Determining Margin Levels using Risk Modelling

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Abstract

This thesis addresses the problem of how a broker should to set margin levels on individual stock or subportfolios using risk modelling. Expected Shortfall is used together with historical data to set margin levels on individual stock. Another method is then derived with the Euler allocation principle to take dependencies into account. A basic comparison is made that shows how a broker can increase the revenue from lending money by taking dependencies into account and set margin levels on subportfolios instead of individual stock.

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Contents

1 Introduction

Using securities as collateral to get credit from a broker is called *margin lending* or *buying on margin* and has been widely used for decades. Margin levels state how much of a security that can be used as credit and range from 0% to close to 100% depending on how risky the security is.

In general, supervising authorities give recommendations or limits for margin levels. In Sweden, *Bankföreningen* gives general recommendations for margin levels but there is no information on how these recommendations are derived [7]. However, since margin levels is a quantitative measure that in some sense should reflect the riskiness of a security, one should be able to use risk modelling to extract those quantitative measures instead of depending on recommendations and regulations. There have been two theses addressing margin levels in Sweden, both [2] and [6] emphasise the importance of both volatility and liquidity, but do not present a reproducible model. The aim of this thesis is to construct a model that can be used by a broker to set margin levels so that the clients/investors can use margin lending.

Since margin levels can be seen as risk measures, brokers should set them so that the brokers risk is acceptable. Acceptable means that the bank is willing and can reserve the buffer capital needed to withstand losses from its operation. In practise however, margin levels are set out of experience, in growing markets margin levels increase since the credit losses as a result of margin buying decrease. Before 1929 buying on margin was widely used on the New York Stock Exchange. After the crash in 1929 the Federal Reserve introduced the *regulation T*, which resulted in margin levels being regulated to a maximum of 50%. Margin levels were regulated not because of brokers risk taking but because high margin levels were thought to have large effects on market volatility. In general, every stock market crash has raised the question whether margin levels were set too high [8].

The margin levels are set by the broker, the riskier the asset the lower the margin level of that asset. Two important variables taken into account when setting margin levels are liquidity and volatility. Liquidity because the broker has to be able to sell off the assets in order to get the given credit back. The volatility is important since the broker has to sell off the assets before the value of the assets drops below the outstanding credit.

Assume that a broker wants to offer margin lending to its customers to charge interest from the outstanding credit. That credit will of course mean that the broker is exposed to credit risk with risky assets as collateral. The problem is to set margin levels such that the total risk is acceptable while maximizing the outstanding credit. The total risk should meet some risk preference set by the broker. In this case the risk preference is that losses generated from buying on margin should not exceed a certain percentage of the outstanding credit. Having a risk preference as a fraction of the outstanding credit is quite natural since interest is also a fraction of the outstanding credit.

Margin levels are useful tools regulating the risk taken by the broker but also monitoring the risk taken by individual investors. Setting margin levels for individual stocks is arguably not the best approach out of a risk perspective since the whole portfolio for each investor is used as collateral. A more natural approach is to set margin levels for each investor's portfolio to reflect the riskiness of that same portfolio. It is however market standard to set margin levels for individual stocks and because of that both margin levels for individual stock and subportfolios will be considered in this report.

This report focuses on how to allocate the total risk taken by a broker on the securities held as collateral. To do that the Euler allocation principle is used with Expected Shortfall as the risk measure. It turns out that the combination of Euler allocation and Expected Shortfall is straightforward if using Monte Carlo simulation.

First of all, any model has to fulfil the risk preference of the broker. Second, the model should increase the outstanding credit while meeting the risk preference, in other words the broker wants to increase incomes while keeping risks on an acceptable level. Also, the model should take *concentration risks* into account, an investor that contributes with a lot of risk should be easily identified. This is more or less how to lend money in a more fair way where people who take higher risks either pay higher interest or are allowed to borrow less money.

Initially, buying on margin will be explained in detail before a short walk-through on basic risk modelling used in this thesis and the Euler allocation. After that the method of using risk modelling to calculate the risk taken by the broker and how to set margin levels will be presented with two different approaches. One simple model without allocation will be compared to a more complex model using the Euler allocation principle. We will see that the simple model has flaws but should not be disregarded. Using a model with Euler allocation is more fair and natural, but the complexity of the model might hinder implementation.

2 Buying on margin

Buying securities on margin is to borrow money from a broker to buy securities while having other securities as collateral. This gives investors the possibility to buy more securities than normal and in that way increase profits (or losses). Since securities are risky assets one cannot borrow the full value of the securities used as collateral. Margin levels state how much of the collateral that can be used to buy more securities and range from 0% to levels close to 100%.

Example 1.1 Suppose an investor has invested 1 SEK in Stock A that has a margin level of 80% and want to buy more on margin. The credit available to the investor is $1 \cdot 0.8$ SEK $= 0.8$ SEK which the investor buys more shares for. The new shares can be used as collateral to buy even more shares and the available credit is now $1 \cdot 0.8 \cdot 0.8$ SEK $= 0.64$ SEK. The investor can always buy whatever security he wants but the available credit will be different with respect to the margin level.

If the investor chooses to buy the same security however the value of the total position is a geometric sum, $\sum_{i=0}^{n} a \cdot \beta^i$, where a is the initial capital, β is the margin level of the stock and n is the number of purchases. If $n \to \infty$ the geometric sum equals $\sum_{i=0}^{\infty} a \cdot \beta^i = \frac{a}{1-\beta}$ which is the maximum value of the position when buying one security on margin. This results in a multiplier effect since the total value of the position can be as large as $\frac{1}{1-\beta}$ times the initial capital. Since the maximum value of the position can easily be calculated some brokers allow investors to take any position up to the maximum allowed in the first purchase.

Example 1.2 With an initial capital of 100 SEK an investor can take a position of $\frac{100 \text{ SEK}}{1-0.8} = 500$ SEK in stock A, the 80% margin level results in a multiplier effect of 5. Since the initial capital was 100 SEK the borrowed amount from the broker is 400 SEK. Notice that the ratio between the borrowed amount and value of the total position is $\frac{400 \text{ SEK}}{500 \text{ SEK}} = 0.8$ which is exactly the margin level. As a result one can say that the margin level states how much of the total value of the position that can be bought with borrowed money. The amount of capital (in this case 20% of the total position) that the investor has to have in the position is called the *margin requirement*.

The investor now has a larger position than the initial capital which means that profits and losses will be greater than an investment made with only the initial capital. Suppose the share price goes up 20%, the value of the position is now 600 SEK. If the investor sells all shares for the value of 600 SEK and the broker gets back the 400 SEK credit, the investor now has a capital of 200 SEK. With an initial capital of 100 SEK the investors capital has increased 100% to 200 SEK. The 20% return on the stock price resulted in a $5 \cdot 20\% = 100\%$ profit for the investor as a result of the multiplier effect. On the contrary if the share price drops 20% the value of the position is 400 SEK which is equal to the amount borrowed initially and the investor has lost the whole initial capital. If the share price drops more than 20% the investor will have a debt to the broker since all initial capital is gone.

Since the broker faces the risk of having an outstanding credit without any collateral the investor has to open a margin account which the broker has the right to take control over if the margin requirement is not met. If there is not enough capital in the margin account to meet the margin requirement the investor gets a *margin call*. The investor can either make a cash deposit, sell securities or deposit other securities to meet the margin requirement. With every margin account there is a credit limit set by the broker to assure that risks taken by individual investors are limited. Each investor is credit rated and gets a credit limit that states the maximum amount that the investor can borrow from the broker. Hence, even if the investor has a capital of 100

SEK and can take a position of 500 SEK a credit limit of 100 SEK would limit him to only take a 200 SEK position. In other words the total value of the position can not exceed the sum of the capital and credit limit.

Example 1.3 Suppose the investor with the initial capital of 100 SEK takes a position of 500 SEK in stock A and the share price drops 10%. The value of the position is now 450 SEK and the margin level states that a maximum of 80% of the position can be bought with borrowed money. The credit used by the investor is still 400 SEK but the maximum allowed credit is now $450 \cdot 0.8$ SEK = 360 SEK, the investor is borrowing 400 SEK − 360 SEK = 40 SEK too much and gets a margin call of 40 SEK. The investor can deposit 40 SEK in cash to reduce the taken credit to 360 SEK, sell securities or deposit more securities.

Selling shares to meet the margin requirement does not mean selling shares for the value of 40 SEK (remember that 80% of those shares are bought on margin). Only 20% of the sold value is the investor's own capital, which means that the investor has to sell shares for the value of $\frac{40 \text{ SEK}}{0.2} = 200$ SEK to meet the margin requirement. The value of the position decreases to 250 SEK. The drop in share price resulted in a 50 SEK loss which leaves the investor with 50 SEK in remaining capital. With 50 SEK in capital the allowed position is $5 \cdot 50$ SEK = 250 SEK and is consistent with the previous calculation. In both cases the investor has lost $5 \cdot 10\% = 50\%$ of the initial capital.

If the investor decides to deposit securities to meet the margin requirement then it is not the value of the securities that should equal the 40 SEK. The investor has to deposit securities such that the available credit from the deposited securities equal 40 SEK. So if the investor wants to deposit securities that have a margin level of 50% then the value of the position should be $\frac{40 \text{ SEK}}{0.5} = 80$ SEK.

Example 1.4 Suppose an investor has 100 SEK of initial capital and takes a position of 200 SEK in stock A. The investor now wants to invest in stock B which has a margin level of 60%. The 200 SEK position in stock A with margin level 80% allows the investor to take a $200 \cdot 0.8$ SEK = 160 credit, since 100 SEK in credit was used to buy the position in stock A there is a net credit of 60 SEK available for the investor. The investor takes the maximum position possible in stock B which is $\frac{60 \text{ SEK}}{1-0.6} = 150$ SEK. The investor is free to take any position in any stock as long as the margin requirement is met. Now what is the margin requirement for the portfolio of stock A and B? The ratio between the borrowed amount and value of the positions, $\frac{250 \text{ SEK}}{350 \text{ SEK}} \approx 0.71$ has to be less or equal to the weighted average of the margin levels, $\frac{200 \text{ SEK}}{350 \text{ SEK}} \cdot 0.8 + \frac{150 \text{ SEK}}{350 \text{ SEK}} \cdot 0.6 \approx 0.71$

The problem is to relate margin levels with the risk taken when using risky assets as collateral. Brokers are well aware of that individual stock liquidity and volatility are two important variables. The portfolio of securities held by the broker as collateral is one risk that could/should be regulated using margin levels. Also the credit limits of individual investors margin accounts and margin levels affect the risks taken by individual investors. Even if the total portfolio of collateral held by the broker is diversified, individual investors might not have diversified portfolios. This results in a higher risk for individual investors defaulting.

Example 2.1 Suppose three investors have margin accounts at the same broker. They all have 100 SEK which they use to take maximum positions. There are only three stocks available on the market, stock A, B and C which have the same margin level. In the tables below show two different situations resulting in the same portfolio of collateral being held by the broker.

Table 1: Situation 1

Situation 1	weight stock A	weight stock B	weight stock C
Investor 1	1/3	1/3	1/3
Investor 2	1/3	1/3	1/3
Investor 3	1/3	1/3	1/3
Portfolio of collateral	1/3	1/3	1/3

Table 2: Situation 2

Situation 2	weight stock A	weight stock B	weight stock C
Investor 1	1	0	0
Investor 2	0	1	0
Investor 3	0	0	1
Portfolio of collateral	1/3	1/3	1/3

The two situations are not equally risky. Suppose stock A decreases in value, then in the first situation then all three investors would get margin calls. If the investors can not pay the margin call the broker can always sell the positions in stock B and C to meet the margin requirements. In the second situation the whole margin call affects one investor and the broker has no other assets than stock A to sell off to meet the margin requirement. The example demonstrates that one can not just consider the total portfolio. Since the total portfolio risk is the aggregated risk from the subportfolios one should consider the risk contributed by each subportfolio. That risk will vary depending on how large the diversification effects are.

3 Problem statement

The aim of this thesis is to find a way to set margin levels given some conditions set by the broker:

- The risk taken by the broker by offering credit should not exceed a certain percentage of the outstanding credit.
- Given that the risk preference is met, margin levels should be set such that the outstanding credit is maximized, leading to higher revenue.
- Given that the risk preference is met individual investors should not exceed a certain risk, meaning that there should be no concentration risk which could be measured as risk over credit taken for each investor.
- Since the margin level regulates how much the share price can fall before the investor gets a debt to the broker, the margin level can be interpreted as a measure of risk where volatility and liquidity plays an important role.
- These conditions should be met at every given time and for a long period of time.

4 Theory

A broker usually holds a total portfolio of collateral which consists of subportfolios held by individual investors. Subportfolios consist of different types of securities but only stocks will be considered in this report. In this section, notation and methods needed for the analysis will be introduced.

4.1 Portfolio and subportfolios

Consider a broker offering investors to buy securities on margin. The broker will have a number of investors (subportfolios) investing in the securities available on the market. Let $w_{i,k}$ denote the value of the position in asset i in the k:th subportfolio. Then if there are n assets available on the market, the value of subportfolio k is

$$\mathrm{S}_k = \sum_{i=1}^{n} w_{i,k} \tag{1}$$

This is simply the sum of the positions in the k:th subportfolio. Consider the case when setting margin levels for each individual stock and let β_i be the margin level of asset i. Then the available credit in the k:th portfolio is

$$\mathrm{C}_k = \sum_{i=1}^{n} \beta_i w_{i,k} \tag{2}$$

Each position is multiplied with its corresponding margin level to get the available credit. However, in practise the available credit is limited by the credit limit which is based on the individual investor. Credit limits will not be considered in the model.

If margin levels are set for subportfolios however, then the available credit for subportfolio k is

$$\mathrm{K}_k = \beta^k \mathrm{S}_k \tag{3}$$

where β^k is the margin level of the k:th subportfolio. It is clear that if margin levels are set for individual stocks then the available credit for a subportfolio is just the weighted average of the available credit for each stock as opposed to the portfolio margin level which is based on the characteristics of the subportfolio.

Looking at the total portfolio, the total position in asset i is $\mathrm{Y}_i = \sum_{k=1}^{m} w_{i,k}$. This is just adding the values of all positions in asset i held by all subportfolios together. Keeping track of the total position in an asset might be important because of the liquidity risk. Holding positions that are larger than the turnover for that asset over a certain time period contributes to the risk and should be considered.

4.2 Loss contributions

Since different subportfolios belong to different investors, profits from one subportfolio does not cancel the loss from another subportfolio. That is why only the losses, not profits, for each subportfolio will be considered. So, losses will be considered as positive numbers and profits as negative, i.e. L= 45 SEK is a loss of 45 SEK. Since an investor gets a debt to the broker if the value of the portfolio is less than the credit taken by the investor, this can be considered as a loss.

Suppose that all subportfolios consist of maximum positions. Then the loss over one time period for portfolio k is $\mathrm{C}_k - \mathrm{S}_k$ when having individual stock margin levels. Again, this is when the loan to the broker exceeds the market value of the collateral. Since a portfolio contributes to the loss only when $\mathrm{C}_k - \mathrm{S}_k > 0$, the loss contribution of portfolio k to the total loss is

$$\mathrm{L}_k = \max(0, \mathrm{C}_k - \mathrm{S}_k) \tag{4}$$

Respectively, if the margin levels are set for subportfolios we get that the loss contribution of subportfolio k is

$$\mathrm{L}_k = \max(0, \mathrm{K}_k - \mathrm{S}_k) \tag{5}$$

The total loss to the broker over one time period is clearly $\mathrm{L} = \sum_{k=0}^{m} \mathrm{L}_k$ independently from how margin levels are set. Note that the available credit can be regulated and affects the probability of a subportfolio generating a loss.

4.3 Risk measures

4.3.1 Properties

To determine the economic capital or buffer capital that a broker should set aside holding a portfolio of collateral, one needs to measure the risk of the portfolio. A regulator, bank or broker decides what risks are acceptable by using a risk measure. Such a risk measure should satisfy some natural properties to be a useful tool when managing risk. Introducing a risk measure ρ then $\rho(\mathrm{L})$ is the buffer capital that a broker should set aside to manage losses generated from the position L. In this case L is the portfolio wide losses generated from the subportfolios and L_k is the loss contribution from the k:th subportfolio or stock depending on how the margin levels are set.

Translation property. The translation property of a risk measure can be written as

$$\rho(\mathrm{L} - \alpha) = \rho(\mathrm{L}) - \alpha \ , \ \alpha \in \mathbf{R} \tag{6}$$

where α is a risk free asset. This means that adding a risk free asset to a risky position decreases the risk with the same amount. It is natural that adding an amount equal to the buffer capital to the position, makes that position acceptable, i.e $\rho(\mathrm{L} - \rho(\mathrm{L})) = \rho(\mathrm{L}) - \rho(\mathrm{L}) = 0$.

Example 3.2 Suppose a position L requires a buffer capital of $\rho(\mathrm{L}) = 100$ SEK, adding that amount of cash to the position L should mean that the position is acceptable. Adding an amount 100 SEK to L results in the position (L − 100 SEK). Since adding a risk free asset such as cash should decrease the risk with the same amount we get that $\rho(\mathrm{L} - 100 \text{ SEK}) = \rho(\mathrm{L}) - 100$ SEK. This results in $\rho(\mathrm{L} - 100 \text{ SEK}) = 0$ since we already know that $\rho(\mathrm{L}) = 100$ SEK. Moreover a position L which results in $\rho(\mathrm{L}) = 0$ is considered acceptable. Remember that the choice of risk measure is made by a regulator or broker and simply states the risk preference that is considered acceptable.

Monotonicity property. If two portfolios, L_1 and L_2, consist of positions that result in losses generated from the first portfolio are always less than the losses from the second one, i.e. $\mathrm{L}_1 \leq \mathrm{L}_2$. Then the buffer capital for the first portfolio, $\rho(\mathrm{L}_1)$, should always be less than the buffer capital for the second portfolio $\rho(\mathrm{L}_2)$. In other words, choosing a less risky position should result in less buffer capital needed to be set aside. This property is called monotonicity and can be written as

$$\mathrm{L}_1 \leq \mathrm{L}_2 \Rightarrow \rho(\mathrm{L}_1) \leq \rho(\mathrm{L}_2) \tag{7}$$

Positive homogeneity property. Another property that should be fulfilled is that if one doubles a position then the buffer capital should double as well. i.e.

$$\rho(\lambda \mathrm{L}) = \lambda \rho(\mathrm{L}) \; , \; \lambda \geq 0 \tag{8}$$

This property holds in markets that are perfectly liquid. However, such an assumption is very strong and will be discussed later on.

Subadditivity property. The last property that should be fulfilled to have a useful measure to manage risk is the subadditivity property which says that diversification should be rewarded. This can be written as

$$\rho(\mathrm{L}_1) + \rho(\mathrm{L}_2) \geq \rho(\mathrm{L}_1 + \mathrm{L}_2) \tag{9}$$

This means that the sum of two stand alone risk for two assets is always greater or equal to the risk of the sum of the two assets. In practise this means that the joint risk is less since both positions will not experience their worst losses in the same time period if they are not perfectly correlated.

A risk measure that fulfils the translation-, monotonicity-, positive homogeneity- and subadditivity property is a coherent risk measure and thus a useful tool for managing risk.

4.3.2 Value at Risk

A risk measure that is widely used is Value at Risk(VaR). The VaR of a position L with a probability α can be written as $\mathrm{VaR}_\alpha(\mathrm{L})$. Now VaR states the buffer capital that should be set aside to cover losses from L with probability α, i.e. L will be smaller than $\mathrm{VaR}_\alpha(\mathrm{L})$ in α of the cases. That means that with the probability $1-\alpha$, losses will exceed the buffer capital. This results in α usually being close to 1 since only very unlikely events should cause losses being greater than the buffer capital set aside.

Another way to describe VaR is that $\mathrm{VaR}_\alpha(\mathrm{L})$ is the smallest loss l such that the probability of L being greater than l is $1-\alpha$. The mathematical definition of VaR can be written as

$$\mathrm{VaR}_\alpha(\mathrm{L}) = \min[l \in \mathbf{R} \; ; \; \mathrm{P}(L \geq l) \leq 1-\alpha] \tag{10}$$

Knowing the distribution of L one can plot the density function and represent $\mathrm{VaR}_\alpha(\mathrm{L})$ as

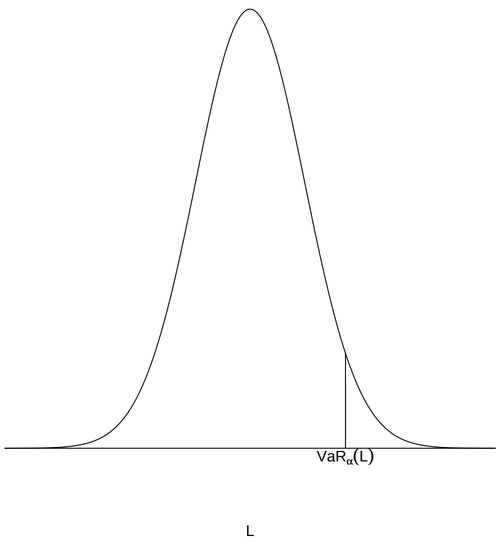

Figure 1: Value at Risk represented with the density function of L

the point where the mass(probability) to the right of $\text{VaR}_\alpha(\text{L})$ is $1-\alpha$.

VaR fulfils all properties of a coherent risk measure except the subadditivity property and is for some cases not coherent. In some cases VaR does not reward diversification. However, for some distributions, such as the normal distribution, VaR fulfils the subadditivity property and is thus a coherent risk measure.

Another drawback is that VaR does not give any information of how losses are distributed when among the $1-\alpha$ worst cases. Are losses greater than $\text{VaR}_\alpha(\text{L})$ concentrated close to $\text{VaR}_\alpha(\text{L})$ or spread out? If high losses are spread out this leads to the tail in the density function being larger. In other words the tail has more mass distributed for large losses and is thus called a heavy-tailed distribution.

4.3.3 Expected Shortfall

Expected Shortfall(ES) is a coherent risk measure and can be written as $\text{ES}_\alpha(\text{L})$ where L is a loss generated from some position and α is a probability. ES states the expected loss given that the loss is among the $1-\alpha$ worst losses. This is a very similar formulation to VaR with the exception that we now consider the expected loss and not smallest lost of the worst $1-\alpha$ cases. This means we can define ES with VaR since $\text{ES}_\alpha(\text{L})$ is the expected loss given that the loss is greater than $\text{VaR}_\alpha(\text{L})$. This can be written as

$$\text{ES}_\alpha(\text{L}) = \text{E}[\text{ L} \mid \text{L} \geq \text{VaR}_\alpha(\text{L})] \qquad (11)$$

This means that ES considers all losses greater than $\text{VaR}_\alpha(\text{L})$ and will better represent the properties of the tail than VaR.

This particular definition of ES is coherent under some smoothness conditions mentioned in [9] and presented in detail in [11]. More on coherent risk measures and their behaviour on data can be read in [3].

4.4 Risk contributions and allocation

The buffer capital for the aggregate losses is $\rho(\mathrm{L})$ and if ρ is subadditive we have that

$$\rho(\mathrm{L}) \leq \rho(\mathrm{L}_1) + ... + \rho(\mathrm{L}_m) \tag{12}$$

Now if the broker set aside the buffer capital $\rho(\mathrm{L}_k)$ to the corresponding subportfolio, the total buffer capital would exceed the buffer capital of the aggregate losses. The buffer capital from the aggregate risk should be allocated to the different subportfolios in a fair way. One way of doing this is to find the risk contribution of each subportfolio to the total portfolio of collateral. Then what is the risk contribution of portfolio k given that the total buffer capital is $\rho(\mathrm{L})$?

Let $\rho(\mathrm{L}_k|\mathrm{L})$ be the risk contribution of subportfolio k to $\rho(\mathrm{L})$. Then a natural requirement which follows from the argument above is that the risk contributions add up to the total risk $\rho(\mathrm{L})$, i.e.

$$\rho(\mathrm{L}) = \sum_{k=1}^{m} \rho(\mathrm{L}_k|\ \mathrm{L}) \tag{13}$$

needs to be satisfied. Also a subportfolio should be rewarded for contributing to the diversification, which means that the risk contribution of a subportfolio should be less than the risk of the subportfolio, i.e. $\rho(\mathrm{L}_k|\ \mathrm{L}) \leq \rho(\mathrm{L}_k)$. In other words, a broker should attract and reward subportfolios that contribute to the diversification.

One allocation principle that fulfils the properties above is the Euler allocation principle. It has its origin in a theorem by Euler stating that

$$f(\boldsymbol{x}) = \sum_{k=1}^{m} x_k \frac{\partial f}{\partial x_k}(\boldsymbol{x}) \tag{14}$$

if f is differentiable and positively homogeneous of order one. Now assume that $f(\boldsymbol{x}) = \rho(\mathrm{L}(\boldsymbol{x}))$ where $\mathrm{L}(\boldsymbol{x}) = x_1\mathrm{L}_1 + ... + x_m\mathrm{L}_m$ and ρ fulfils the properties from the Euler theorem then

$$\rho(\mathrm{L}(\boldsymbol{x})) = \sum_{k=1}^{m} x_k \frac{\partial \rho}{\partial x_k}\mathrm{L}(\boldsymbol{x}) \tag{15}$$

Note that $\mathrm{L}(\mathbf{1}) = \mathrm{L}$ and the relation above can be written as

$$\rho(\mathrm{L}) = \sum_{k=1}^{m} \frac{\partial \rho}{\partial x_k}\mathrm{L}(\boldsymbol{x})\Big|_{\boldsymbol{x}=\mathbf{1}} \tag{16}$$

which gives us the allocation principle where

$$\rho(\mathrm{L}_k|\ \mathrm{L}) = \frac{\partial \rho}{\partial x_k}\mathrm{L}(\boldsymbol{x})\Big|_{\boldsymbol{x}=\mathbf{1}} \tag{17}$$

Remember that $\rho(\mathrm{L})$ has to fulfil the positive homogeneity property (the assumption of perfectly liquid markets) for this to work. Now choosing a coherent risk measure we are sure that the property holds mathematically, however the assumption that the markets are perfectly liquid has to be considered.

If we choose VaR as our risk measure, i.e. $\rho(\mathrm{L}) = \mathrm{VaR}_\alpha(\mathrm{L})$ then the formula for the risk contribution from portfolio k is

$$\rho(\mathrm{L}_k|\mathrm{L}) = \mathrm{E}(\mathrm{L}_k|\ \mathrm{L} = \mathrm{VaR}_\alpha(\mathrm{L})) \tag{18}$$

In words, the risk contribution from portfolio k is the expected loss of that portfolio given that the total loss is $\mathrm{VaR}_\alpha(\mathrm{L})$. So those portfolios that contribute less to the total loss will have smaller risk contributions and vice versa. Note that the contributions are given when the loss is equal to $\mathrm{VaR}_\alpha(\mathrm{L})$, there is no consideration of which portfolios that contribute when the total loss is greater than $\mathrm{VaR}_\alpha(\mathrm{L})$. This drawback is inherited from VaR as a risk measure, the characteristics of the loss distribution for losses greater than $\mathrm{VaR}_\alpha(\mathrm{L})$ are not taken into account.

In section 3.3 we saw that ES does not have the drawbacks of VaR. Using ES as the risk measure and Euler allocation the risk contribution can be written as

$$\rho(\mathrm{L}_k|\mathrm{L}) = \mathrm{E}(\mathrm{L}_k|\ \mathrm{L} \geq \mathrm{VaR}_\alpha(\mathrm{L})) \tag{19}$$

In other words the capital allocated to the portfolios is what loss they contribute with on average when the total loss is greater or equal to $\mathrm{VaR}_\alpha(\mathrm{L})$. Each subportfolio gets allocated the risk contribution that cover the losses that they are expected to contribute with in the $1-\alpha$ worst outcomes.

This theory is presented in detail in [9] and [10]. [9] is strongly recommended for further reading about the Euler allocation principle whereas [10] is slightly more technical and general.

5 Method

5.1 Liquidity Risk / Time Period

Liquidity risk is the risk associated with the market not being able to trade large volumes of shares fast enough. In this case the liquidity risk is holding a position which cannot be sold off over one time period. For instance if risks are calculated for one day time periods and a broker wants to sell a position that cannot be traded in one day but rather 30 days, then the one day risk does not represent the actual risk in the position. However, changing the time period for which the risk is calculated for to 30 days, the broker can sell the position over one time period. If the risk is how much the share price can fall over one time period, one should remember that fractions of the position is sold every day at different prices and the average price will lie between the share price at the beginning and the end of the time period. This is an overestimation of the risk but an assumption that simplifies the problems with liquidity risks.

Assuming that the time period for each stock is chosen such that the total position in that stock can be liquidated in one time period. Then of course the time period will be a measure of the liquidity of the position and stocks will have different time periods associated with them. This results in a problem, if there are different time periods for different stocks how is the risk of a portfolio of more than one stock calculated? The dependence between the stocks should be included since there should be some diversification effect. But can one find a dependence between time periods that are different?

Consider a portfolio of two stocks where the position in stock A can be liquidated in three days and a position in stock B which can be liquidated in ten days. Again the risk we want to compute is for the time period for which we can liquidate the assets. Assume that the broker starts selling of the assets at day one and for the first three days both stocks will be sold simultaneous. On day four there will be no shares of stock A to sell off since the position could be liquidated in three days. Stock B however is not liquidated yet and the shares will be sold until the tenth day when the last shares are sold. Notice that after day 3 stock A does not contribute to the risk since the position is sold off. Since stock A does not contribute to the risk we do not consider the change in share price for stock A after day 3.

If y_t and z_t are the log returns on day t for stock A and B respectively then the table below represents the process of selling the assets in stock A and B. As you can see the time period for which the assets are liquidated is as long as the longest time period of the stocks in the portfolio. All other stocks will only contribute during their individual time periods.

Table 3: Daily log returns over a liquidation period

Stock/Day	1	2	3	4	5	6	7	8	9	10
A	y_1	y_2	y_3	0	0	0	0	0	0	0
B	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}

With this approach one can calculate the portfolio risk while taking both liquidity and dependence into account. Using simulation to calculate the risk with some risk measure one can simply simulate liquidation periods for a portfolio. To simulate the example given in the table above one can simulate log returns from the joint distribution for day one to ten and then set all y_t to 0 after day 3. With a large number of stocks, setting log returns to 0 will be a waste of computation time but, on the other hand, the method is very easily implemented.

There are many different ways to take liquidity risks into account, [5] does is it by introducing a stochastic supply curve which leads to a *liquidity cost*. The method requires an estimation of that supply curve which would add unnecessary complexity to this thesis. Another approach is presented in [1] were a value function of the position is introduced. That method uses *Convex Risk Measures* as opposed to *Coherent Risk Measures* which do not fulfil the properties needed to apply Euler allocation. The article also emphasises the lack of consensus market practise on how to manage liquidity risks.

5.2 Simple model

If one only considers individual stocks and apply margin levels according to individual stock characteristics, then it is natural that the buffer capital given by a risk measure should set the margin requirement for each individual stock. Remember that the margin requirement was the brokers buffer for selling off the assets before the loaned amount exceeds the market value of the collateral and the buffer capital given by ES states how much the value will decrease on average in the $1-\alpha$ worst cases. Setting the margin requirement equal to the buffer capital (given in percent) results in the broker being able to sell of the assets before the loaned amount exceeds the market value of the collateral in most of the cases. The methods used in this section, among other methods for calculating risk, are presented in [4] which provides theory and examples as well as algorithms.

Given some position in some stock were $d \in \mathbb{N}$ is the liquidation period given in days and X is the stochastic variable stating the return of the stock over the liquidation period d, i.e.

$$X = \frac{\mathrm{S}^d}{\mathrm{S}^0} - 1 \tag{20}$$

where S^t is the value of the position at time t, we get that the margin requirement should equal the buffer capital given by $\rho(-\mathrm{X})$, i.e.

$$(1-\beta) = \rho(-\mathrm{X}) \;\Rightarrow\; \beta = 1 - \rho(-\mathrm{X}) \tag{21}$$

where β is the margin level of the stock.

Example 4.1 Suppose stock A has a liquidation period of 5 days and the 5 day return, X is defined as above, i.e. $X = \frac{\mathrm{S}^5}{\mathrm{S}^0} - 1$. Furthermore, the buffer capital for a worst case scenario given by some risk measure ρ is calculated for stock A and results in $\rho(-X) = 34\%$. That means that the position can decrease with 34% for that worst case scenario and the margin level for that stock should be $\beta = 100\% - 34\% = 66\%$.

For a broker holding a portfolio of collateral this results in one margin level for each stock (remember that the total market value in a stock held by the broker as collateral is Y_i). That market value together with the daily turnover for that stock gives us the liquidation period $d_i \in \mathbb{N}$ for stock i, i.e

$$\left\lceil \frac{\mathrm{Y}_i}{\text{Daily turnover in stock } i} \right\rceil = d_i \tag{22}$$

where the result is rounded upwards to get whole days. The corresponding stochastic variable for the return over a time period of d_i is X_i for stock i and with the same reasoning as before, the margin level for stock i is written as

$$\beta_i = 1 - \rho(-\mathrm{X}_i) \tag{23}$$

Now this model only considers the risk in the stock and there is no reward for a investor to hold a well diversified subportfolio. Not even the total portfolio of collateral can affect margin levels by diversification since the model only considers individual stock characteristics. But there are some benefits of this simple model besides being simple and easily implemented.

The largest benefit with this model is that the problem of illiquid stocks can be solved very easily. Since every stock is analysed individually without respect to other stocks there is no consideration of dependencies. The assumption is that all stocks will experience their worst outcomes at the same time which can be seen as a worst case scenario. A worst case scenario that is not that far fetched, when a market crashes all stocks seem to experience their worst outcomes at the same time.

This model lacks any reward for diversification since the risk is calculated out of stand alone characteristics. Even if one calculates margin levels for individual stocks there should be some diversification effect from holding a diversified portfolio of collateral, meaning that the total risk taken by the broker is less than the sum of the risks in each stock. Since the model does not consider the composition or size of subportfolios it does not give any indication of high/low risk subportfolios.

5.2.1 Calculations

The log returns are extracted from a vector of historical closing prices. Assume the data set of closing prices consist of 250 data points in a vector, i.e.

$$\begin{bmatrix} s_1 & s_2 & s_3 & \dots & s_{249} & s_{250} \end{bmatrix} \tag{24}$$

where s_i is the closing price of day i where s_1 is the closing price one year ago (assuming 250 trading days in one year). To get the log returns of that set, the logarithm of is taken for each component and the difference between each day is calculated. This results in the vector of log returns,

$$\begin{bmatrix} \ln(s_2) - \ln(s_1) & \ln(s_3) - \ln(s_2) & \ln(s_4) - \ln(s_3) & \dots & \ln(s_{250}) - \ln(s_{249}) \end{bmatrix} \tag{25}$$

where $\ln(s_2) - \ln(s_1)$ is the log return of day 2 and so on. Notice that the vector now consists of 249 data points. Calculating the log returns instead of actual returns is to simplify calculations. There is no loss in information. From the set of log returns the estimates *expected value* $\hat{\mu}$ and *variance* $\hat{\sigma}^2$ are calculated. The daily log returns of all stocks are assumed to be independent identically t-distributed with three degrees of freedom,

$$Z_i \sim t_3(\hat{\mu}_i, \hat{\sigma}_i^2) \tag{26}$$

where Z_i is the stochastic daily log return of stock i and the estimates $\hat{\mu}_i$ & $\hat{\sigma}_i^2$ are extracted from the historical daily log returns of stock i. Now stock i has a liquidation period of d_i days and therefore d_i-day log returns need to be simulated. Since the daily log returns are independent identically distributed(IID) variables the d_i-day log return is simply the sum of d_i daily log returns, i.e.

$$u_{1,i} = z_{1,i} + z_{2,i} + z_{3,i} + z_{4,i} + \dots + z_{d_i,i} \tag{27}$$

where $u_{1,i}$ is a sample of a d_i-day log return of stock i. A large sample of N d_i-day log returns are simulated resulting in the vector

$$\begin{bmatrix} u_{1,i} & u_{2,i} & u_{3,i} & u_{4,i} & \dots & u_{N-1,i} & u_{N,i} \end{bmatrix} \tag{28}$$

This vector represents the distribution of the d_i-day log returns of stock i. Now converting log returns to returns by

$$x_{k,i} = (e^{u_{k,i}} - 1) \qquad \forall \quad k \tag{29}$$

where $x_{k,i}$ is the k:th sample return of stock i and $u_{k,i}$ the corresponding log return. Note that $x_{k,i}$ is a sample of X_i which is the stochastic return for stock i over a time period of d_i days. Now to calculate $\rho(-X_i)$ the worst $1-\alpha$ returns are extracted from the sample and the average return in that subset gives us $\rho(-X_i)$ which is the ES of the position Y_i.

Finally, given a position with market value Y_i in stock i with some daily turnover the margin level β_i is calculated by setting $\beta_i = 1 - \rho(-X_i)$ where $\rho(-X_i)$ is calculated as in the algorithm in this section.

5.3 Euler allocation model

The idea with using Euler allocation to determine margin levels is to take advantage of the dependences between securities on the market. This is done by determining the risk contribution instead of the stand alone risk of a subportfolio. With the same train of thought as in the simple model, instead of using the stand alone risk, we now use risk contributions to set margin levels. If every subportfolio gets a margin level given by the risk contribution then the margin levels are set with respect to dependences as well.

Similar to the simple model the margin requirement should equal the risk contribution (instead of stand alone risk) of the position, i.e.

$$\beta^k = 1 - \rho(-X_k| - \sum_{k=1}^{m} X_k) \tag{30}$$

In this way the risk preference is always met and since the risk contributions add up to the total risk the method takes dependences into account.

5.3.1 Calculations

The calculations to get the risk contributions are similar to the ones done in the previous section with the difference that dependences are included. Instead of simulating daily log returns for each stock separately the daily log returns are simulated from the joint distribution of all stocks on the market resulting in the vector of daily log returns

$$\boldsymbol{Z} = \begin{bmatrix} Z_1 & Z_2 & Z_3 & \dots & Z_n \end{bmatrix} \qquad \boldsymbol{Z} \sim t_3(\hat{\boldsymbol{\mu}}, \frac{\nu}{\nu-2}\hat{\Sigma}) \tag{31}$$

where Z_i is the daily log return of stock i, $\hat{\boldsymbol{\mu}}$ is the vector of estimated expected values $\hat{\boldsymbol{\mu}} = \begin{bmatrix} \hat{\mu}_1 & \hat{\mu}_2 & \hat{\mu}_3 & \dots & \hat{\mu}_n \end{bmatrix}$ and $\hat{\Sigma}$ is the estimated dispersion matrix of the joint distribution and can be written in terms of the estimated covariance matrix as

$$\widehat{\Sigma} = \frac{\nu - 2}{\nu} \cdot \widehat{\text{Cov}}(\boldsymbol{Z}) \quad \text{where} \quad \widehat{\text{Cov}}(\boldsymbol{Z}) = \begin{bmatrix} \hat{\sigma}_1^2 & \hat{\rho}_{2,1}\hat{\sigma}_2\hat{\sigma}_1 & \dots & \hat{\rho}_{n,1}\hat{\sigma}_n\hat{\sigma}_1 \\ \hat{\rho}_{2,1}\hat{\sigma}_1\hat{\sigma}_2 & \hat{\sigma}_2^2 & \dots & \hat{\rho}_{n,2}\hat{\sigma}_n\hat{\sigma}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\rho}_{1,n}\hat{\sigma}_1\hat{\sigma}_n & \dots & \dots & \hat{\sigma}_n^2 \end{bmatrix} \tag{32}$$

where ν is the degrees of freedom thus $\nu = 3$ here and $\widehat{\text{Cov}(\boldsymbol{Z})}$ is the estimated covariance matrix of $\boldsymbol{Z}$ which contains the correlations and standard deviations of all stocks. Here, $\rho_{i,j} = \rho_{j,i}$ and is the correlation of stock i and j which is estimated from the historical daily log returns of each pair of stocks.

To get samples of $\boldsymbol{Z}$ the following calculations are made. First n (the number of stocks on the market) independent samples from a standard normal distribution are simulated in the vector $\boldsymbol{V}$. Then to get a sample of $\boldsymbol{Z}$ the following operation is made

$$\boldsymbol{Z} = \hat{\boldsymbol{\mu}} + A\sqrt{\frac{\nu}{\chi_\nu^2}}\boldsymbol{V} \tag{33}$$

where $\hat{\boldsymbol{\mu}}$ is the vector of estimated expected values, $\boldsymbol{V}$ is the vector of independent $\text{N}(0,1)$ variables and χ_ν^2 is a random chi squared variate independent of $\boldsymbol{V}$. A is the Cholesky decomposition of the dispersion matrix Σ which can be written as

$$\Sigma = AA^T \tag{34}$$

Now $\boldsymbol{z}_j = \begin{bmatrix} z_{j,1} & z_{j,2} & z_{j,3} & \dots & z_{j,n} \end{bmatrix}$ is a sample of daily log returns $\boldsymbol{Z}$ for all stocks where the dependence is included. Same as before, the d_i-day log returns need to be calculated for each stock. Since d_i is different for each stock and all stocks are simulated for each sample of $\boldsymbol{Z}$ it is the largest liquidation period d_i which sets the limit of how many daily log returns that need to be calculated. Let $d_{max} = \max(d_1, d_2, \dots, d_n)$ be the limit, then $d_{\max}$ samples of $\boldsymbol{Z}$ are simulated resulting in the matrix

$$\begin{bmatrix} z_{1,1} & z_{2,1} & z_{3,1} & \dots & z_{d_{\max},1} \\ z_{1,2} & z_{2,2} & z_{3,2} & \dots & z_{d_{\max},2} \\ z_{1,3} & z_{2,3} & z_{3,3} & \dots & z_{d_{\max},3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{1,n} & z_{2,n} & z_{3,n} & \dots & z_{d_{\max},n} \end{bmatrix} \tag{35}$$

Now the d_i-day log return needs to be calculated. This is done by summing the first d_i daily log returns of row i for all rows in the matrix above. This results in the vector of d_i-day log returns

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{d_1} z_{k,1} \\ \sum_{k=1}^{d_2} z_{k,2} \\ \sum_{k=1}^{d_3} z_{k,3} \\ \vdots \\ \sum_{k=1}^{d_n} z_{k,n} \end{bmatrix} \tag{36}$$

Now $\boldsymbol{u}$ is a sample vector of log returns for each stock over the corresponding liquidation period d_i. Generating a large sample N of these d_i-day log returns results in the matrix

$$\begin{bmatrix} u_{1,1} & u_{2,1} & u_{3,1} & \dots & u_{N,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & \dots & u_{N,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{1,n} & u_{2,n} & u_{2,n} & \dots & u_{N,n} \end{bmatrix} \tag{37}$$

This matrix contains log returns where $u_{j,i}$ is the j:th sample of the i:th stocks d_i-day return. Converting log returns to returns follows the same formula (29) as before. Hence the following is done

$$m_{k,i} = (e^{u_{k,i}} - 1) \quad \forall \quad k, i \tag{38}$$

The matrix of returns is now

$$\boldsymbol{M} = \begin{bmatrix} m_{1,1} & m_{2,1} & m_{3,1} & \dots & m_{N,1} \\ m_{1,2} & m_{2,2} & m_{3,2} & \dots & m_{N,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{1,n} & m_{2,n} & m_{2,n} & \dots & m_{N,n} \end{bmatrix} \tag{39}$$

Remember that these are are the d_i-day returns for all the stocks in the market and not the subportfolios profits/losses. To get that we need to multiply the matrix $\boldsymbol{M}$ of stock returns with the matrix of investors positions $\boldsymbol{W}$ in each stock. Now $\boldsymbol{W}$ is a $m \times n$ matrix, m investors able to invest in n stocks and $\boldsymbol{M}$ is a $n \times N$ matrix where n is of course the number of stocks and N is the number of simulated samples. Multiplying those matrices result in a $m \times N$ matrix $\boldsymbol{J}$ containing the simulated profits/losses

$$\boldsymbol{J} = \boldsymbol{W} \times \boldsymbol{M} \tag{40}$$

where each row in $\boldsymbol{J}$ is the vector of simulated profits/losses of a subportfolio. To get the profits/losses of the total portfolio each column(subportfolio profits/losses) needs to be summed

$$J_l = \sum_{k=1}^{m} j_{l,k} \tag{41}$$

It is now from those total portfolio profits/losses we extract the worst $1-\alpha$ profits/losses and their corresponding column in $\boldsymbol{J}$. Lets call the the sub-matrix of $\boldsymbol{J}$ containing the worst $1-\alpha$ profit/loss of the total portfolio $\boldsymbol{J'}$. Each row k in $\boldsymbol{J'}$ represents the losses in subportfolio k given that the total portfolio loss $\sum_{k=1}^{m} j_{l,k}$ is among the worst $1-\alpha$ total portfolio losses. Notice that we are interested in returns and not losses and remember that the value of the k:th subportfolio is S_k. To get X_k, the d_i-day return of subportfolio k we divide each row k in $\boldsymbol{J'}$ its corresponding subportfolio value S_k to get samples of X_k

$$x_{l,k} = \frac{j_{l,k}}{\mathrm{S}_k} \quad \forall \quad l, k \quad \text{where} \quad j_{l,k} \in \boldsymbol{J'} \tag{42}$$

This sample represents the distribution $\boldsymbol{X} = \begin{bmatrix} X_1 & X_2 & X_3 & \dots & X_n \end{bmatrix}$ in the worst $1-\alpha$ portfolio losses and it is now possible to calculate

$$\rho(-X_k| - \sum_{k=1}^{n} X_k) = \mathrm{E}(-X_k| - \sum_{k=1}^{n} X_k \geq \mathrm{VaR}_\alpha(-\sum_{k=1}^{n} X_k)) \tag{43}$$

This is simply taking the average of each row k

$$\rho(-X_k| - \sum_{k=1}^{n} X_k) = \frac{\sum_{l=1}^{N(1-\alpha)} x_{l,k}}{N(1-\alpha)} \tag{44}$$

where $N(1-\alpha)$ is the number of samples in the worst $1-\alpha$. We have now calculated the risk contribution of each portfolio and all that is left is to set the margin level of portfolio k to

$$\beta^k = 1 - \rho(-X_k| - \sum_{k=1}^{m} X_k) \tag{45}$$

One should notice that the weight matrix $\boldsymbol{W}$ could be changed to a diagonal $n \times n$ matrix with the market value Y_i one the diagonal,$\mathrm{Y}_i = w_{i,i}$. Each subportfolio then consists of the total position in one stock and the margin level is thus calculated for individual stock but with dependencies included.

5.4 Aggregate risk

The requirement that, in the long run, losses should not exceed a fraction γ of the outstanding credit is difficult to model. The requirement must be met in the long rung but on a day to day basis as well. The problem is that there is no way of knowing how the total portfolio of collateral is going to be distributed among the investors. The requirement is therefore changed to a simple alternative. The new requirement is to be able to sell off the entire portfolio of collateral without any losses with a high probability, in this case ES with a confidence level of 99%. That means that at any given time point one should be able to sell off the total portfolio of collateral and only experience losses when losses exceed the average loss in the 1% worst events. To do this the aggregate risk $\rho(\mathrm{L})$ needs to be calculated. Remember that L is the loss that the broker experiences when the portfolio value decreases more than the margin requirement. The margin levels are set so that losses are limited but there is still some risk left which is represented by $\rho(\mathrm{L})$. This risk is adjusted by making margin levels more/less conservative by adjusting the confidence level α when setting margin levels. The confidence level for the brokers risk $\rho(\mathrm{L})$ is however fixed to 99%. ES is chosen as a risk measure and we state that if

$$\frac{\rho(\mathrm{L})}{\text{Outstanding credit}} \leq \gamma \tag{46}$$

the requirement is met. Now γ is set close to zero since one wants to have a low risk in relation to a large outstanding credit. The outstanding credit is a known variable and is the sum of all subportfolios outstanding credit.

Since there are two different models the calculations of the brokers risk $\rho(\mathrm{L})$ differs between the models. In the following two sections the calculations will be presented with the notation found in the calculations for the corresponding model.

5.4.1 Calculations: Aggregate risk for Simple model

Remember that margin levels in the Simple model are set such that the broker experiences no losses when the return in each stock is greater than the average return among the worst $1-\alpha$ returns. That implies that the broker does experience losses when the return is worse than the average return among the worst $1-\alpha$ cases, i.e

$$\mathrm{L}_k \geq 0 \quad \text{when} \quad -X_k \geq \rho(-X_k) = 1 - \beta_k \tag{47}$$

From this relation it is easy to see that the lower the margin level β_k is the larger $-X_k$ needs to be for the broker to experience losses from stock k. The distribution of L_k is easily extracted from the calculations already made in the Simple model. Remember that that a large sample of the distribution of X_k was simulated in the Simple model. That, together with the relation above makes the calculations easy. First the worst 1% returns are extracted from the sample of X_k. The loss L_k can then be written as

$$\mathrm{L}_k = \max[0, \mathrm{Y}_k\beta_k - \mathrm{Y}_k(1 + X_k)] \tag{48}$$

where $\mathrm{Y}_k\beta_k$ is the loan and $\mathrm{Y}_k(1 + X_k)$ is the market value of stock k after one liquidation period. This calculation is then made on the worst 1%. The risk $\rho(\mathrm{L}_k)$ is then calculated by taking the average of those L_k:s.

By the assumption that all stocks experience their worst outcomes at the same time, the total risk $\rho(\mathrm{L})$ is

$$\rho(\mathrm{L}) = \sum_{k=1}^{n} \rho(\mathrm{L}_k) \tag{49}$$

If the requirement in equation (45) is not fulfilled or is not close to γ then the confidence level α in the Simple model needs to be adjusted.

5.4.2 Calculations: Aggregate risk for Euler model

The aggregate risk in the Euler model is also easy to calculate since most calculations are already made for setting the margin levels. Similar to the Simple model the margin levels have been set such that the broker does not experience losses when the aggregate loss generated from the subportfolios is less than the average loss in the worst 1%. The profit/loss matrix of all the subportfolios are already simulated and ready in the matrix $\boldsymbol{J}$. In this matrix each row is the simulated profit/loss of a subportfolio and each column is a sample of a liquidation period for all subportfolios. To get the matrix of losses $\boldsymbol{L}$ which contains the loss contributions from each subportfolio we do the following

$$l_{j,k} = \max(0, \beta^k \mathrm{S}_k - (1 + j_{j,k})\mathrm{S}_k) \quad \forall \quad l, k \tag{50}$$

where $\beta^k \mathrm{S}_k$ is the loan of portfolio k and $(1 + j_{j,k})\mathrm{S}_k$ the market value of the collateral after one liquidation period. What remains is to extract the aggregate loss which is the column sum in $\boldsymbol{L}$, i.e

$$\sum_{k=1}^{m} l_{j,k} \tag{51}$$

and from that vector take the average of the worst 1%, which will equal $\rho(\mathrm{L})$.

6 Data analysis

Here, some of the stocks that will be used in the following comparison of the models will be presented. The stocks in the market are chosen because they represent different characteristics such as liquid/illiquid, high/low volatile stocks. Here, some of their characteristics will be presented. The data consists of closing prices from February 2008 to March 2009.

The Bio Invent stock will represent a stock where the position has a long liquidation period. The historical distribution for the daily log returns for Bio Invent is shown in figure 2 with a fitted normal and t3 distribution.

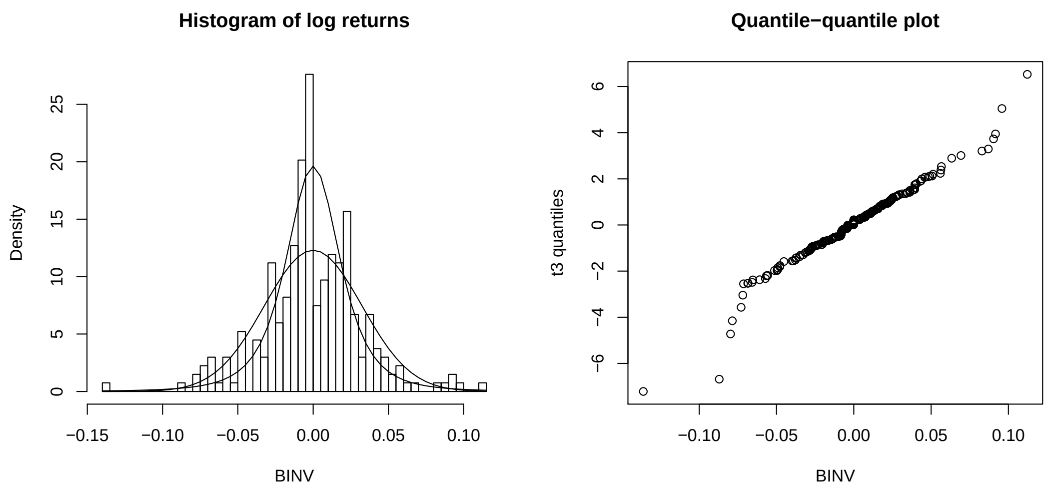

Figure 2: Histogram and QQ-plot of daily log returns for the Bio Invent stock

The t3 distribution seems to give a better fit than the normal distribution. From the QQ-plot it is seems that the empirical tail is lighter than that of the t3 distribution. This means that the risk is in some way overestimated. Since the same distribution is assumed for all stocks it is better to be a bit conservative and choose a distribution with a heavier tail.

A stock where the share price does not change every day is Concordia. This results in many log returns being equal to zero as one can see in figure 3.

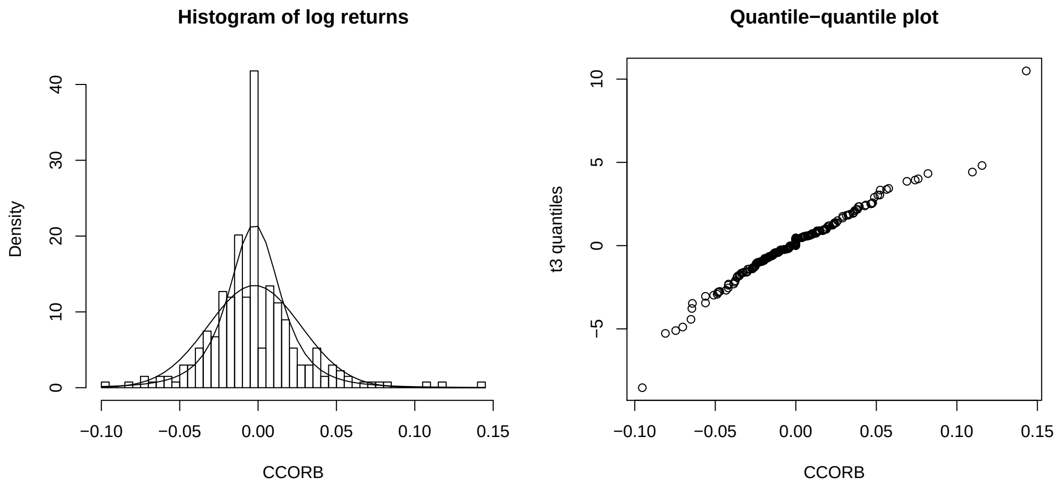

Figure 3: Histogram and QQ-plot of daily log returns for the Concordia stock

Again the t3 distribution seems to fit the historical distribution better. The many log returns equal to zero might be a result of the stock not being traded every day. That does not mean that the liquidity is low since we define liquidity as turnover related to the position held by the broker. Again, the QQ-plot shows that the t3 distribution gives a good fit to the historical distribution.

Another stock with similar characteristics is the BioPorto stock shown in figure 4.

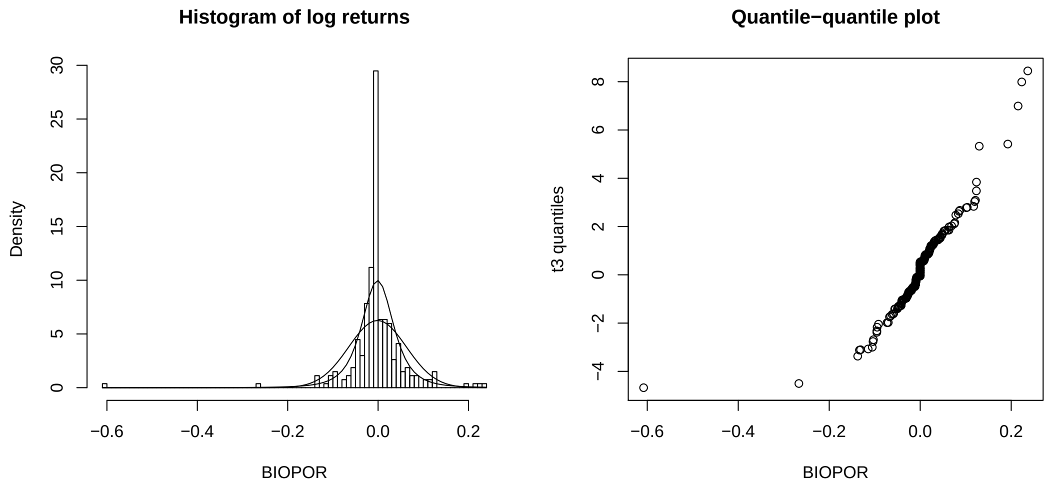

Figure 4: Histogram and QQ-plot of daily log returns for the BioPorto stock

Again, it is difficult to capture the historical distribution. Also, the distribution can change dramatically since stocks with low trading are prone to new information in the market. The most traded stocks seem to have more consistent historical distributions which makes them easier to deal with. Here, the QQ-plot shows that the t3 distribution is a good match if one disregards the worst two historical log returns. This is a good example of how difficult it is to find a good fitted distribution. Overall, the t3 distribution gives a good fit.

A stock which is relatively traded a lot is Nokia. The distribution of the historical daily log returns is shown in figure 5.

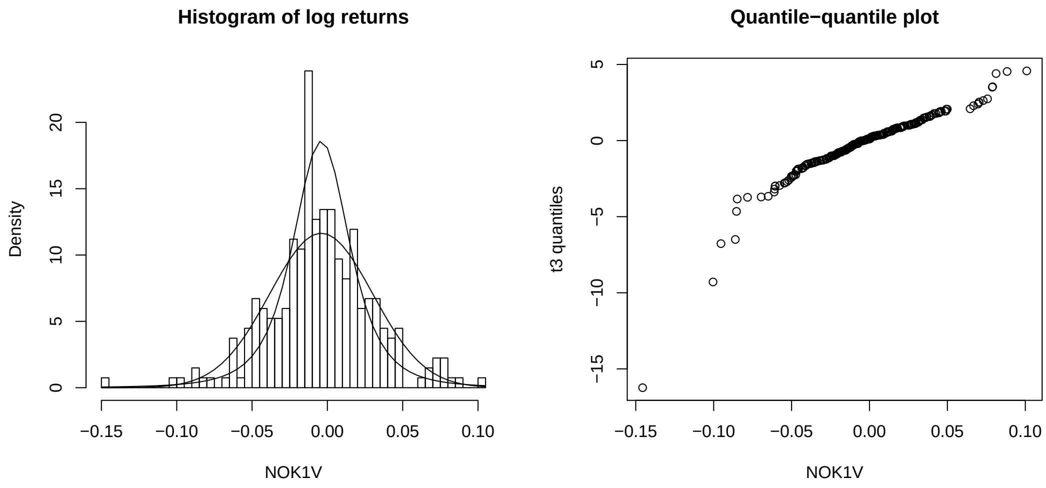

Figure 5: Histogram and QQ-plot of daily log returns for the Nokia stock

The t3 distribution seems somewhat heavier-tailed than the historical distribution. An unavoidable problem when assuming the same distribution to all stocks available. The argument for using the t3 distribution is simply to overestimate the risk rather than underestimate it by assuming a light tailed distribution.

7 Comparison and results

To show some examples of the models we choose some stocks that that will represent the stocks in our portfolio of collateral. The table below shows nine stocks traded in the Nordic market and their corresponding liquidation periods. The liquidation periods are set with the assumption that 100 SEK can be sold in that stock over the liquidation period. The comparison will be based on one year of data, i.e. about 250 data points for each stock.

Table 4: Liquidation periods for the stock on the market

Stock	Short name	Days to liquidate
Bio Invent	BINV	30
Concordia B	CCOR B	15
ABB B	ABB B	3
Boliden	BOL	3
Vestas Wind Systems	VWS	5
Bioporto	BIOPOR	8
SAS	SAS DKK	5
Neurosearch	NEUR	5
Nokia	NOK1V	3

The historical data of these stocks will be used for a chosen portfolio of collateral. That portfolio of collateral will be represented in two scenarios, one where each investor only invests in one stock, and one where investors have more diversified portfolios. Hence, the total portfolio of collateral will be the same but the distribution among the subportfolios will differ. Margin levels will then be calculated with each model for both portfolios using ES and a confidence level α which will be adjusted so that the condition in (46) is fulfilled when γ is set to 4%.

7.1 Portfolio composition

7.1.1 Undiversified portfolio

First we calculate the margin levels for the undiversified portfolio shown in table 5.

Table 5: Investors positions in the available stock on the market

Stock/Portfolio	1	2	3	4	5	6	7	8	9
BINV	100	0	0	0	0	0	0	0	0
CCOR	0	100	0	0	0	0	0	0	0
ABB	0	0	100	0	0	0	0	0	0
BOL	0	0	0	100	0	0	0	0	0
VWS	0	0	0	0	100	0	0	0	0
BIOPOR	0	0	0	0	0	100	0	0	0
SAS	0	0	0	0	0	0	100	0	0
NEUR	0	0	0	0	0	0	0	100	0
NOKIA1V	0	0	0	0	0	0	0	0	100

Each investor has a position of 100 SEK in one of the stocks so the total value of the collateral is 900 SEK. All investors will have maximum positions in their portfolios. This results in high concentration risks since the investors do not diversify at all. Calculating margin levels with the simple model one gets the following results.

Table 6: Results using the Simple model

Short name	Simple model β
BINV	44%
CCOR B	58%
ABB B	75%
BOL	64%
VWS	57%
BIOPOR	44%
SAS DKK	36%
NEUR	63%
NOK1V	74%

These results are given by the calculations for the simple model. With the ES for each stock we calculate the margin level. The risk ρ(L) over outstanding credit is 3.9%. Since the portfolios only consist on one stock the loaned amount is simply the portfolio value multiplied with the corresponding margin level. The total value of the collateral is 900 SEK and the outstanding credit is 514 SEK or 57 % of the collateral.

Calculating margin levels using the Euler allocation model with risk contributions one gets the following results.

Table 7: Results using the Euler model with the undiversified portfolio

Short name	Euler β
BINV	65%
CCOR B	71%
ABB B	84%
BOL	77%
VWS	70%
BIOPOR	65%
SAS DKK	63%
NEUR	72%
NOK1V	86%

Each subportfolio will have to hold the calculated risk contribution as the margin requirement. The risk over outstanding credit is 4.0%. The total outstanding loan is 657 SEK or 73% of the collateral. Since each portfolio consists of one stock the portfolio margin level is the same as the stock margin level. The dependence among the stocks is taken into account as opposed to in the simple model.

7.1.2 Diversified portfolio

Suppose investors hold more diversified portfolios, like the ones in table 8. Investors still have larger positions in one stock but 60% of the capital is equally distributed among the other stock in the market.

Table 8: Investors positions in the available stock on the market

Stock/Portfolio	1	2	3	4	5	6	7	8	9
BINV	40	7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5
CCOR	7.5	40	7.5	7.5	7.5	7.5	7.5	7.5	7.5
ABB	7.5	7.5	40	7.5	7.5	7.5	7.5	7.5	7.5
BOL	7.5	7.5	7.5	40	7.5	7.5	7.5	7.5	7.5
VWS	7.5	7.5	7.5	7.5	40	7.5	7.5	7.5	7.5
BIOPOR	7.5	7.5	7.5	7.5	7.5	40	7.5	7.5	7.5
SAS	7.5	7.5	7.5	7.5	7.5	7.5	40	7.5	7.5
NEUR	7.5	7.5	7.5	7.5	7.5	7.5	7.5	40	7.5
NOKIA1V	7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5	40

Calculating margin levels only for the Euler allocation method, since the simple model yields the same results whatever the portfolio composition, we should get some diversification effect. Using the same method as in the previous section, margin levels are set to

Table 9: Results using the Euler model with the diversified portfolio

Portfolio	Euler β
1	83%
2	85%
3	88%
4	86%
5	85%
6	83%
7	81%
8	85%
9	89%

The total risk over outstanding credit is 4.0% so the requirement is met. The total outstanding credit is 765 SEK or 85% of the market value of the collateral.

7.2 Change in liquidity

Here the consequences of changes in liquidity will be briefly studied. The liquidation days will be changed for some securities to see how the changes affect the previously calculated margin levels. This will be done for the diversified portfolio using both the simple model and allocation model.

7.2.1 Overall decrease in liquidity

First we multiply the liquidation period for all stocks with 3. This represents a general decrease in the turnover with $\frac{2}{3}$. Quite large differences but how does the new liquidation periods affect portfolio margin levels? The new portfolio margin levels together with the previously calculated margin levels from the simple model are presented in the table below.

Table 10: Results using the Simple model and general decrease in stock liquidity

Short name	β, old liquidity	β, new liquidity	Δ percentage points
BINV	44%	23%	-21
CCOR B	58%	31%	-27
ABB B	75%	52%	-23
BOL	64%	37%	-27
VWS	57%	38%	-19
BIOPOR	44%	24%	-20
SAS DKK	36%	12%	-24
NEUR	63%	39%	-24
NOK1V	74%	60%	-14

Remember that the confidence level α has been adjusted so that the risk over outstanding credit is less than 4%. The outstanding credit over market value of the collateral has decreased from

57% to 35%.

Using the Euler model results in the following margin levels

Table 11: Results using the Euler model and general decrease in stock liquidity

Portfolio	β, old liquidity	β, new liquidity	Δ percentage points
1	83%	71%	-12
2	85%	74%	-11
3	88%	78%	-10
4	86%	76%	-10
5	85%	73%	-12
6	83%	70%	-13
7	81%	69%	-12
8	85%	73%	-12
9	89%	80%	-9

The total outstanding credit over market value of the collateral has now decreased from 85% to 74%.

7.2.2 Decrease in single stock liquidity

We now change the liquidation period for SAS DKK from 5 to 30. The other liquidation periods stay the same as before. Using the simple model, the margin level for SAS DKK changes from 36% to 9% and the total outstanding credit over market value of the collateral decreases from 57% to 54%. The new portfolio margin levels are presented in table 12.

Table 12: Results using the Euler model and decrease in single stock liquidity

Portfolio	β, old liquidity	β, new liquidity	Δ percentage points
1	83%	83%	0
2	85%	84%	-1
3	88%	86%	-2
4	86%	85%	-1
5	85%	84%	-1
6	83%	81%	-2
7	81%	69%	-12
8	85%	84%	-1
9	89%	87%	-2

The total outstanding credit over market value of the collateral has decreased from 85% to 83%.

8 Discussion

8.1 Portfolio composition

The results from the comparison of portfolio composition clearly shows that one can increase margin levels by taking dependences into account. Ignoring dependences is equivalent to assuming that all securities experience their worst scenarios at the same time. That assumption is conservative, it will not lead to higher losses but to lower incomes, this since well diversified subportfolios will not be rewarded. The simple model ignores the composition of subportfolios and therefore ignores any diversification effect in a subportfolio.

When taking dependences into account we saw that the margin levels increased dramatically. The simple model had an overall outstanding credit of 57% of the market value of the collateral whereas the Euler allocation model had an overall outstanding credit of 73% for the undiversified portfolio. That is an increase with 28% of the outstanding credit with the same risk level. Increasing the outstanding credit increases revenue with the same percentage.

Table 13: Results from Simple and Euler model with the undiversified portfolio

Short name	Simple model β	Euler model β	Δ percentage points
BINV	44%	65%	21
CCOR B	58%	71%	13
ABB B	75%	84%	9
BOL	64%	77%	13
VWS	57%	70%	13
BIOPOR	44%	65%	21
SAS DKK	36%	63%	27
NEUR	63%	72%	9
NOK1V	74%	86%	12

The margin levels are still set for each stock, since each subportfolio only consist of one stock, and we see that the diversification effect is quite large. There is an overall increase in margin level for each stock as a result of the subaddativity property. In practise however, one must be able to argue why the dependence structure is applicable. Here only 9 stocks are available in the market but brokers usually offer margin lending on thousands of securities. Extracting a dependence structure for that many securities might be a difficult task. It is however clear that margin levels should be higher when offering margin lending on more stocks since the risks are allocated to the different stocks.

More interesting is that when calculating margin levels for diversified subportfolios the overall outstanding credit is 85% of the market value of the collateral. The overall outstanding credit is 49% higher than with the simple model and 16% higher than the Euler allocation model when setting margin levels on stocks. Calculating portfolio margin levels or stock margin levels with the Euler allocation model is equally complex, so choosing one over the other can not be motivated by complexity of the model. The main difference is that when setting stock margin levels with the Euler model the risk is allocated to stocks and not portfolios. So the stock margin levels will not change unless the overall weights in the stocks change. In other words, the distribution among subportfolios does not matter as long as the total portfolio of collateral stays the same.

When setting portfolio margin levels the distribution among the subportfolios has some affect as opposed to stock margin levels. Suppose all positions are held in one subportfolio and the other subportfolios are empty. Then the portfolio margin levels will change but not the stock margin levels. So even though there is little change in the total outstanding credit the portfolio margin levels is affected by the composition of the subportfolios.

Take portfolio 7 for example. With the subportfolio margin levels the margin level of portfolio 7 is set to 81%. If using the stock margin levels from the Euler model the average margin level for portfolio 7 is 70%, simply multiplying the weights in the stock with the stock margin level as in (2). That difference is due to the diversification effect in the subportfolio as opposed to the Euler stock margin levels which only considers the diversification in the total portfolio of collateral.

The simple model yields conservative results since there is no estimation of the dependence structure. It also lacks any information about the composition of securities in the total portfolio or in the subportfolios. The Euler model does take dependencies into account and the reward for that is significant, a 28% increase for stock margin levels and a 49% increase for subportfolio margin levels. Another argument for using subportfolio margin levels instead of stock margin levels is that the subportfolio margin levels can be seen as risk measures of the subportfolio on to the total portfolio. Again, it is investors subportfolios that experience losses and debts to the broker and not individual stocks.

8.2 Change in liquidity

The overall change in liquidity results in quite large differences in margin levels with the simple model. The outstanding credit decreases from 57% to 35% of the market value of the collateral. That is a decrease of 39%. Those 12 percentage points in decrease can be seen as excess risk. Suppose margin levels are set with the simple model and the liquidity then drops with $\frac{2}{3}$. Then the margin levels in the comparison are set too high and the difference could result in a debt to the broker.

The same decrease in liquidity for the Euler model results in a much lower decrease in outstanding credit. The outstanding credit over market value of the collateral changes from 85% to 74%, that is a decrease of 13%, which is much lower compared to the decrease for the Simple model. Again, it is the reward from the diversification effects and dependences that gives these results.

As for the decrease in a single stock liquidity the results are quite natural. The decrease in liquidity is dramatic for the Simple model resulting in a decrease from 36% margin level to 9%. That is a drop of 75% in the stock margin level. As for the same decrease in liquidity using the Euler model one can see that all portfolios experience drops in the portfolio margin level. This is natural since all portfolios have positions in the stock with the drop. But more importantly, portfolio 7 which has the largest position in SAS DKK, experiences the largest decrease in portfolio margin level.

9 Conclusions

In this thesis the problem of how to set margin levels has been addressed. The main assumption has been that margin levels should reflect the risk taken by the broker and that common risk modelling based on historical data can be used to calculate margin levels. Two different methods have been derived both using the same data but with different approaches. Assuming a market of 9 tradable stocks and 9 subportfolios we saw that a broker can increase the revenue from lending money by setting margin levels out of subportfolio risk contributions instead of individual stock margin levels. The main difference between the methods is that the one yielding larger outstanding credit takes diversification effects into account. Since that is a result of the dependence structure of the tradable stocks, the method is more complex.

The analysis made in this thesis is quite basic since not much has been written on the topic. Other methods with other assumptions could answer some natural questions. How are the margin levels affected if the liquidation periods d_i are stochastic variables instead of static coefficients? How are margin levels affected by the number of securities available on the market and the number of investors who use margin lending in their portfolios? The methods derived in this thesis are both based on Monte Carlo simulation, which is quite time consuming, so the possibility of an analytic model would be very interesting to study.

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