-2.524	0.012*
EURUSD	-0.086	0.040	-2.170	0.030*
EURGBP	-0.078	0.048	-1.639	0.102
EURSEK	-0.351	0.059	-5.938	0.000***
USDJPY	-0.183	0.042	-4.325	0.000***
USDCAD	-0.121	0.052	-2.309	0.021*
AUDUSD	-0.105	0.056	-1.871	0.062
2005-05-14–2009-01-27				
GBPUSD	0.002	0.055	0.043	0.966
EURUSD	-0.060	0.044	-1.356	0.176
EURGBP	-0.029	0.046	-0.639	0.523
EURSEK	-0.013	0.096	-0.139	0.889
USDJPY	0.029	0.089	0.330	0.742
USDCAD	0.008	0.096	0.088	0.930
AUDUSD	0.314	0.177	1.773	0.077

Table 3: The estimated average risk premium and tests of the null-hypothesis of zero average for the domestic variance swap rate for the subsamples. (*** $p < .001$,** $p < .01$,* $p < .05$)

	(a)		(b)	(c)
GBPUSD	-0.000	1.133	0.999	05-Jan-1998
EURUSD	-0.000	1.131	0.999	04-Jan-1999
EURGBP	0.000	1.095	0.999	04-Jan-1999
EURSEK	0.000	1.105	0.999	04-Jan-1999
USDJPY	-0.000	1.164	0.997	05-Jan-1998
USDCAD	0.000	1.083	1.000	08-Feb-1999
AUDUSD	-0.000	1.102	1.000	05-Jan-1998

Table 4: (a): the intercept and the slope of the regression of the variance swap rate on the ATM volatility squared. (b): Coefficient of determination (R^2) of the regression. (c): date from which ATM quotes are available.

determination the regressions account for almost all of the variation in the variance swap rate. This suggests that the ATM volatility alone can be used to compute a very good approximation of the variance swap rate. The table also reports the dates after which ATM volatility quotes are available. The sample period can be extended by about three years if the estimated regressions for each currency pair are used to construct approximate variance swap rates for the period where butterfly and risk reversal quotes are unavailable. In reality the average convexity of the smile during this period may have been lower than during the period for which the coefficients were calculated, but it may also have been higher.

A very conservative estimate can be obtained using the lowest observed ratio between the variance swap rate and the ATM volatility squared. This ratio is found to be 1.041. Consequently the series of estimated variance swap rates is extended by using the ATM volatility squared multiplied by this number. Using this longer series of estimates the hypothesis tests are repeated. The results are reported in table 5. For most currency pairs the results are similar with somewhat higher significance, but for the pound against the euro and against the dollar the difference is considerable.

	Estimate	Std. Error	t value	Pr(>|t|)
GBPUSD	-0.110	0.030	-3.686	0.000***
EURUSD	-0.100	0.024	-4.110	0.000***
EURGBP	-0.085	0.028	-2.998	0.003**
EURSEK	-0.126	0.051	-2.472	0.014*
USDJPY	-0.071	0.045	-1.580	0.114
USDCAD	-0.085	0.047	-1.824	0.068
AUDUSD	0.039	0.072	0.551	0.582

Table 5: The estimated average risk premium and tests of the null-hypothesis of zero average for the domestic variance swap rate where the sample is extended using the ATM approximation. (*** $p < .001$,** $p < .01$,* $p < .05$)

The currency pairs for which there were highly significant variance risk premiums are those that did not experience large positive returns on variance swaps during the sample period, suggesting that perhaps they may be affected by *the peso problem* (see figures 8-14). Bondarenko (2003) discusses the peso problem as a possible explanation for why index put options appear to be overvalued and describes it as “a situation when a rare and influential event could have reasonably happened but did not happen in the sample”. This means that if investors correctly estimate the probability of rare negative events, it is likely that they appear to overprice put options if such an event did not actually happen in the sample.

The problem is with the method of statistical inference which relies on assumptions about the asymptotic distribution of the estimate that may not be sound when the distribution whose expected value is to be estimated, is very skewed. As stated by Bekaert et al. (1997) the peso problem is an issue when catastrophic events are possible but unlikely. Because the events are improbable, they are unlikely to be observed. Because they are catastrophic they, have a substantial effect on equilibrium prices and returns.

In this case this means that the significance of the results for EURUSD, GBPUSD and EURGBP may be exaggerated because there were no extreme positive returns, which are clearly possible. This lowers the estimated variance of the estimate and the estimate itself, thereby inflating significance. From the histograms it is clear that the distributions of variance swap returns is very skewed and is characterized by infrequent large positive values.

Looking at returns on variance swaps on USDJPY and AUDUSD in the extended sample period it appears the recent large positive returns are not so uncommon as the original seven years of data suggest, giving less credence to the second explanation of why no significant premium was observed.

Given the conflicting results of tests of the null-hypothesis for different sample periods and different currency pairs it is difficult to reach a definite conclusion on the existence of a variance risk premium in currency options. Do the differing results between currency pairs reflect some intrinsic qualities or are they merely due to chance? Should we conclude that there is a variance risk premium in EURUSD and GBPUSD options but not in AUDUSD options?

5.2 Correlation with stock market returns

In table 6 the results of the regressions of U.S. dollar denominated variance swap returns on MSCI world excess returns are reported. For all currencies negative beta coefficients are estimated. However they are insignificant with the exception of EURUSD and USDJPY, where the hypothesis of zero correlation (and zero beta) can be rejected at the .05 level. Because the average market excess return was negative during the sample period and the beta coefficients are negative the alpha coefficients are lower than the average returns. However neither the alpha coefficients nor the average returns are significantly different from zero.

5.3 The foreign variance swap rate

Table 2 reports the hypothesis test of zero average variance risk premium for the foreign variance swap rate; e.g. for EURSEK the notional amount is in euro and not in SEK. The results are nearly identical to those for the domestic rate in table 1, because the difference between the two rates is very small. Table 7 reports some summary statistics for the relative difference. As noted in section 3 the difference is determined by the slope of the smile. As can be seen from the table the difference is greatest for USDJPY for which the smile is often steep; USD calls have lower volatility meaning that the risk reversal is negative. Indeed, the risk reversal accounts for

	α	β	$\widehat{\mathrm{RP}}$
GBPUSD			
Estimate	-0.021	-2.175	-0.019
P(>|t|)	0.654	0.268	0.687
EURUSD			
Estimate	-0.054	-2.272	-0.054
P(>|t|)	0.129	0.013*	0.145
USDJPY			
Estimate	-0.071	-4.505	-0.065
P(>|t|)	0.211	0.019*	0.283
USDCAD			
Estimate	-0.045	-4.997	-0.040
P(>|t|)	0.483	0.241	0.542
AUDUSD			
Estimate	0.096	-11.738	0.121
P(>|t|)	0.382	0.200	0.334

Table 6: Regression of U.S. dollar denominated variance swap returns on the MSCI World Index. Values under $\widehat{\mathrm{RP}}$ are sample averages of variance swap returns. Significance tests are performed using the heteroskedasticity consistent estimator of the standard errors suggested by Long and Ervin (2000), (*** $p < .001$,** $p < .01$,* $p < .05$).

	(a)	(b)		(c)
GBPUSD	-0.063	-0.483	0.200	0.999
EURUSD	0.045	-0.237	0.322	0.999
EURGBP	0.132	-0.030	0.403	0.999
EURSEK	0.121	-0.070	0.581	0.998
USDJPY	-0.512	-1.714	0.037	0.995
USDCAD	0.006	-0.172	0.164	0.999
AUDUSD	-0.264	-0.960	0.100	0.999

Table 7: (a): the average relative difference between the foreign and domestic variance swap rates in percent. (b): 5 and 95 percentiles of the difference. (c): Coefficients of determination (R^2) of regressions of the relative difference on the 25 delta risk reversal.

almost all of the variation in the relative difference.

6 Robustness of the estimated variance swap rate

One peculiar feature of the vanna-volga smile is that the implied volatility starts to decrease far from the money. In fact, it can be shown that the limits at zero and infinity is the ATM volatility (Castagna and Mercurio, 2007). One might expect that to put a downward bias on the variance swap rate if this phenomenon does not reflect reality, but as noted earlier, since deep-out-of-the money options are cheap, they have little impact on the variance swap rate. Nevertheless, the magnitude of this possible bias should be quantified. In order to accomplish this the vanna-volga method is assumed to produce accurate volatilities in the 10 delta put to 10 delta call range and the SVI parametrization is fitted to these volatilities.

In the SVI (*Stochastic Volatility Inspired*) parametrization due to Gatheral (2004), the total implied variance, defined as the volatility squared multiplied by the time to maturity, $v = t\sigma^2$ is given by

$$v(k) = a + b\left((k-m)\rho + \sqrt{(k-m)^2 + s^2}\right)$$

where $k = \ln \frac{K}{F}$. a, b, ρ, m and s are parameters that determine the shape of the smile. It can be shown that the variance increases asymptotically linearly in k. As shown by Lee (2004), the implied volatility cannot grow faster than this if the risk neutral distribution is to have finite variance.

10 and 25 delta put and call volatilities and the ATM volatility are extracted from the vanna-volga smile, and the SVI parametrization is fitted to them. Using the SVI smile the variance swap rate is computed from (5). Sometimes, when the smile is very steep, the SVI fit gives negative implied volatilities far from the money. These observations are excluded. The difference between the two methods for the remaining observations are reported in 8. Clearly the difference is not

	(a)	(b)	(c)
GBPUSD	1.88	0.78 3.66	0.00
EURUSD	1.78	0.70 3.52	0.00
EURGBP	2.14	1.04 3.93	2.40
EURSEK	2.49	1.00 4.64	3.74
USDCAD	1.86	0.98 3.66	0.00
AUDUSD	1.80	0.95 3.46	4.35

Table 8: (a): average relative difference between the variance swap rate using the SVI smile and using the vanna-volga smile in percent, $\frac{\mathrm{KV}_{\mathrm{SVI}}}{\mathrm{KV}_{\mathrm{VV}}} - 1$. (b): the 5 and 95 percentiles of the relative difference. (c): the percentage of observations where there was no successful fit.

negligible. The relative error in the estimated variance swap rate due to inaccuracy in the prices of options beyond 10 delta may be as much as a few percent. The proposed method may considerably underestimate the variance swap rate.

In light of this finding, the estimated average variance swap returns may be underestimated too. Consequently the hypothesis tests are repeated using this higher estimate of the variance swap rate. The results are reported in table 9. There are no major differences in the conclusions.

	Estimate	Std. Error	t value	Pr(>|t|)
GBPUSD	-0.067	0.035	-1.905	0.057
EURUSD	-0.089	0.029	-3.017	0.003**
EURGBP	-0.072	0.034	-2.153	0.031*
EURSEK	-0.181	0.064	-2.815	0.005**
USDCAD	-0.071	0.055	-1.292	0.197
AUDUSD	0.092	0.095	0.966	0.334

Table 9: The estimated average risk premium and tests of the null-hypothesis of zero average using the the SVI estimate (*** $p < .001$,** $p < .01$,* $p < .05$)

7 Conclusion

The vanna-volga method is used to estimate variance swap rates on exchange rates from 25 delta risk reversal and butterfly quotes. It is shown that the variance swap rate consistent with vanna-volga prices is given as a closed form expression.

The estimated variance swap rates are then compared to realized variance in an attempt to determine the existence of a variance risk premium. The null-hypothesis of zero variance risk premium cannot be rejected in most cases. Conflicting results of hypothesis tests are obtained in two sub-samples investigated. By extending the sample using a conservative approximation of the variance swap rate highly significant negative variance risk premiums are found in a few currency pairs.

It is found that the proposed method possibly underestimates the variance swap rate by as much as a few percent even if the vanna-volga method is accurate in the 10 delta put to 10 delta call range.

It is shown that the variance swap rate of a foreign denominated variance swap is proportional to the domestic gamma swap rate, and that the difference between the two variance swap rates is determined by the slope of the smile. In practice the difference is found to be very small.

Furthermore, currency variance swap returns are found to be correlated with stock market returns in a few cases.

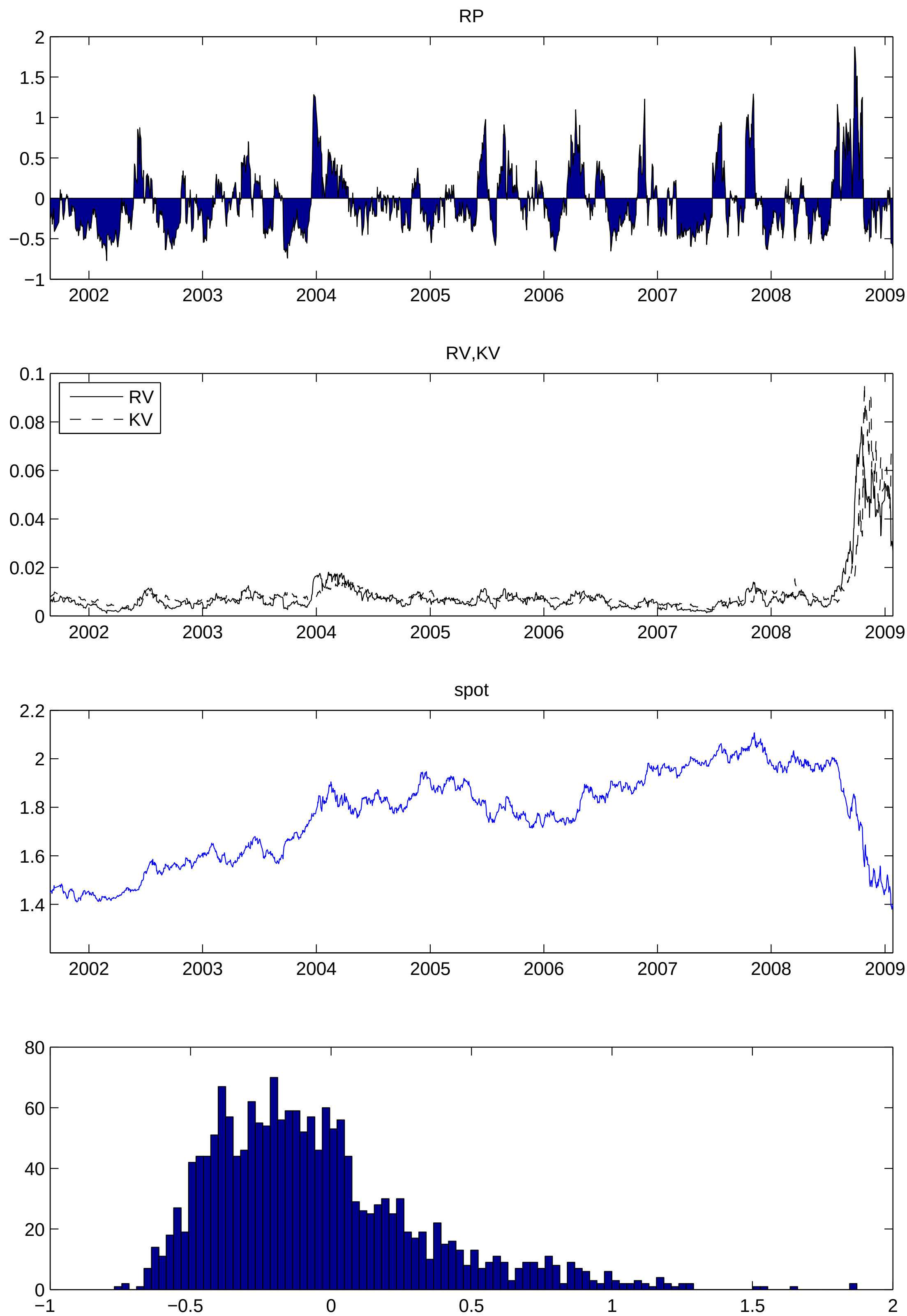

Figure 1: GBPUSD. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP.

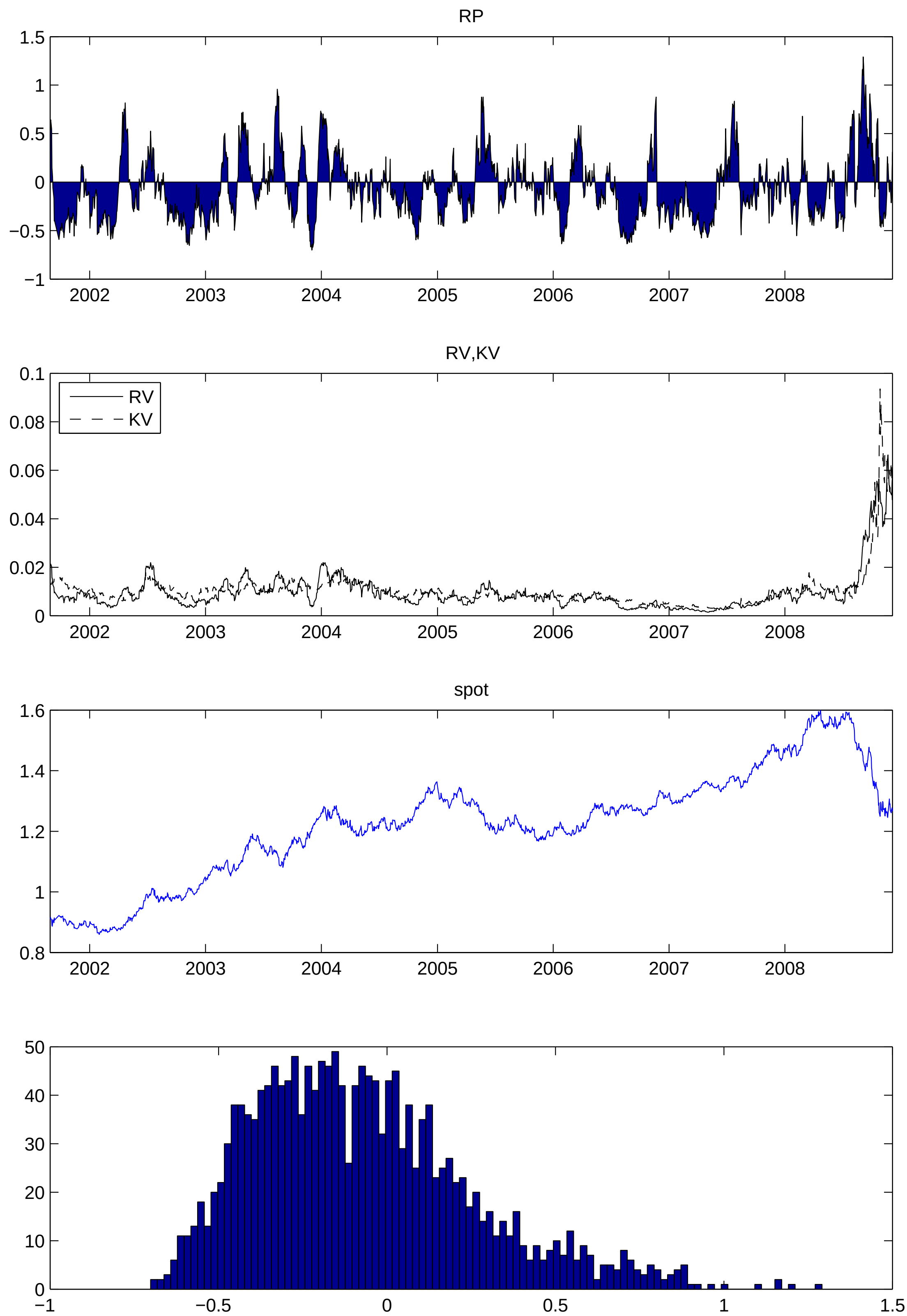

Figure 2: EURUSD. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP.

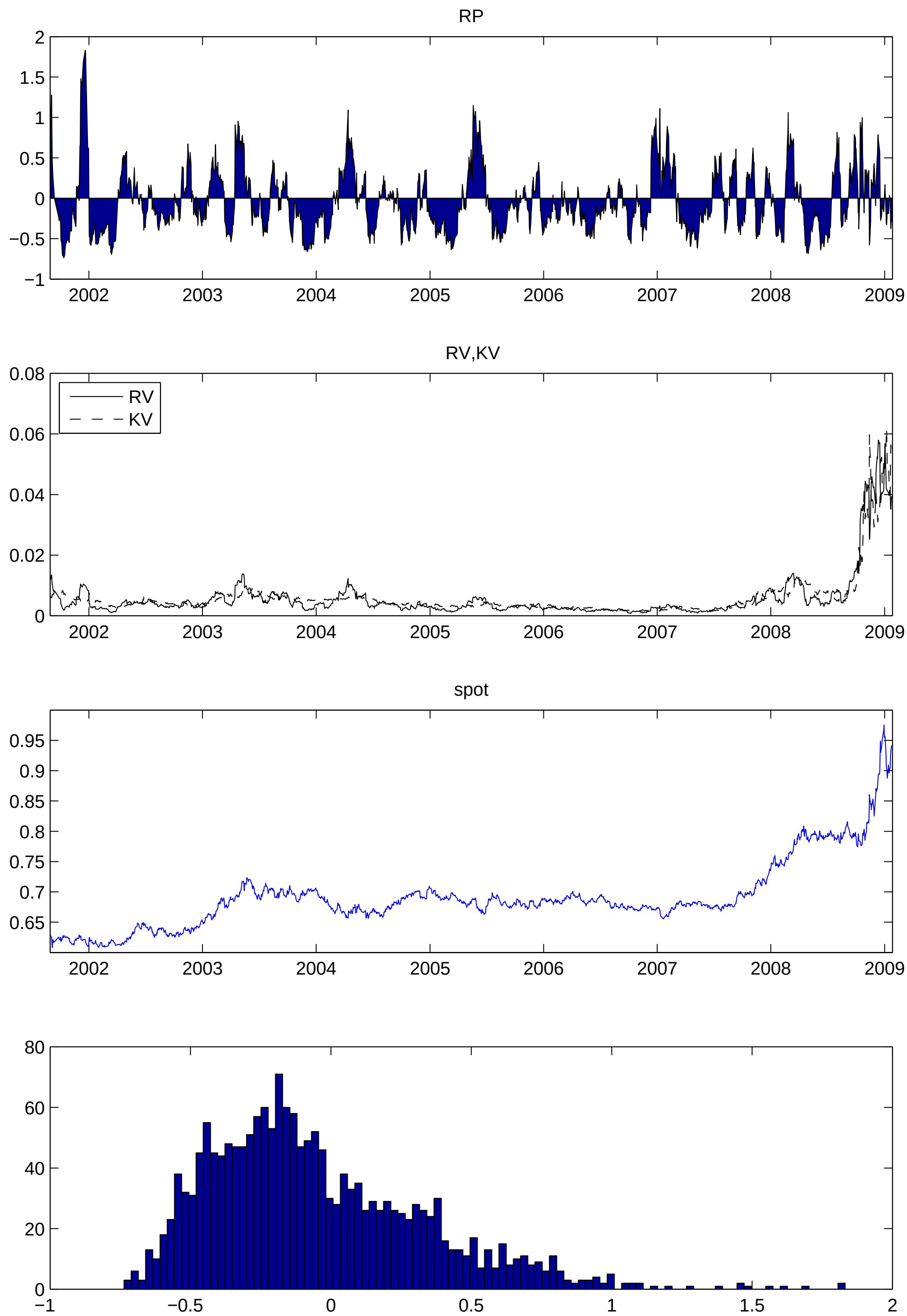

Figure 3: EURGBP. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP.

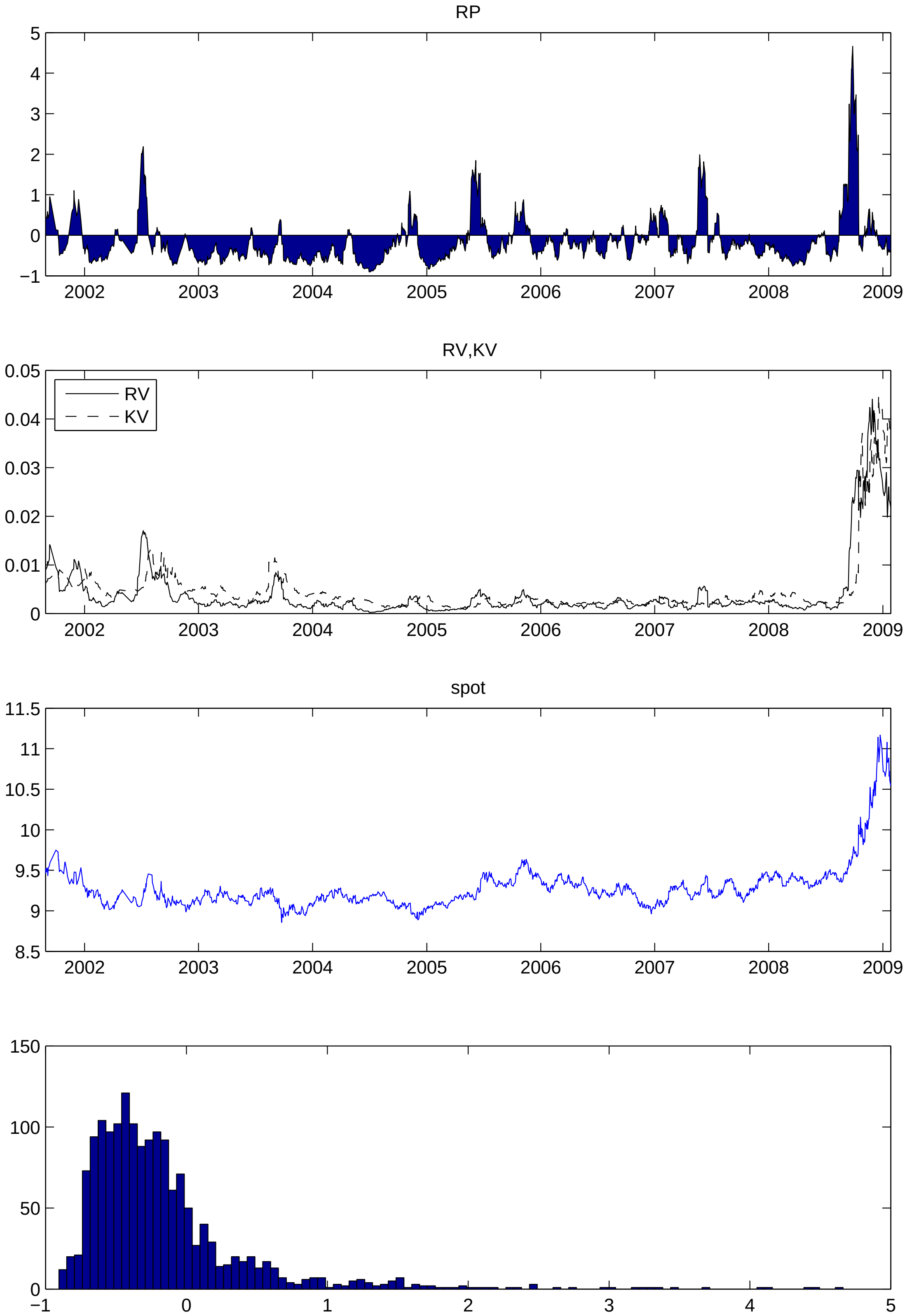

Figure 4: EURSEK. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP.

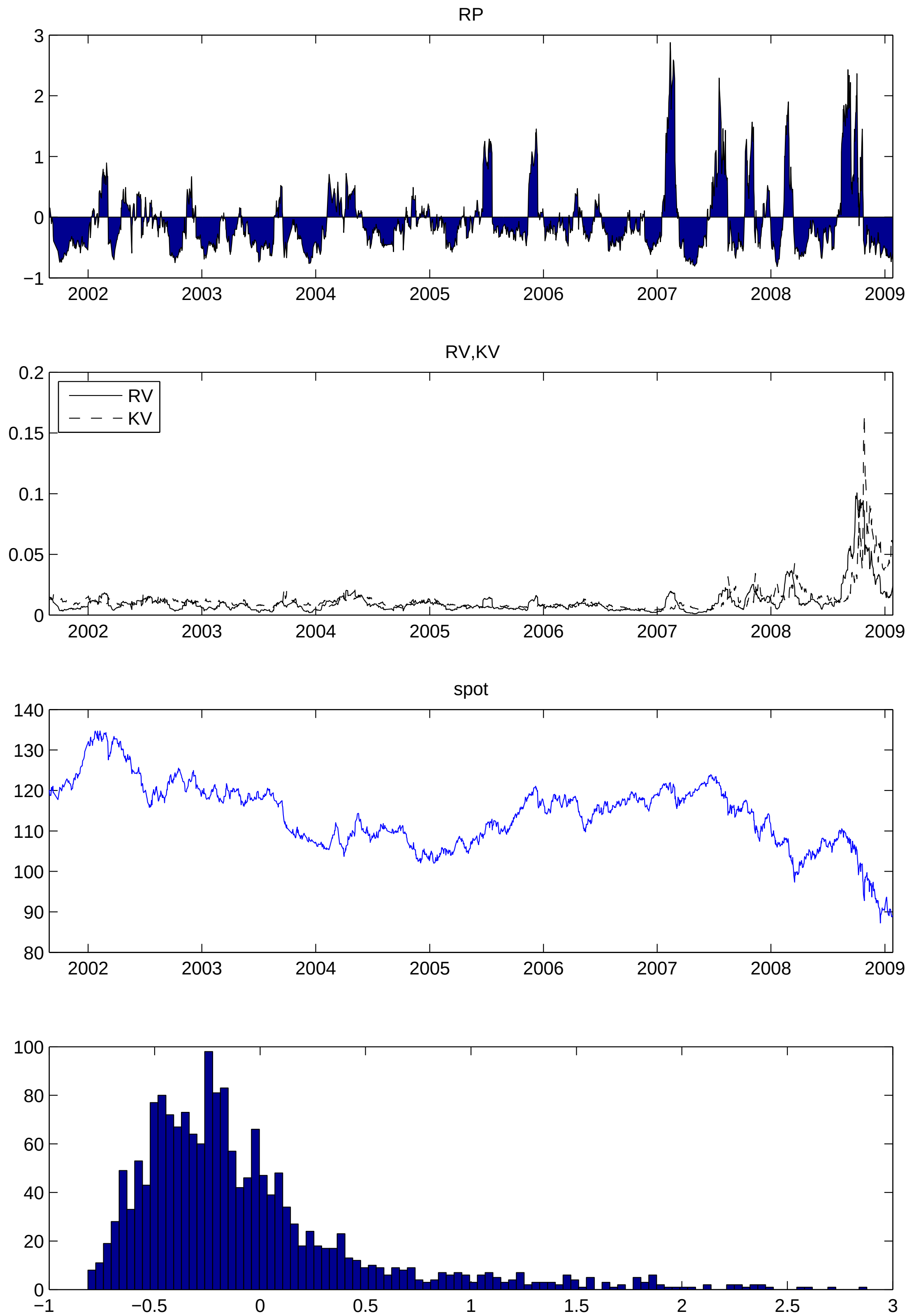

Figure 5: USDJPY. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP.

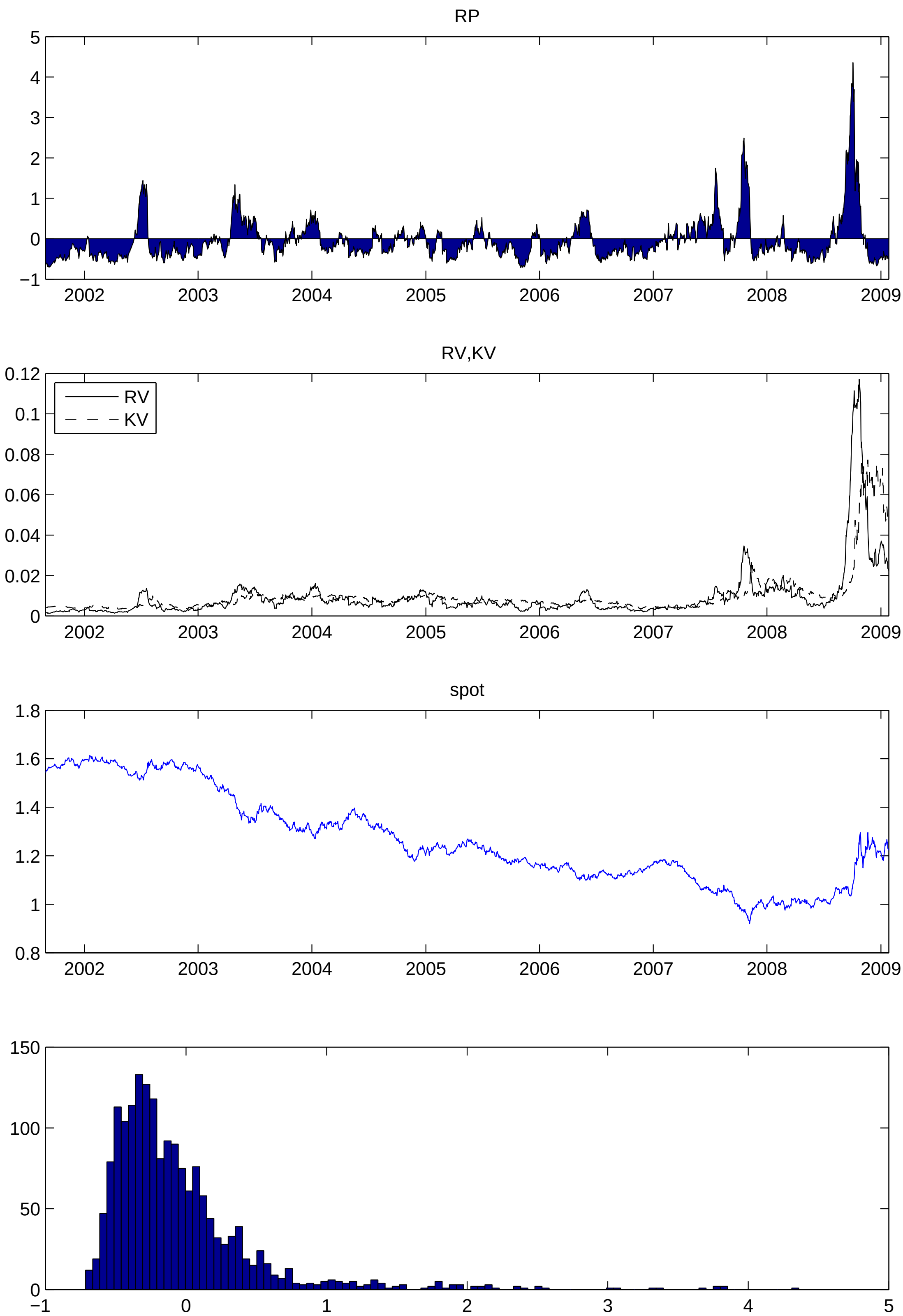

Figure 6: USDCAD. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP.

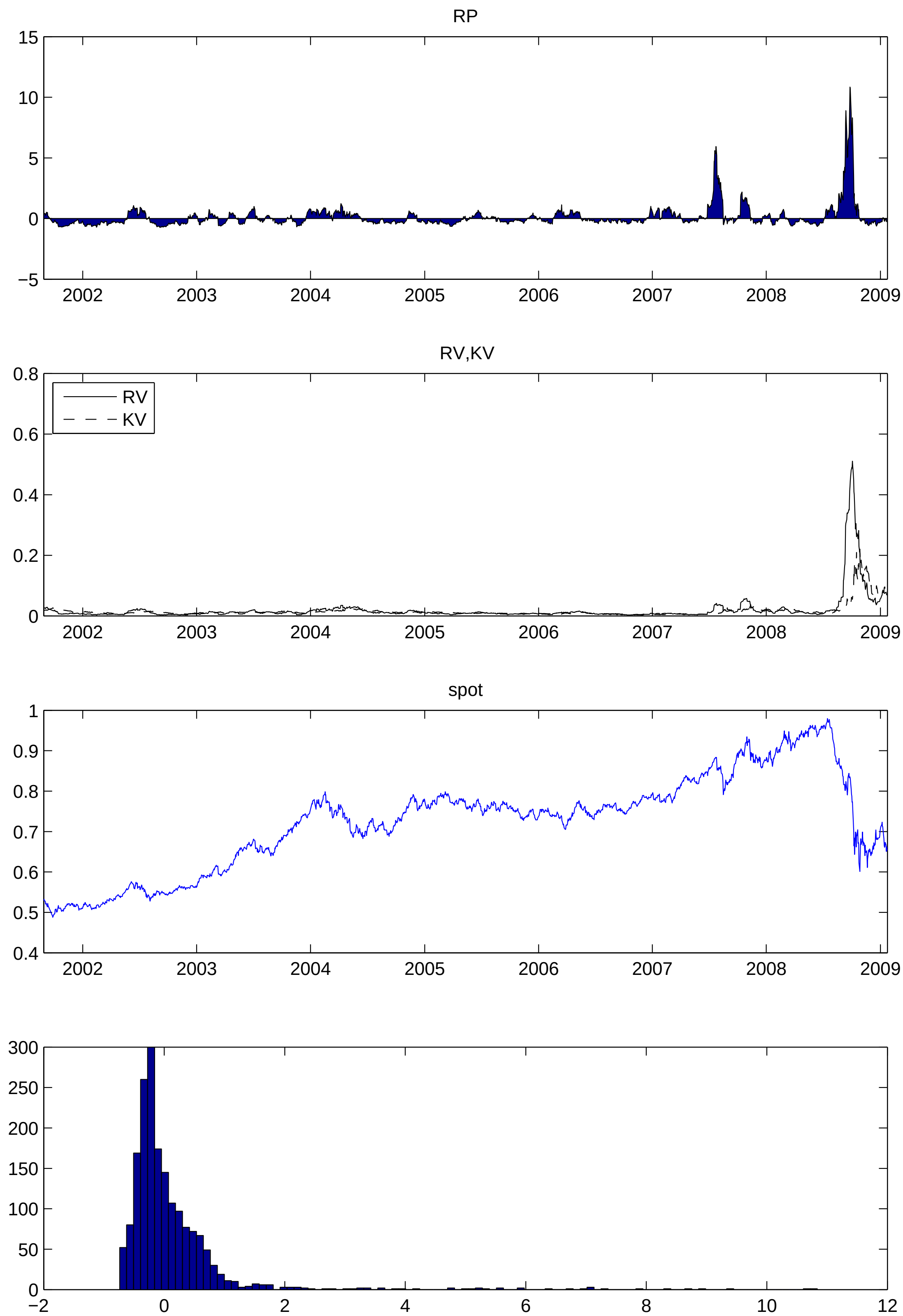

Figure 7: AUDUSD. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP.

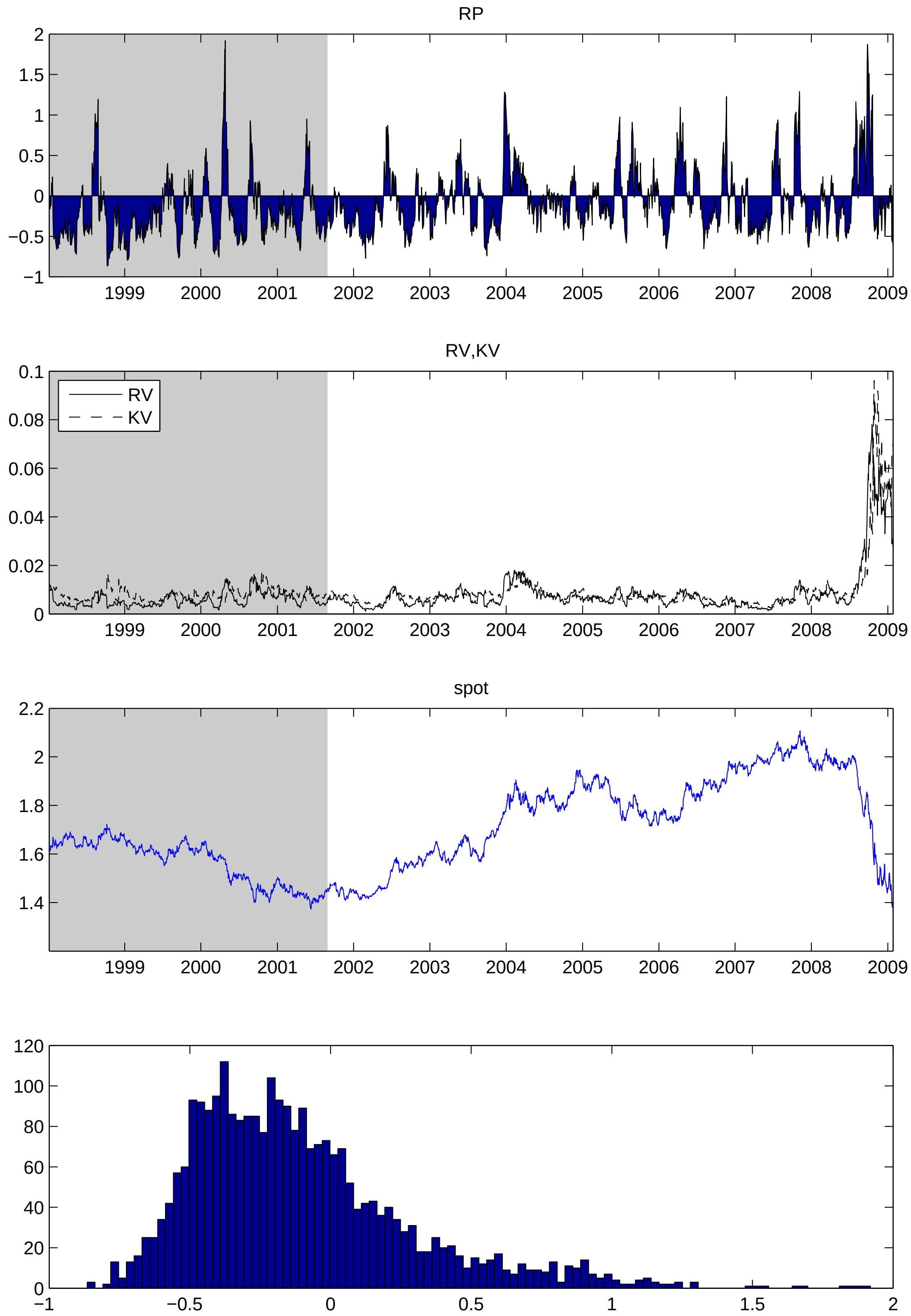

Figure 8: GBPUSD. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP. In the shaded area the approximation based on the ATM volatility is used.

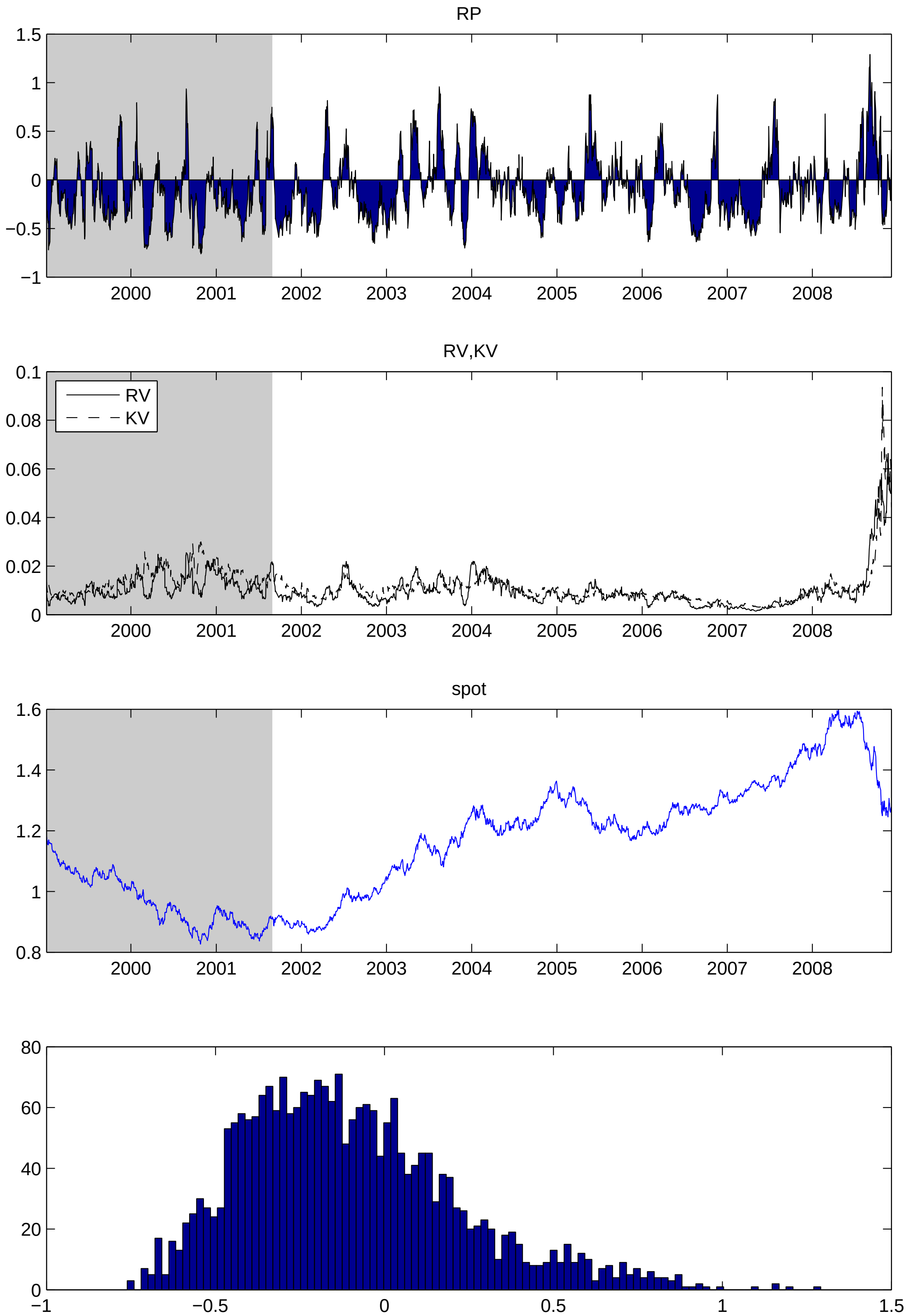

Figure 9: EURUSD. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP. In the shaded area the approximation based on the ATM volatility is used.

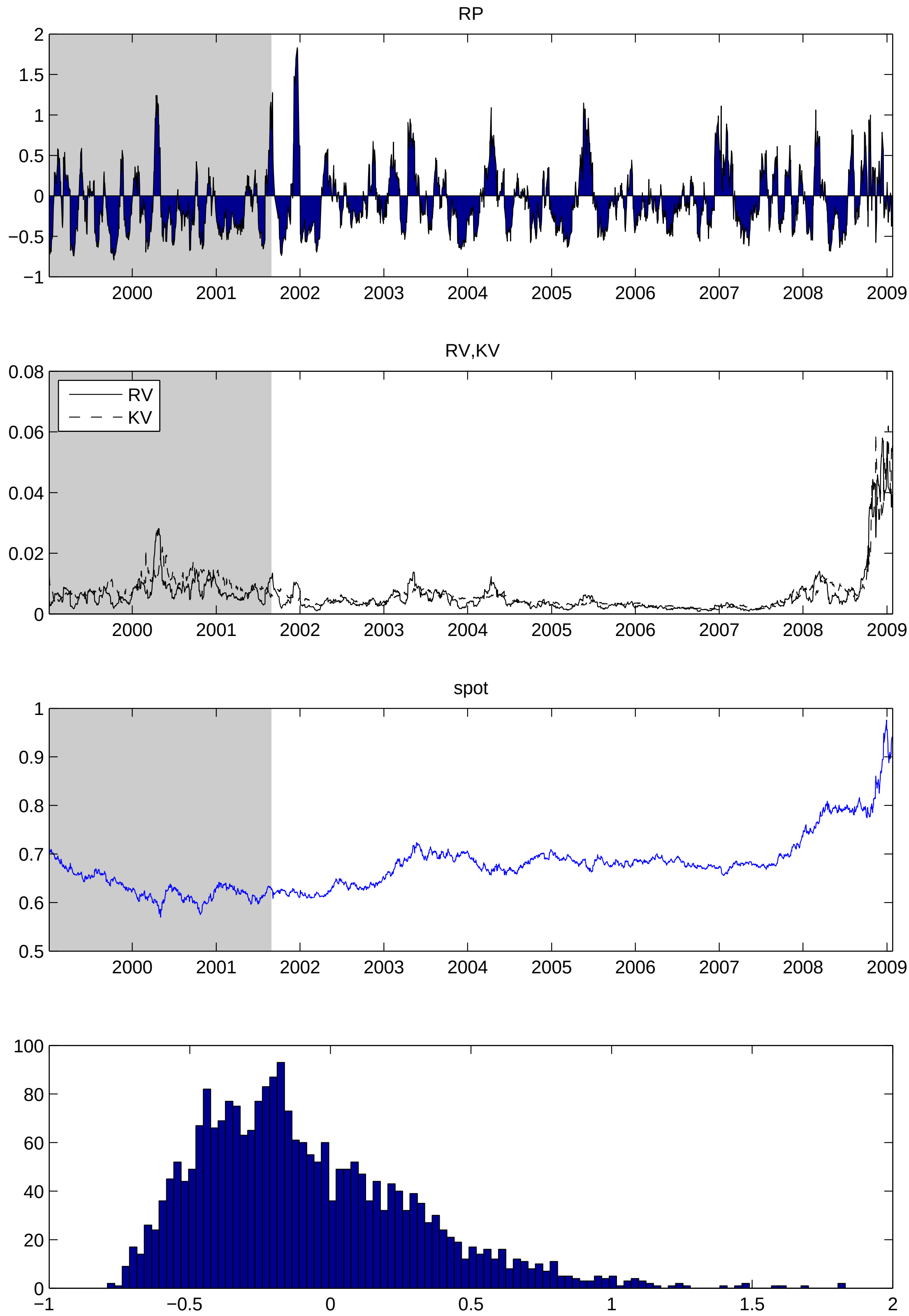

Figure 10: EURGBP. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP. In the shaded area the approximation based on the ATM volatility is used.

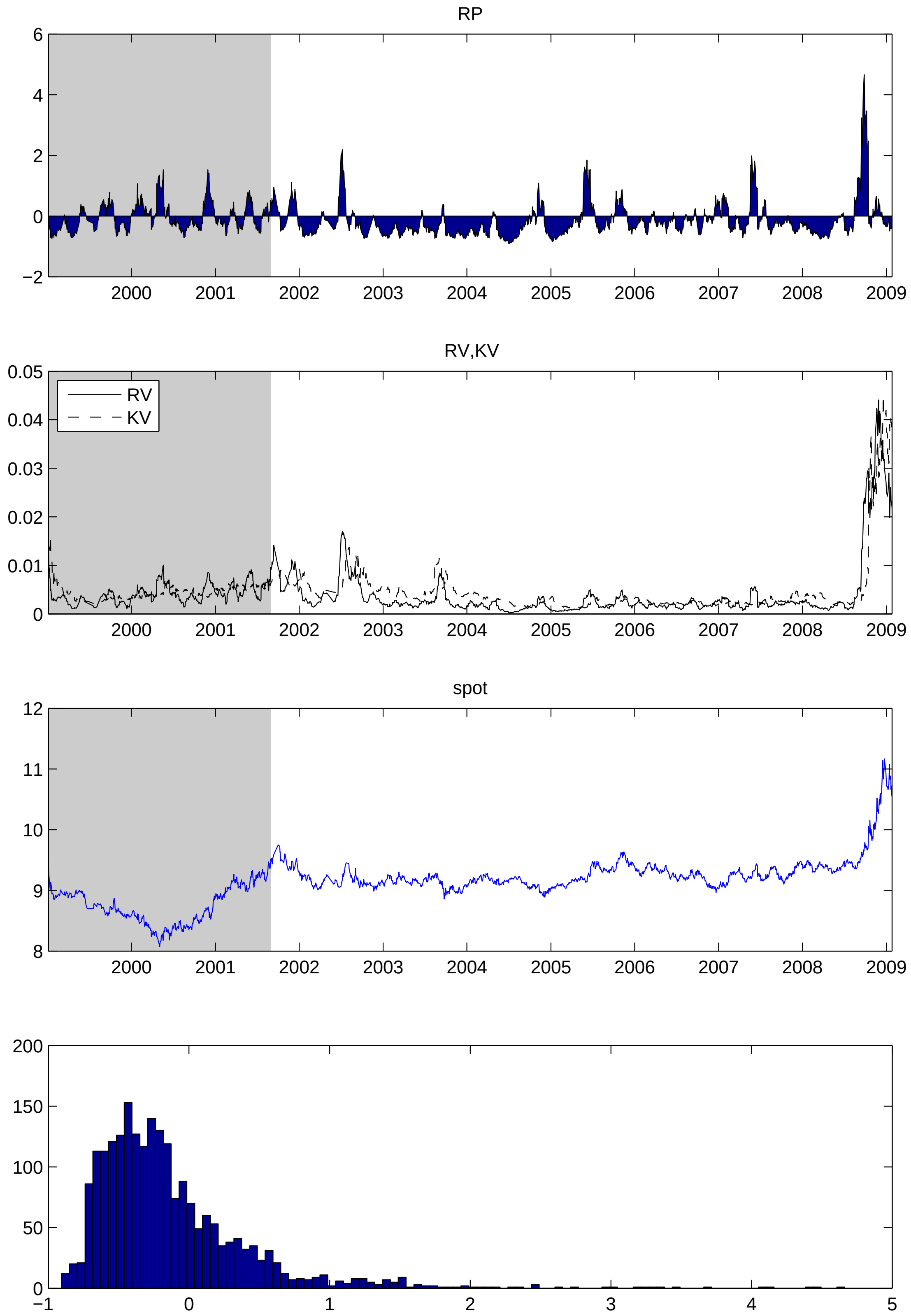

Figure 11: EURSEK. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP. In the shaded area the approximation based on the ATM volatility is used.

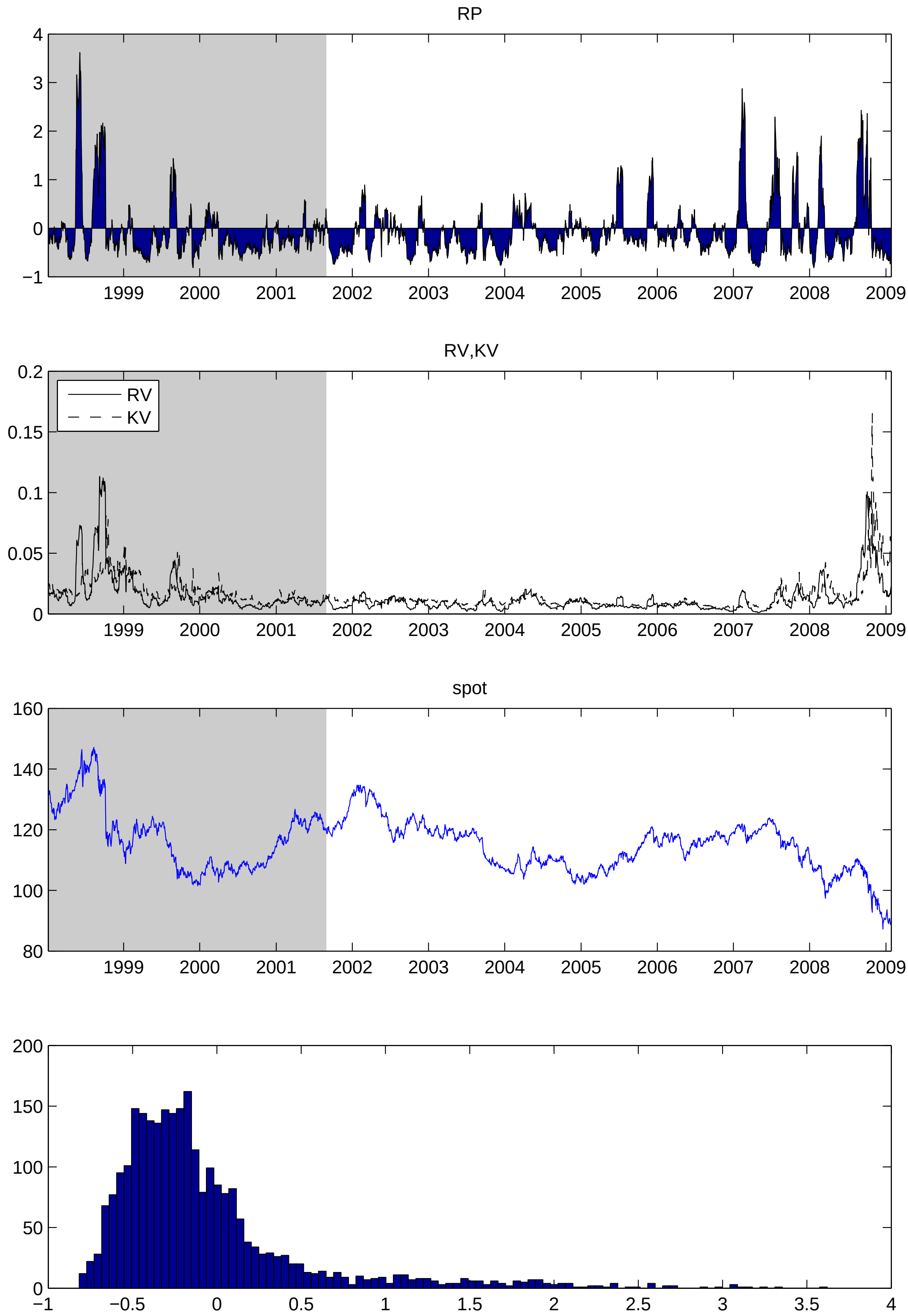

Figure 12: USDJPY. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP. In the shaded area the approximation based on the ATM volatility is used.

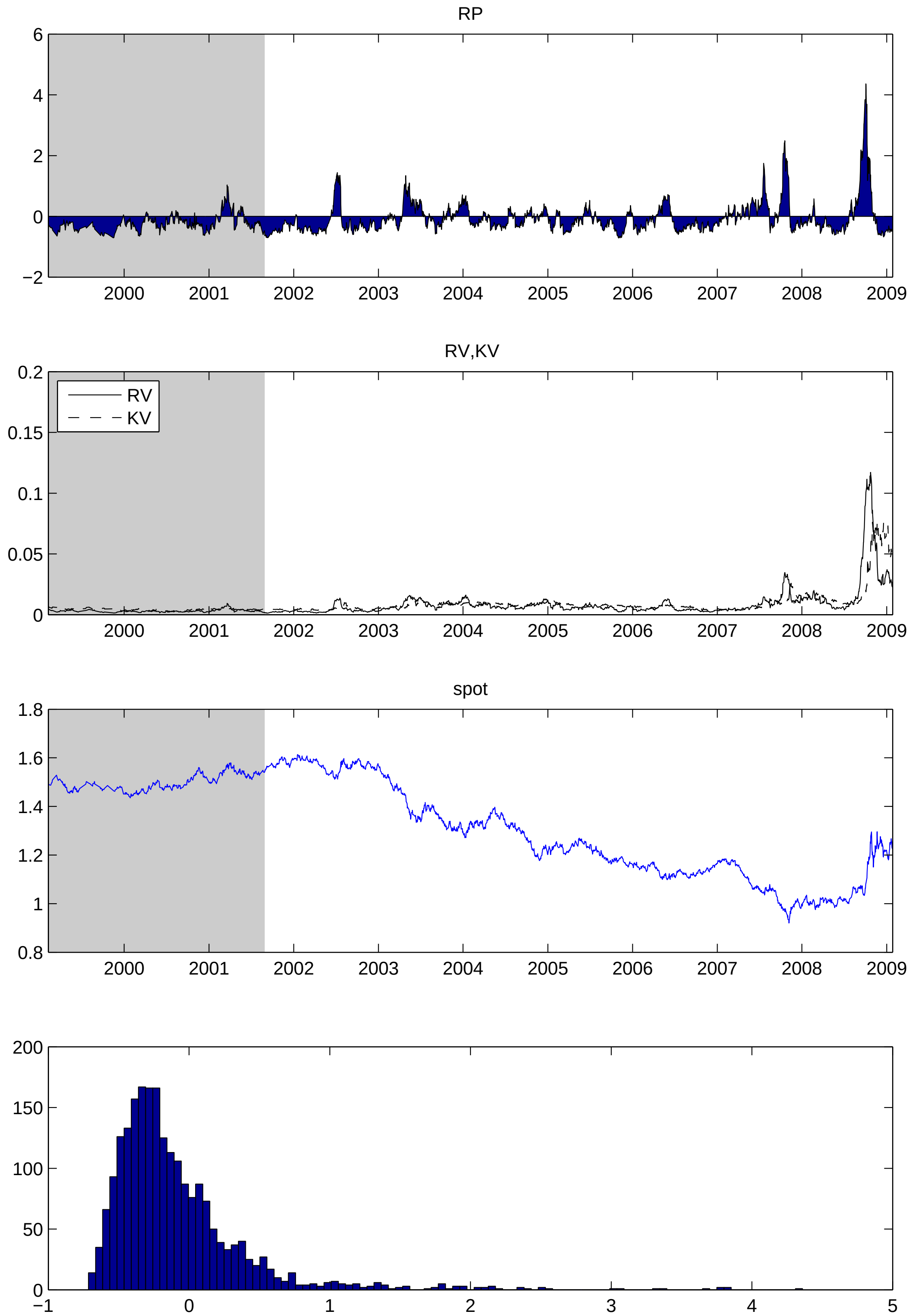

Figure 13: USDCAD. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP. In the shaded area the approximation based on the ATM volatility is used.

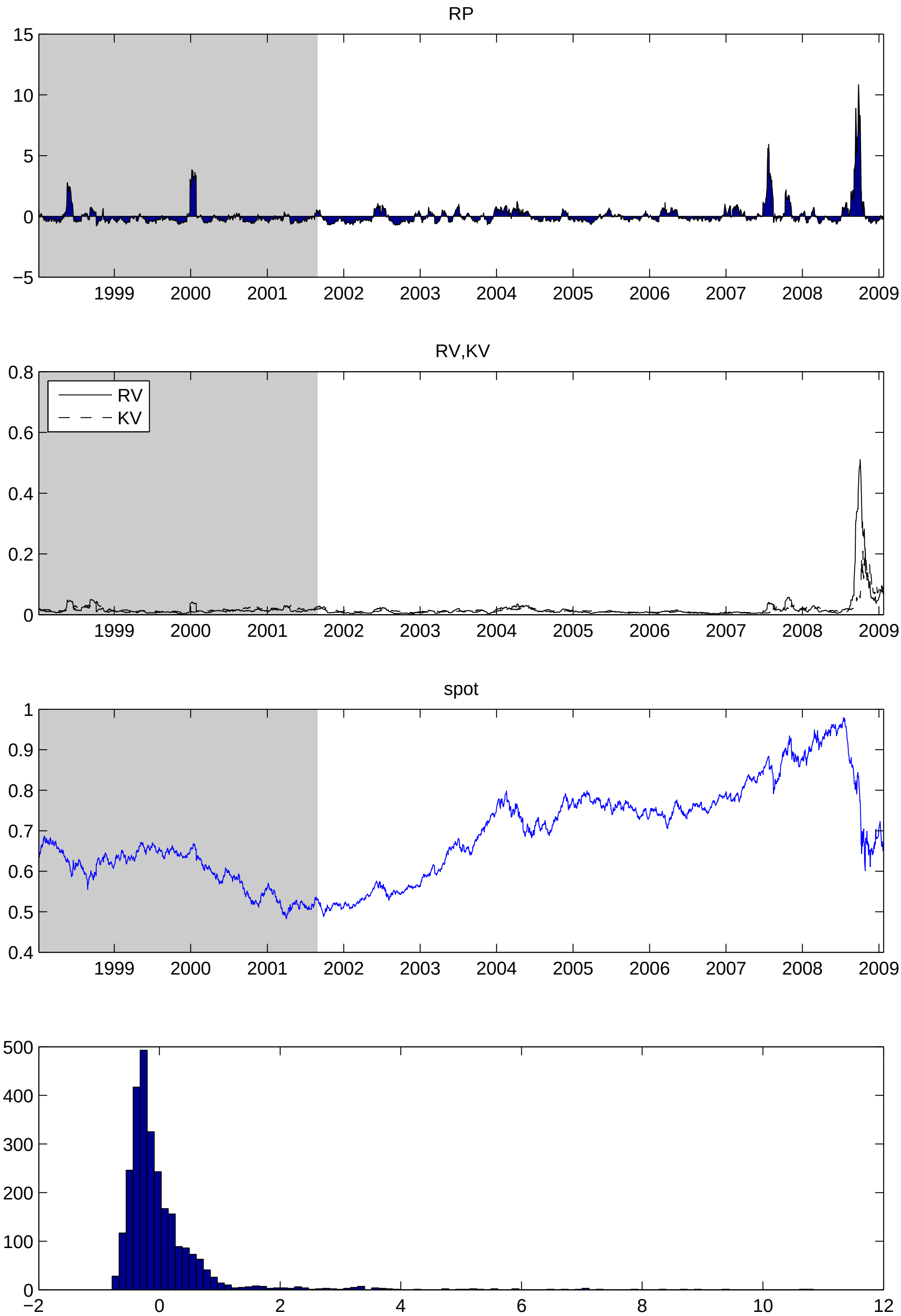

Figure 14: AUDUSD. Variance swap returns (RP), variance swap rates (KV), realized variance (RV), the spot exchange rate and histogram of RP. In the shaded area the approximation based on the ATM volatility is used.

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