Pricing FX barriers with local volatility surface

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Abstract

This paper utilizes local volatility surface to price FX one touch barrier options for currency pair USD/SEK. A functional surface based on discrete market data for the implied volatility surface is created. The data is further used to compute the local volatility surface based on the famous Dupire (1994) model. The paper further investigates the pricing discrepancies between options prices under implied volatility surface using the analytical pricing formula for one touch barrier options proposed by Reiner & Rubenstein (1991) with the finite discretization method by Crank & Nicholson (1947) using local volatility surface. We found that the discrepancies are small between the analytical and numerical priced barrier.

Chapter 1

Introduction

Options today are widely used by financial institutions and corporate for different purposes, to hedge, speculate or used to structure different solutions for advanced trading strategies. As the financial markets develop, more advanced option types are created to satisfy client needs, mainly referred as exotic options. The market for exotic options has expanded dramatically in the past decades with increased volume and liquidity. The prices are becoming two-way observable in the market and the bid-ask spread is constant tightening. This paper focuses one type of exotic option, barrier option.

Barrier options are mainly traded OTC and preferred by practitioners due to it is cheaper compared to vanilla options. A barrier option is similar to a vanilla option, but has a barrier level set. Set accordingly to client preferences, the option can become active or worthless when the underlying stock price hits the barrier level. There are several pricing formulas or approaches for the pricing of barrier options. However they all require that the volatility for the underlying asset is known. The volatility is the only Black & Scholes (1973) input that is not directly inspected from the market, but can be implied by the market quoted vanilla option prices. Dupires (1994) argues that if the Black & Scholes (1974) model were perfect, the implied volatility would be constant for all maturities and strikes. But this is not the case. Implied volatility varies with maturities and strikes. The shape is often like a smile, hence sometimes referred as the “volatility smile”.

Currently, the market convention for pricing exotic options is to compute the price under normal Black & Scholes environment using the implied volatility for at-the-money vanilla options (Jex, Henderson & Wang, 1999). However there exist a notable discrepancy between their traded value and the theoretical one. Also, given the market liquidity for vanilla options are only set to limited certain maturities, OTC-traded exotic options sometimes need to be complying and not perfectly tailored. In this paper we create new set of volatilities using Dupire (1994), conventionally referred as local volatilities, to price options and exotics that is compatible with the market for an arbitrarily given maturity and strike. We will be using barrier options based on currency pair USD/SEK and investigate how Dupire’s model complies with FX options by comparing the options priced under implied volatility with the local one. In addition, it is also of interest to measure the pricing discrepancies between the barrier call prices, priced analytically respective numerically using finite difference method.

Chapter2

2.1 Volatility

The Black & Scholes (1973) has two fundamental assumptions. It first assumes a risk neutral valuation, that is, the value of the priced contingent claim does not depend on investors risk preferences, hence preference-free. This implies that a stock option (or options based on any other underlying asset such as FX rates and commodities) is valued based on the underlying asset presumes a risk free return. The second feature is the stochastic process governed by the asset price evolves log-normally, followed by a constant volatility σ. The process is described by the following stochastic equation:

$$dS/_S = \mu(t)dt + \sigma dZ \tag{2.1}$$

Where $\mu(t)$ is the risk free drift and Z is a stochastic process with mean zero and variance dt. Unfortunately, market prices of options are not exactly consistent with the theoretical computed prices from Black & Scholes (1973). This is due to the existence of a volatility varying with strikes and maturities, often referred as the volatility smile. The volatility smile can be regarded as adjustment for second order effects where the Black & Scholes model insufficiently ignores. But due to the extensive success of the Black & Scholes pricing formula, option traders and market makers today quote volatilities so that the resulted price equal to the theoretical one, causing an effect commonly referred as “the wrong number in the wrong formula to get the right price.” (Rebonato, 1999).

There have been numerous attempts to amend this and to extend the Black & Scholes formula to account for the volatility smile. One approach proposed by Hull and White (1987) imposes a random process for the volatility itself. Another approach by Merton (1976) takes account of the discontinuities, or jumps, of the underlying asset development. These methods do not retain the completeness feature of the model, which is of importance as it allows for arbitrage pricing and hedging.

The concept of local volatility model was originated by Derman & Kani (1994) with discrete time steps. The model was further brought into a continuous-time equation by Dupire (1994). The idea is to have a spot model that conceals the completeness and yet is compatible with the market observed prices. Mathematically, we want to find a risk neutral process for the spot in the form of

$$dS/_S = \mu(t)dt + \sigma(S,t)dZ \tag{2.2}$$

The above model should be built upon arbitrage free prices of European call prices $C(K,T,S_0)$. It was shown since Breeden & Litzenberger (1973) that a risk-neutral probability distribution can be extracted by market quoted European option prices. This is important because of the Black & Scholes model assumes a log-normal probability distribution with a constant volatility. But the presence of volatility smiles implicates that market implied probability distribution deviates from being log-normal, as shown by European options with different strikes are dependent upon which implied probability distribution a certain maturity imposes (Jex, Henderson & Wang, 1999).

We will now show that there is a unique spot process consistent with the market extracted probability distribution function and that the unique state-dependent diffusion coefficient $\sigma(S,t)$is the local volatility, compatible with market European option prices. Assume that the risk-neutral process for S is given by equation 2.2, then we have the following:

Theorem 2.1, **Dupires local volatility formula**.

Given that the underlying asset is governed by stochastic differential equation (2.2), the local volatility $\sigma(K,t)$ is:

$$\sigma(K,t)^2 = \left\{\frac{\partial C}{\partial T} + \ (r(T) - q(T))K\frac{\partial C}{\partial K} + q(T)C\right\} / \ \left\{\frac{1}{2}\frac{\partial^2 C}{\partial K^2}K^2\right\} \qquad (2.3)$$

Proof.

We write the European call option price $C(K,T,S_0)$ as the following:

$$C(K,T,S_0) = D\int_K^{\infty}(S-K)\varphi(S,T,S_0,K)dS \qquad (2.4)$$

Here $\varphi(S,T,S_0,K)$ is the probability density of the underlying asset at maturity satisfying the Fokker-Plank equation and D is the deterministic discount factor:

$$\frac{\partial\varphi}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S^2}\varphi S^2\sigma^2 - \frac{\partial}{\partial S}(r(t) - q(t))S\sigma \qquad (2.5)$$

$$D = exp\left\{-\int_0^T r'^{(t)}dt\right\} \qquad (2.6)$$

We will now show that we can use equation (2.5) and (2.6) to write the local volatility $\sigma(S,t)$ in terms of partial derivatives of a standard European call option price.

Differentiating $C(K,T,S_0)$ with respect to K gives:

$$\frac{\partial C}{\partial K} = D\frac{\partial}{\partial K}\int_K^\infty (S-K)\varphi(S,T,S_0,K)dS$$

$$= -D\frac{\partial}{\partial K}\int_\infty^K (S-K)\varphi(S,T,S_0,K)dS$$

$$= -D((K-K) - \int_\infty^K \frac{\partial}{\partial K}(S-K)\varphi(S,T,S_0,K)dS\ )$$

$$= D\int_\infty^K \varphi(S,T,S_0,K)dS \tag{2.7}$$

Differentiating $C(K,T,S_0)$ with respect to K twice and using (2.7) gives:

$$\frac{\partial^2 C}{\partial K^2} = D\ \varphi(S,T,S_0,K) \tag{2.8}$$

Notice that equation (2.8) states that given market prices of European call options, we have recovered the risk-neutral probability density.

Differentiating (2.4) with respect to T and using Fokker-Planck equation (2.5) yields:

$$\frac{\partial C}{\partial T} = -r(T)C + \int_K^\infty (S-K)\left\{\frac{\partial}{\partial T}\varphi(S,T,S_0,K)\right\}dS\ =$$

$$-\mathrm{r(T)C} + \mathrm{D}\int_K^\infty (S-K)\left\{\frac{1}{2}\frac{\partial^2}{\partial \mathrm{S}^2}\varphi(S,T,S_0,K)S^2\sigma^2 - \frac{\partial}{\partial \mathrm{S}}(r-q)S\sigma\right\}dS \tag{2.9}$$

Integrating by parts twice and using equation (2.7) & (2.8) yields:

$$\frac{\partial C}{\partial T} = -r(T)C + D\frac{\sigma^2 K^2}{2}\varphi(S,T,S_0,K) + D\int_K^\infty (r-q)S\varphi(S,T,S_0,K)dS$$

$$= -r(T)C + D\frac{\sigma^2 K^2}{2}\varphi(S,T,S_0,K) + D\int_K^\infty (r-q)S\varphi(S,T,S_0,K)dS$$

$$= -r(T)C + D\frac{\sigma^2 K^2}{2}\varphi(S,T,S_0,K) + D\int_K^\infty (r-q)(S-K+K)\varphi(S,T,S_0,K)dS$$

$$= -r(T)C + D\frac{\sigma^2 K^2}{2}\varphi(S,T,S_0,K) + D\int_K^\infty (r-q)(S-K+K)\varphi(S,T,S_0,K)dS$$

$$= -r(T)C + D\frac{\sigma^2 K^2}{2}\varphi(S,T,S_0,K) + (r(T)-q(T))\left(C - K\frac{\partial C}{\partial K}\right)$$

$$= -r(T)C + \frac{\sigma^2 K^2}{2}\frac{\partial^2 C}{\partial K^2} + \big(r(T) - q(T)\big)\left(C - K\frac{\partial C}{\partial K}\right) \qquad (2.10)$$

In which we can rearrange to get Dupires equation. For a more detailed derivation of this proof, including full integration and the Fokker-Planck equation, we refer to Kohn (2000) and Derman & Kani (1994).

Equation (2.4) derives local volatility in terms of partial derivatives of European call option prices. However, as FX call option prices are quoted in implied volatility, it may be more practical to relate local volatility directly to implied volatility.

Let σ_{imp} represent the implied volatility, reversed by the Black & Scholes model, and then we have the following:

$$C(K, T, S_0) = C_{BS}\big(K, T, S_0, \sigma_{\text{imp}}\big) \qquad (2.11)$$

It is no surprise that the partial derivatives of a European option can be analytically formulated. Hence given equation (2.11), the local volatility can also be related directly to market implied volatility.

Corollary 2.1, **Dupires local volatility formula in terms of implied volatility**

Given that the underlying asset is governed by stochastic differential equation (2.2), assuming that the local volatility $\sigma(K,t)$ is differentiable with respect to T and twice differentiable with respect to K, $\sigma(K,t)$ can be represented as:

$$\sigma(K,T)^2 = \frac{\frac{\sigma_{imp}}{T} + 2\big(r(T) - q(T)\big)K\frac{\partial \sigma_{imp}}{\partial K} + 2\frac{\partial \sigma_{imp}}{\partial T}}{K^2\left\{\frac{\partial^2 \sigma_{imp}}{\partial K^2} - d_1\sqrt{T}\left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2 + \frac{1}{\sigma_{imp}}\left(\frac{1}{K\sqrt{T}} + d_1\frac{\partial \sigma_{imp}}{\partial K}\right)^2\right\}} \qquad (2.12)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r(T) - q(T) + \frac{1}{2}\sigma_{imp}^2\right)T}{\sigma_{imp}\sqrt{T}} \qquad (2.13)$$

$$d_2 = d_1 - \sigma_{imp}\sqrt{T} \qquad (2.14)$$

2.2 Functional form of volatility

Given that we can connect local volatility directly to implied volatility using equation (2.12), we need to find a functional form for the implied volatility. To do this, we extract real market quoted implied volatility and then fit the data to a well-chosen function using least square minimization.

Below is the market implied volatilities for USD/SEK currency pair dated 2009-11-26, extracted from real financial platform. The rates are hence used by traders when quoting for option prices. This table will be used later to construct the implied volatility surface.

	ATM		25D Call USD		25D Put USD		10D Call USD		10D Put USD	
Exp	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
1D	15.595	22.288	15.676	24.125	14.386	22.901	13.115	29.01	10.679	26.888
1W	14.74	17.27	15.106	18.264	14.143	17.347	14.63	20.388	12.908	18.775
2W	15.08	16.965	15.551	17.894	14.572	16.963	15.442	19.698	13.7	18.06
3W	15.445	16.94	16.009	17.861	14.995	16.895	16.173	19.53	14.341	17.797
1M	15.535	16.725	16.175	17.645	15.087	16.603	16.527	19.185	14.552	17.306
2M	16.1	17.26	16.924	18.344	15.734	17.219	17.665	20.23	15.502	18.203
3M	16.735	17.92	17.751	19.192	16.462	17.986	18.902	21.498	16.543	19.317
6M	17.155	18.195	18.33	19.572	16.965	18.313	19.834	22.064	17.333	19.789
1Y	17.36	18.27	18.624	19.686	17.259	18.451	20.437	22.333	17.935	20.115
18M	16.68	17.93	17.913	19.347	16.586	18.234	19.549	22.103	17.039	20.059
2Y	16.29	17.655	17.458	19	16.159	17.964	18.941	21.684	16.525	19.84
3Y	15.59	16.985	16.798	18.337	15.367	17.218	18.404	21.128	15.601	19.007
5Y	13.68	15.115	15.059	16.603	12.993	14.895	16.574	19.294	12.672	16.17
7Y	13.673	15.026	15.098	16.501	13.237	15.034	16.75	19.203	12.844	16.149

Table 2.1 – Implied volatility in terms of delta for USD/SEK, 2009-11-26

As we can see, the y-axis represents the maturity date for different call / put options based on this currency pair. However one would expect the x-axis to be the strike of different options, but this is not the case. As for currency options, the implied volatility surface extract is quoted in delta. In order to convert it to strikes, we use the following formula for call and put options.

$For\ FX\ call\ options$:

$$\Delta_c = N(d_1) \tag{2.15}$$

$For\ FX\ put\ options$:

$$\Delta_p = N(d_1) - 1 \tag{2.16}$$

Worth to mention is the risk free rate associated when pricing European options based on currency pairs. Generally when dealing with options based on equities, only one risk free rate is involved. However when the underlying equity pays dividend, it will have an effect on the underlying equity price, which in turn will have an effect on the option pricing. For this reason, the dividend rate is also accounted when pricing equity options paying dividend. When dealing with FX currency options, there is always a domestic risk free rate as a foreign risk free rate. Both of them need to be accounted. The similarity in computational concern involves replacing the dividend rate with the foreign risk free rate. In this case, the Swedish currency SEK is the domestic currency. Therefore the corresponding LIBOR for SEK is used, whilst LIBOR for USD is used as the foreign risk free rate.

USD			**SEK**	
s/n-o/n	0.00186		**s/n-o/n**	0.00230
1w	0.00215		**1w**	0.00353
2w	0.00222		**2w**	0.00358
1m	0.00234		**1m**	0.00358
2m	0.00244		**2m**	0.00400
3m	0.00254		**3m**	0.00490
4m	0.00308		**4m**	0.00568
5m	0.00392		**5m**	0.00635
6m	0.00476		**6m**	0.00705
7m	0.00565		**7m**	0.00800
8m	0.00665		**8m**	0.00860
9m	0.00749		**9m**	0.00930
10m	0.00833		**10m**	0.00960
11m	0.00915		**11m**	0.00990
12m	0.01003		**12m**	0.01013

Table 2.2 Libor for USD and SEK respectively

Using the above formula with correct risk free rates and corresponding maturities, one could extract the strikes, K. Notice that the ATM strike doesn't necessarily need to be exact as the current spot rate due to the value of these options at a certain time before maturity (Wilmott, 2007). The transformed table is represented by strikes:

	Strike									
	7.011755	7.018202	7.660313	7.714898	6.465716	6.431922	8.413183	8.605003	5.993472	5.87493
	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
1D	15.60%	22.29%	15.68%	24.13%	14.39%	22.90%	13.12%	29.01%	10.68%	26.89%
1W	14.74%	17.27%	15.11%	18.26%	14.14%	17.35%	14.63%	20.39%	12.91%	18.78%
2W	15.08%	16.97%	15.55%	17.89%	14.57%	16.96%	15.44%	19.70%	13.70%	18.06%
3W	15.45%	16.94%	16.01%	17.86%	15.00%	16.90%	16.17%	19.53%	14.34%	17.80%
1M	15.54%	16.73%	16.18%	17.65%	15.09%	16.60%	16.53%	19.19%	14.55%	17.31%
2M	16.10%	17.26%	16.92%	18.34%	15.73%	17.22%	17.67%	20.23%	15.50%	18.20%
3M	16.74%	17.92%	17.75%	19.19%	16.46%	17.99%	18.90%	21.50%	16.54%	19.32%
6M	17.16%	18.20%	18.33%	19.57%	16.97%	18.31%	19.83%	22.06%	17.33%	19.79%
1Y	17.36%	18.27%	18.62%	19.69%	17.26%	18.45%	20.44%	22.33%	17.94%	20.12%
18M	16.68%	17.93%	17.91%	19.35%	16.59%	18.23%	19.55%	22.10%	17.04%	20.06%
2Y	16.29%	17.66%	17.46%	19.00%	16.16%	17.96%	18.94%	21.68%	16.53%	19.84%
3Y	15.59%	16.99%	16.80%	18.34%	15.37%	17.22%	18.40%	21.13%	15.60%	19.01%
5Y	13.68%	15.12%	15.06%	16.60%	12.99%	14.90%	16.57%	19.29%	12.67%	16.17%
7Y	13.67%	15.03%	15.10%	16.50%	13.24%	15.03%	16.75%	19.20%	12.84%	16.15%

Table 2.3 – Implied volatility in terms of strikes for USD/SEK, 2009-11-26

As the above table shows, the market prices of European options do not appear as a nice and continuous function, but rather as discrete data points, with only a few strikes corresponding to a certain 10-delta, 25-delta and ATM represented. In order to implement Dupires equation, whether in the form of (2.3) or (2.12), a smooth interpolation between market data points is needed to find the functional form of the implied volatility surface. For this, the formula proposed by Duma & Whaley (1998) is used:

$$\sigma_{imp}(X,T) = a_0 + a_1 X + a_2 X^2 + a_3 T + a_4 T^2 + a_5 TX \tag{2.17}$$

where X is defined as moneyness, $X = \frac{K}{Se^{(r-q)T}}$.

The corresponding derivatives, with respect to strike K, and time to maturity T, can either be computed analytically or numerically. Using *lsqcurvefit* in Matlab, fitting the data to the above nonlinear equation in a least square sense, yields the following result:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0.8208 \\ -1.2680 \\ 0.6353 \\ -0.0832 \\ 0.0712 \\ 0.0116 \end{bmatrix}$$

We plot the original discrete points implied volatility surface as shown in Figure 2.1, the functional implied volatility surface in Figure 2.2 and using equation (2.12) to implement a local volatility surface, shown in Figure 2.3.

Equation (2.12), the relationship between local and implied volatility was hard to implement in practice. In addition, calculating partial derivatives for fitted functional form of the implied volatility makes the local volatility sensitive for the choose of functional form for implied volatility and the interpolation method. For interested readers in this field, we refer to Dumas & Whaley (1998) and Sehgal & Vijayakumar (2008). However, once the implementation is complete, we can price path-dependent options in a way consistent with the market observed vanilla option prices. In addition, the local volatility makes it possible to hedge these exotic options, given that we know the volatility to use. Although the hedging aspect of implementing local volatility is not discussed in this paper, the subject area is of high interest for practitioners and interested readers may look into Carr & Chou (1997).

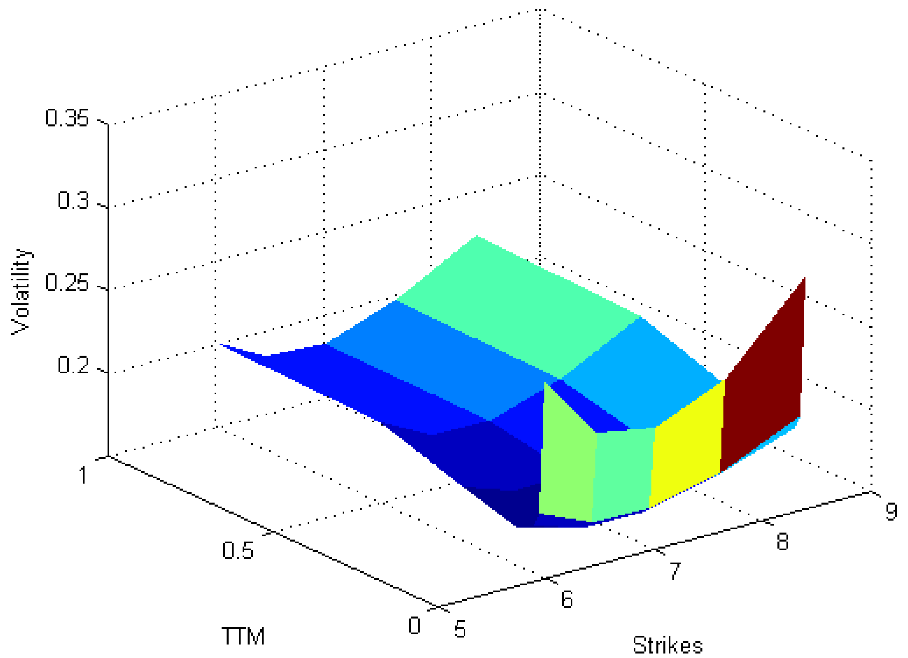


Figure 2.1 - Implied volatility surface

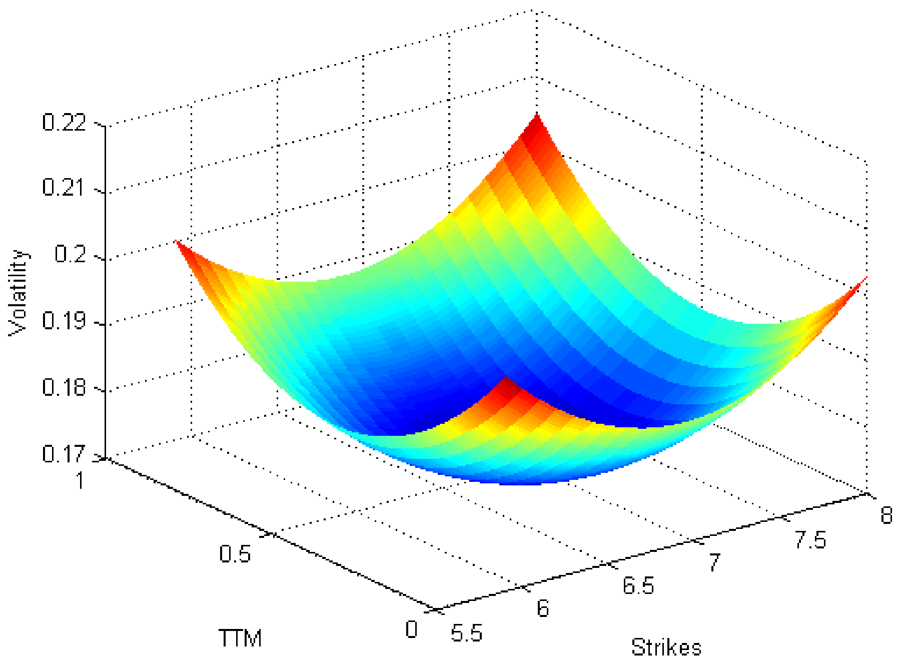


Figure 2.2 – Functional form of implied volatility surface

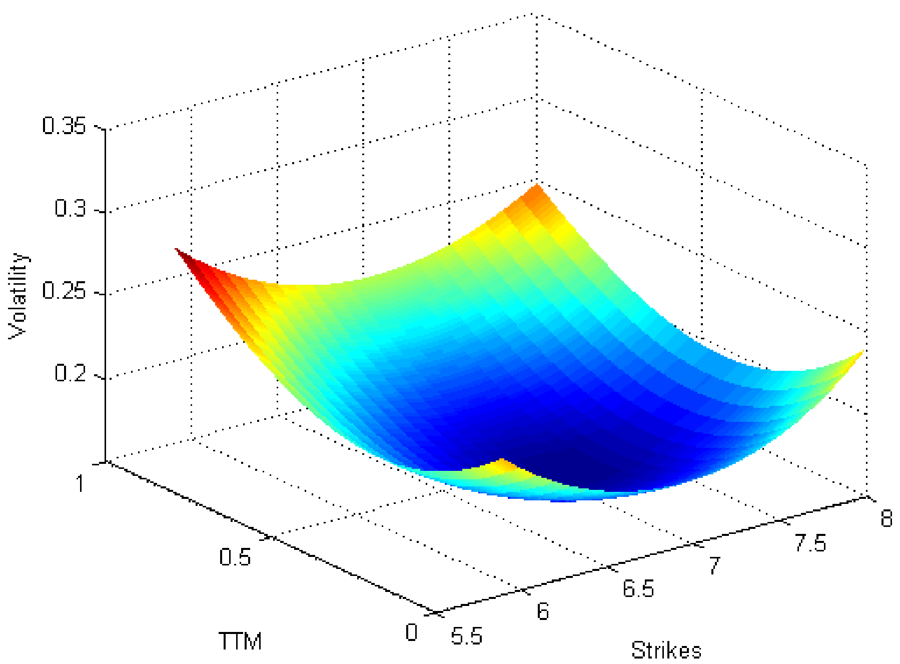


Figure 2.3 – Local volatility surface

Chapter 3

3.1 Barrier options

Barrier options are path-dependent options that can appear in different complex forms and settings. Although they are all represented by the common property that the option is either knocked-out, i.e. the option becomes worthless when the asset price hits the barrier, or knocked-in, meaning the option becomes active when the underlying option hits the barrier. In this paper we focus on basic single one touch barrier option, i.e. options that is either knocked-in or knocked-out the first time the underlying asset price hits the barrier.

Given that the call and put settings are available for each of the groups, we have a total of eight different kinds of one-touch barrier options. In addition to the exotic setting of barrier levels, barrier options can sometimes carry a rebate, which is paid out to the holder of the contract when a barrier is reached. In this paper we do not involve any rebates, although they can easily be integrated into the pricing formulas, whether analytically or numerically.

The one-touch barrier options can be divided into following groups:

Up-and-in options: barrier levels are set above the initial underlying asset price, and the option becomes active when the underlying asset price hits the barrier and attains the feature of a vanilla option. If the barrier is not hit during the entire time horizon, the options mature without payoff.

Up-and-out options: barrier levels are set above the initial underlying asset price, and the option becomes worthless when the underlying asset price hits the barrier, giving the holder zero payoff. If the barrier is not hit during the entire time horizon, the options matures with the payoff as a vanilla option.

Down-and-in options**:** barrier levels are set beneath the initial underlying asset price, and the option becomes active when the underlying asset price hits the barrier and attains the feature of a vanilla option. If the barrier is not hit during the entire time horizon, the options mature without payoff.

Down-and-out: barrier levels are set beneath the initial underlying asset price, and the option becomes worthless when the underlying asset price hits the barrier, giving the holder zero payoff. If the barrier is not hit during the entire time horizon, the options mature with the payoff as a vanilla option.

This paper has chosen to price down-and-out options for the result analysis. The local volatility implemented in Chapter 2 will be used for the numerical computation of barrier options using finite difference methods. In addition, we calculate the barrier prices by analytical formulas, using implied volatility, acting as a comparison object for the numerical computed prices. For interested readers, there are also other ways to compute a barrier option price, such as using a Monte Carlo algorithm or replicating a barrier using a static portfolio of vanilla options (Carr & Chou, 1997).

The analytical pricing formulas for up-and-out call options are derived in the next section. The proofs are similar for all single one touch barriers.

3.2 Analytical formula

We will be pricing down-and-out call options based on the analytical formula presented by Rubinstein & Reiner (1991) and finite difference methods. Before we do that, we will first deliver a full proof of the analytical formula for a down-and-out call. The proof for other one-touch barrier options can be derived in a similar manner. Beforehand, we need some mathematical tool. Please note that the all proofs are standard and can be found in Björk (2009).

Definition 3.1

$For\ any\ y \in R, the\ hitting\ time\ of\ y\ for\ the\ process\ X\ is$

$$\tau(X, y) = inf\{t \geq 0 \mid X(t) = y\} \tag{3.1}$$

$The\ process\ X\ absorbed\ at\ y\ is\ defined\ as$

$$X_y(t) = X(\tau \wedge \mathrm{t}) \tag{3.2}$$

$The\ running\ maximum\ and\ minimum\ processes, M_X(t) and\ m_X(t)\ are\ defined$

$$M_X(t) = \sup X(s)\,, \qquad for\ 0 \leq s \leq t \tag{3.3}$$

$$m_X(t) = \inf X(s)\,, \qquad for\ 0 \leq s \leq t \tag{3.4}$$

Let's first consider a general down-and-out contract. The contract has payoff V in maturity, if the underlying asset price stays above a barrier H during the entire contract period up to the time of maturity. H is set in a way such that $H > S_0$. However if the asset price at some time point before maturity hits the barrier H, then the contract becomes lifeless and nothing is paid to the holder of the contract. Let's denote the contingent claim as following:

$$V = \Phi\big(S(T)\big) \tag{3.5}$$

In addition, the pricing function is represented by $F(t, s; T, \Phi)$. Then we can define the down-and-out type of this claim as following:

Definition 3.2.

Take as given a contract $V = \Phi(S(T))$ *with maturity* T, V_{DO} *is defined by:*

$$V_{DO} = \begin{cases} \Phi\big(S(T)\big), & if\ S(t) > H\ for\ all\ t\ \in [0, T] \\ 0, & if\ S(t) \leq H\ for\ some\ t\ \in [0, T] \end{cases} \tag{3.6}$$

To be able to price V_{DO}, we will need the function $\Phi_D(x)$. This function can be thought as the original claim function $\Phi(x)$ cut off below the level H.

Definition 3.3

For a fixed function Φ, the function Φ_D is defined by

$$\Phi_D(x) = \begin{cases} \Phi(x), & x > 0 \\ 0, & x \leq 0 \end{cases} \quad or \ \Phi_D(x) = \Phi(x) * I(x > H) \tag{3.7}$$

It is worth notifying that the pricing function $F(T, s; T, \Phi)$ follows linearity in terms of Φ-argument. In addition, function (3.7) is also linear. Indeed, we state these qualities as a lemma.

Lemma 3.1

$for\ real\ numbers\ a\ and\ b, and\ all\ functions\ \Phi\ and\ \Psi, we\ have$:

$$F(T, s; T, a\Phi + b\Psi) = a\, F(T, s; T, \Phi) + bF(T, s; T, \Psi) \tag{3.8}$$

$$(\mathrm{a}\Phi + \mathrm{b}\Psi)_D = a\Phi_{\mathrm{D}} + \mathrm{b}\Psi_D \tag{3.9}$$

Proof

For the pricing function F, the linearity follows from the risk-neutral valuation with the linearity of the expectation operator. The linearity for the payoff function is obvious.

With the help of above introductions, we now have the main result of our analytical pricing formula. The formula insists that the price for the down-and-out version of our claim $\Phi(S(T))$ reduces to the pricing of the claim $\Phi_D(S(T))$ without implications of a barrier. Reversely, if we know how to price a standard European claim with contract function $\Phi_D(S(T))$, then we can price the down-and-out version of the contract $\Phi(S(T))$. One can think of barrier option as a type of path-dependent option. It is essential for the trader to have a knowledge regarding the path of the underlying asset movement to a certain point. In the case of barrier option, we are only interested of the asset dynamic up till a time where it hits the barrier. Hence we need to assign a distribution for this process. With the help of our earlier presented mathematical results, we make a proposition for the dynamic of the underlying process absorbed at a point β:

Proposition 3.1

The density function $f_\beta(x, t, a)$ of the absorbed process $X_\beta(t)$ is

$$f_{\beta}(x,t,a) = \varphi\left(x, \mu t + \alpha, \sigma\sqrt{t}\right) - \exp\left(-\frac{2\mu(\alpha-\beta)}{\sigma^2}\right)\varphi\left(x, \mu t - \alpha + 2\beta, \sigma\sqrt{t}\right) \quad (3.10)$$

The support of this density is in the interval (β, ∞) *if* $\alpha > \beta$ *and* $(-\infty, \beta)$ *if* $\alpha < \beta$. *Here* $\varphi(x, \mu, \sigma)$ *denotes the density of a normal distribution with mean* μ *and variance* σ^2 *and* $X(t)$*follows the stochastic dynamics given by equation (2.1).*

We are now ready for the pricing theorem.

Theorem 3.1 (Pricing down-and-out contracts)

Consider a claim at maturity paying $= \Phi(S(T))$, *the corresponding down-and-out pricing function, with barrier* $H < S$ *is denoted by* $F_{DO}(t, s; T, \Phi)$ *has the price at time* t:

$$F_{DO}(t,s;\Phi) = F(t,s;\Phi_D) - \left(\frac{H}{s}\right)^{\frac{2r}{\sigma^2}-1} * F\left(t, \frac{H^2}{s}; \Phi_L\right) \quad (3.11)$$

Proof

We set, without loss of generality, $t = 0,\ s(0) > H$. *Let* S_H *denote our underlying asset dynamic with the possibility to be absorbed at H. Then we use the brute force of risk neutral valuation, with Q denoting the martingale measure guaranteeing the arbitrage free property*

$$F_{DO}(0,s;\Phi) = e^{-rT}E_{0,s}^{Q}[V_{DO}] = e^{-rT}E_{0,s}^{Q}\left[\Phi\left(S(T)\right) * I\{inf_{0\le t\le T}S(t) > H\}\right]$$

$$= e^{-rT}E_{0,s}^{Q}\left[\Phi_D\left(S_H(T)\right) * I\{inf_{0\le t\le T}S(t) > H\}\right] = e^{-rT}E_{0,s}^{Q}\left[\Phi_D\left(S_H(T)\right)\right] \quad (3.12)$$

The last expectation

$$E_{0,s}^{Q}\left[\Phi_D\left(S_H(T)\right)\right] = \int_H^{\infty} \Phi_D(x)g(x)dx \quad (3.13)$$

where we have used the notation $g(x)$ *as density for the stochastic process* $S_H(T)$. *In order to evaluate the right hand side of above equation, we log our stochastic process which is governed by*

$$S(T) = e^{(lns + \left(r - \frac{1}{2}*\sigma^2\right)T*\sigma W(T)))} = e^{X(T)} \quad (3.14)$$

$$dX(t) = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dZ_t \quad (3.15)$$

and equation (3.13) becomes

$$E_{0,s}^{Q}\left[\Phi_D\left(S_H(T)\right)\right] = \int_{lnH}^{\infty} \Phi_D(e^x) f(x) dx \tag{3.16}$$

Here, $f(x)$ *denotes the density for the stochastic process* $X_{lnH}(T)$*, which we can use proposition 3.1 to evaluate as*

$$f(x) = \varphi\left(x, \left(r - \frac{1}{2}\sigma^2\right)T + lns, \sigma\sqrt{t}\right) - \left(\frac{H}{s}\right)^{\frac{2r}{\sigma^2}-1} \varphi\left(x, \left(r - \frac{1}{2}\sigma^2\right)T + \ln\left(\frac{H^2}{s}\right), \sigma\sqrt{t}\right) \tag{3.17}$$

Hence,

$$E_{0,s}^{Q}\left[\Phi_D\left(S_H(T)\right)\right] = \int_{lnH}^{\infty} \Phi_D(e^x) f(x) dx =$$

$$\begin{aligned}
&\int_{lnH}^{\infty} \Phi_D(e^x)\varphi\left(x, \left(r - \frac{1}{2}\sigma^2\right)T + lns, \sigma\sqrt{t}\right) dx \\
&- \left(\frac{H}{s}\right)^{\frac{2r}{\sigma^2}-1} \int_{lnH}^{\infty} \Phi_D(e^x)\varphi\left(x, \left(r - \frac{1}{2}\sigma^2\right)T + \ln\left(\frac{H^2}{s}\right), \sigma\sqrt{t}\right) dx \\
&= \int_{lnH}^{\infty} \Phi_D(e^x) f(x) dx \\
&= \int_{-\infty}^{\infty} \Phi_D(e^x)\varphi\left(x, \left(r - \frac{1}{2}\sigma^2\right)T + lns, \sigma\sqrt{t}\right) dx \\
&- \left(\frac{H}{s}\right)^{\frac{2r}{\sigma^2}-1} \int_{-\infty}^{\infty} \Phi_D(e^x)\varphi\left(x, \left(r - \frac{1}{2}\sigma^2\right)T + \ln\left(\frac{H^2}{s}\right), \sigma\sqrt{t}\right) dx \\
&= E_{0,s}^{Q}\left[\Phi_L\left(S(T)\right)\right] - \left(\frac{H}{s}\right)^{\frac{2r}{\sigma^2}-1} E_{0,\frac{H^2}{2}}^{Q}\left[\Phi_L\left(S(T)\right)\right]
\end{aligned} \tag{3.18}$$

which gives us the desired result.

Given the linearity of the pricing function F and our claim, it is natural for the pricing function $F_{DO}$to be linear. For a down-and-out-call, F is calculated using the classical Black & Scholes formula.

Corrollary 3.1

$for\ real\ numbers\ a\ and\ b, and\ all\ functions\ \Phi\ and\ \Psi, we\ have$:

$$F_{DO}(T, s; T, a\Phi + b\Psi) = a\, F_{DO}(T, s; T, \Phi) + bF_{DO}(T, s; T, \Psi) \tag{3.19}$$

3.3 finite difference methods

Any contingent claim with underlying asset S will have a value function $V(S,t)$ satisfying the Black & Scholes differential equation

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} = rV \tag{3.20}$$

with value function

$$V(S,t) = F(S,t) \tag{3.21}$$

In the case for a standard European Call, the payoff function is

$$V(S,T) = \max(S_T - K, 0) \tag{3.22}$$

Obviously, a European call satisfies the Black & Scholes differential equation. However, when dealing with exotic options, the payoff sometimes is path-dependent, for example in an Asian option and barrier option. Hence an analytical solution to the differential equation is hard to derive and may not even exist. It is then plausible to compute option prices numerically, by approximating the above partial derivatives with finite differences. The method is generally referred as finite difference method.

Let suppose that the life of the option is T, we can then divide it into N equally spaced intervals of length $\Delta t = T/N$. We can do the same for the underlying asset price, by dividing it into M steps, each with interval $\Delta S = (S_{max} - S_{min})/M$. Equation (3.20) can now be approximated by

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r-q)j\Delta S\frac{f_{i,j+1} - f_{i+1,j-1}}{2\ \Delta S} + \frac{1}{2}(j\Delta S)^2\sigma^2\frac{f_{i,j+1} + f_{i,j-1} - 2}{\partial S^2}f_{i,j} = rf_{i,j} \tag{3.23}$$

For $j = 1,2 \ldots M-1$ and $i = 1,2 \ldots N-1$.

Equation (3.23) represents the approximated equation (3.22) when we want to use the implicit finite difference method, as we can solve for $f_{i+1,j}$ implicitly using $f_{i,j+1}, f_{i,j}, f_{i,j-1}$. The reverse holds for explicit finite difference method, where we solve for $f_{i,j}$ using $f_{i+1,j+1}, f_{i+1,j}, f_{+1i,j-1}$. Note that explicit method is easier to implement, as we solve (3.23) using recursion, whilst we must solve $M-1$ equations simultaneously for the implicit finite difference method. In this paper, we use the Crank & Nicholson (1947) approach, which uses the average of explicit and implicit method. The Crank-Nicholson method is unconditionally stable, produces least error compared to the implicit and explicit methods (Morton & Mayers 2005).

Solving equation (3.20) for options valuations, boundary conditions are needed. For a down-and-out call, with S_{min} as the down-and-out barrier, the boundary conditions are

$$f_{N,j} = \max(S_{min} + j\Delta S - K, 0)\,, \qquad j = 0,1,\ldots,M \tag{3.24}$$

$$f_{i,0} = 0, \qquad i = 0,1,\ldots,N \tag{3.25}$$

$$f_{i,M} = 2f_{i,M-1} - f_{i,M-2} \qquad i = 0,1,\ldots,N \tag{3.26}$$

Equation (3.24) follows immediately from the payoff function from a call option at maturity, equation (3.25) gives the option value at barrier and equation (3.26) follows from that for large values of S, the payoff is at most linear (Wilmott, 2007).

3.4 The result

For the results, we have priced a down-and-out barrier call and a vanilla call. The down-and-out barrier call is priced using both the analytical formula, introduced by Rubenstein & Reiner (1991), and the finite difference method by Crank-Nicholson. The vanilla call is priced using the classical Black & Scholes formula.

The volatility used in pricing the barrier option and vanilla option analytically is implied volatility, whilst the local volatility surface is used for the finite difference approach. The implied volatility surface used is the same as in Chapter 2, dated 23 November 2009 for USD/SEK. The spot price at this date was $S = 6.9524$. We price the options with in the money strike 6 up and include out of the money strike 8. For barrier options, the barrier level H is set to 5.9. The maturities used vary from 1 week up to 1 year. For the risk free rates, we use Libor for USD and SEK, each one with maturities shown in table 2.2.

The results are shown in the following figures.

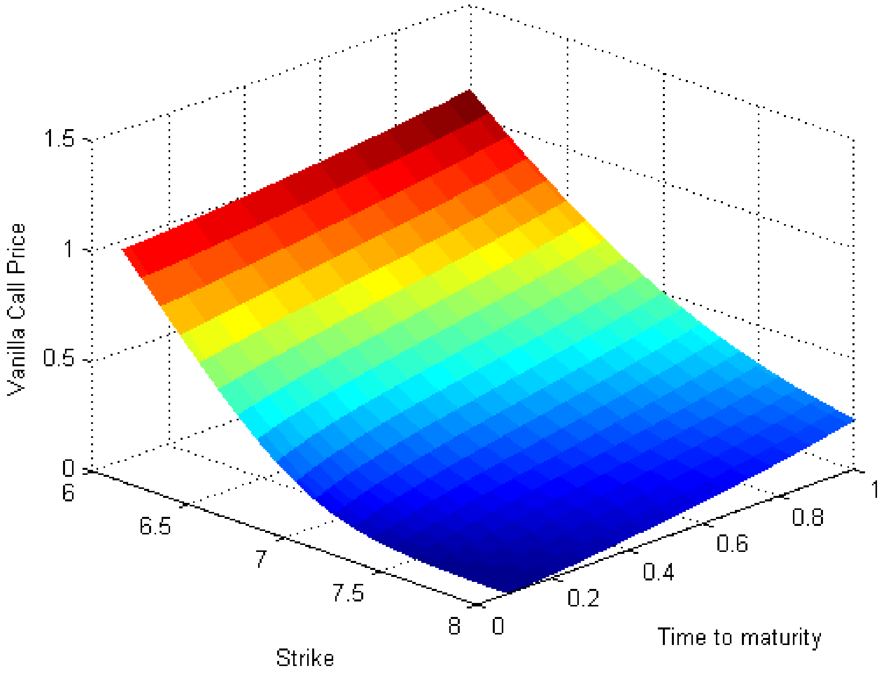


Figure 1 – Vanilla call prices, computed using Black & Scholes formula

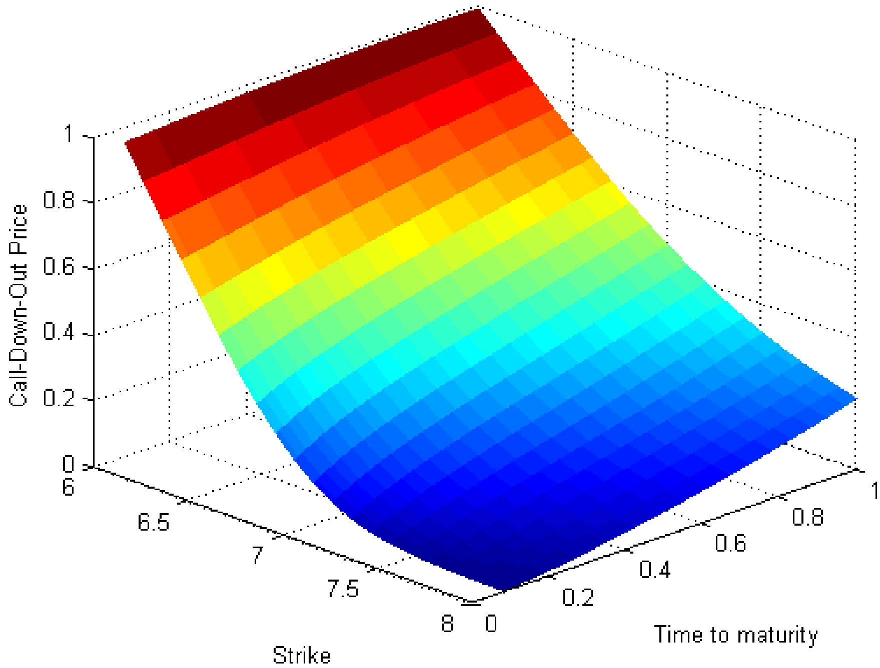


Figure 2 – Analytical down-and-out barrier call prices using implied volatility

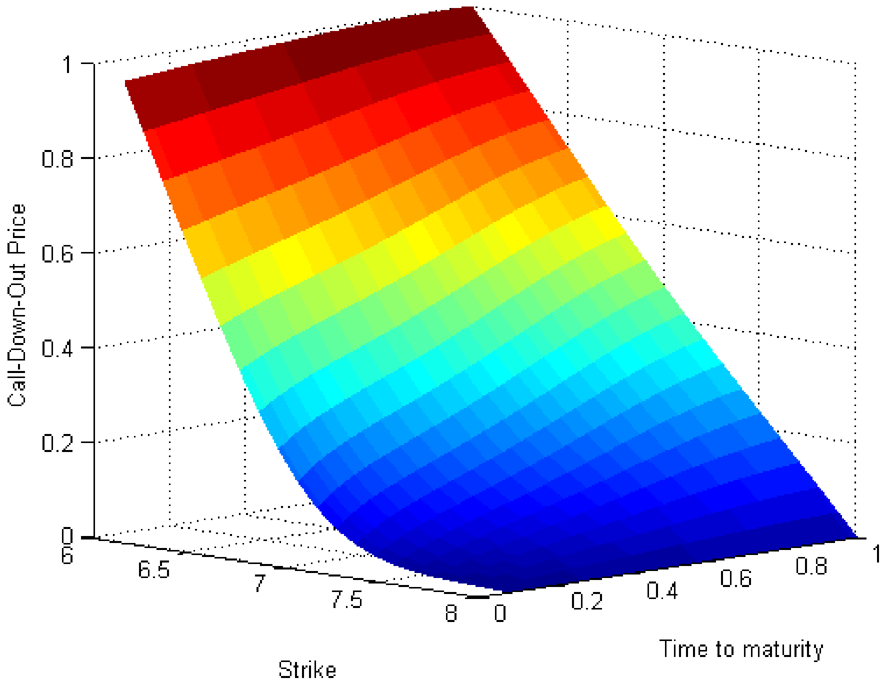


Figure 3 – Crank-Nicholson down-and-out barrier call prices using local volatility

Intuitively, we expect the price of the down-and-out barrier call to be lower than the vanilla call. To confirm this, we plot the price discrepancies and the discrepancies in percent in the following figures, using

$$Discrepancy = Vanilla - Barrier$$

$$Discrepancy\ in\ \% = \frac{Vanilla - Barrier}{Vanilla}$$

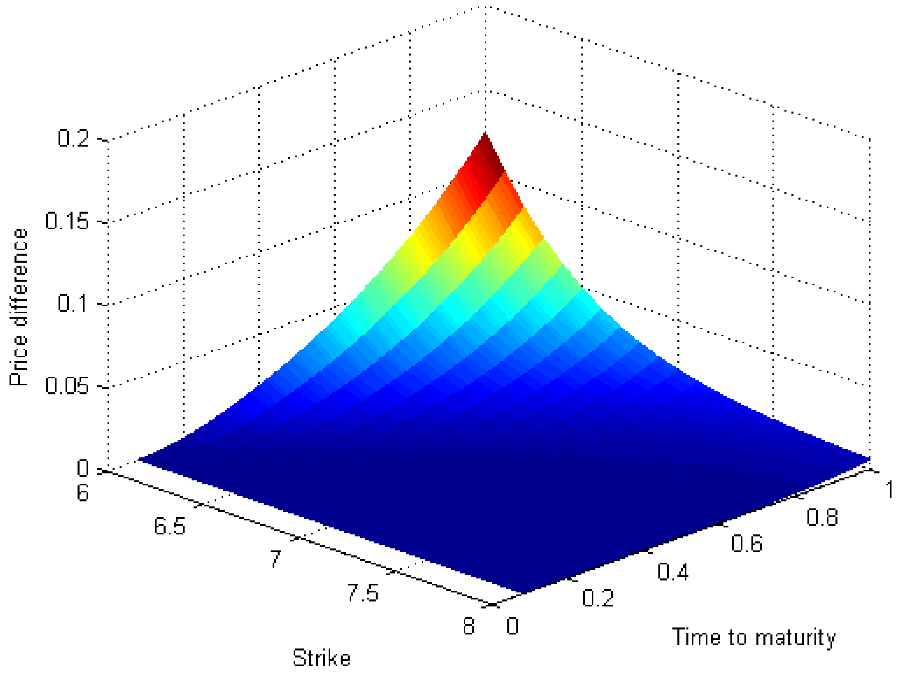


Figure 4 – Price discrepancies between vanilla call and barrier call (analytical)

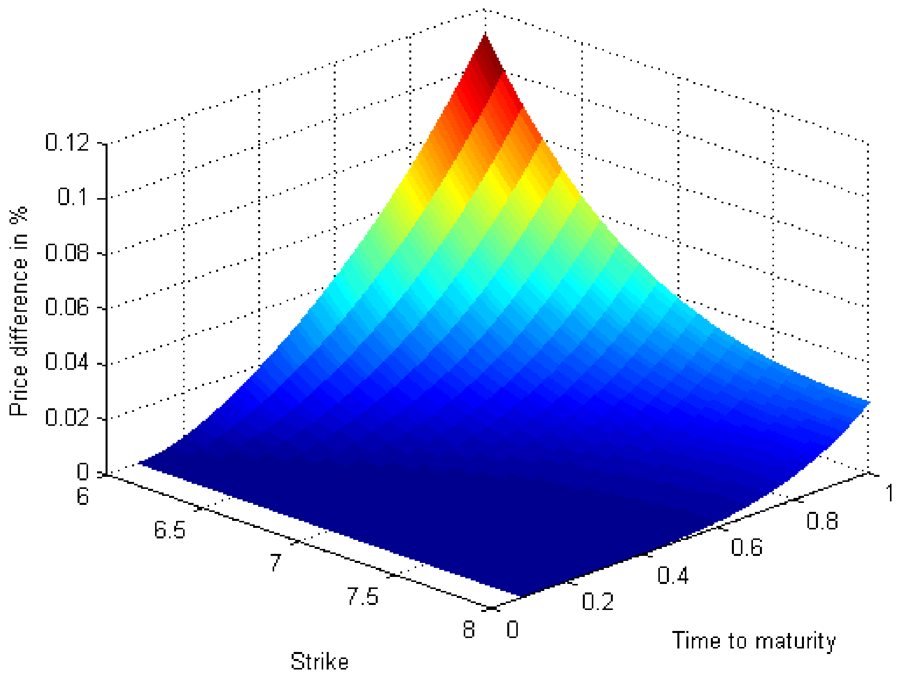


Figure 5 – Price discrepancies (%) between vanilla call and barrier call (analytical)

Picture 4 and 5 confirms the intuition and indicate an overall lower price for the barrier option for all strikes and maturities. The biggest difference was found with strikes near barrier and longer maturities. For shorter time to maturities, the prices were similar. We investigate whether this is the same for barrier call computed numerically. The discrepancies between the vanilla call and the down-and-out barrier call, computed using Crank-Nicholson discretization, are found in figure 6.

Compared to figure 3 and 4, we can see that the discrepancy between the vanilla call and the barrier call, computed using Crank-Nicholson discretization method, is as heaviest when dealing with out of money strikes and strikes near the barrier. In addition, the discrepancies reveal an increasing effect when time to maturity is increased. The largest discrepancies in percent were found for out-of-the money strike around $K = 8$, where both of the options exhibit a price near zero. Once again, the vanilla call priced higher than the barrier call in general except for shorter time to maturities, where the barrier call priced similar to the vanilla one. This is intuitive, as the barrier feature of the barrier call diminishes, i.e. it is less likely for the barrier to be hit when time to maturity decreases, hence priced much more similar as a vanilla call.

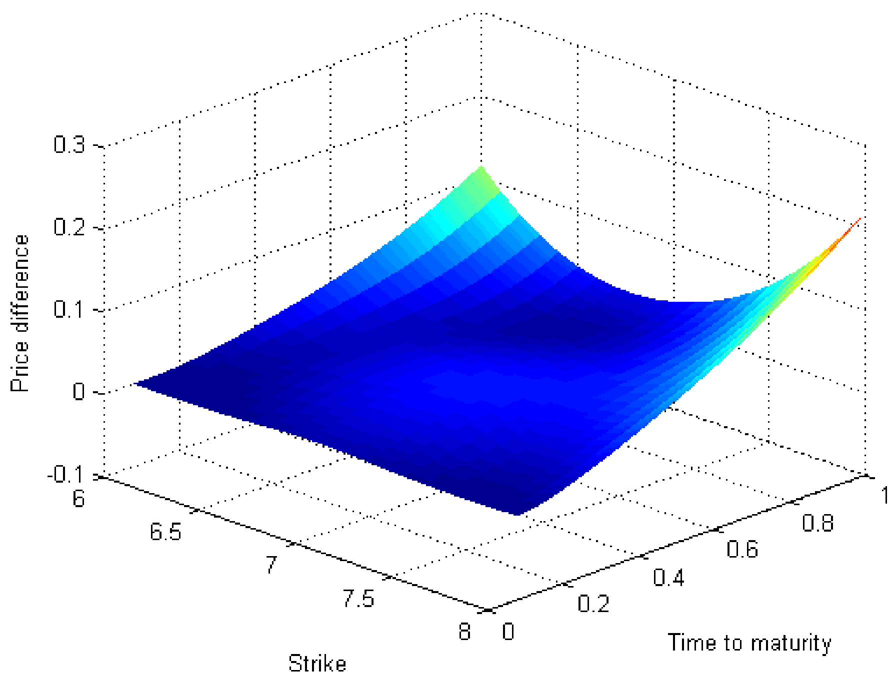


Figure 6 – Price discrepancies between vanilla call and barrier call (Crank-Nicholson)

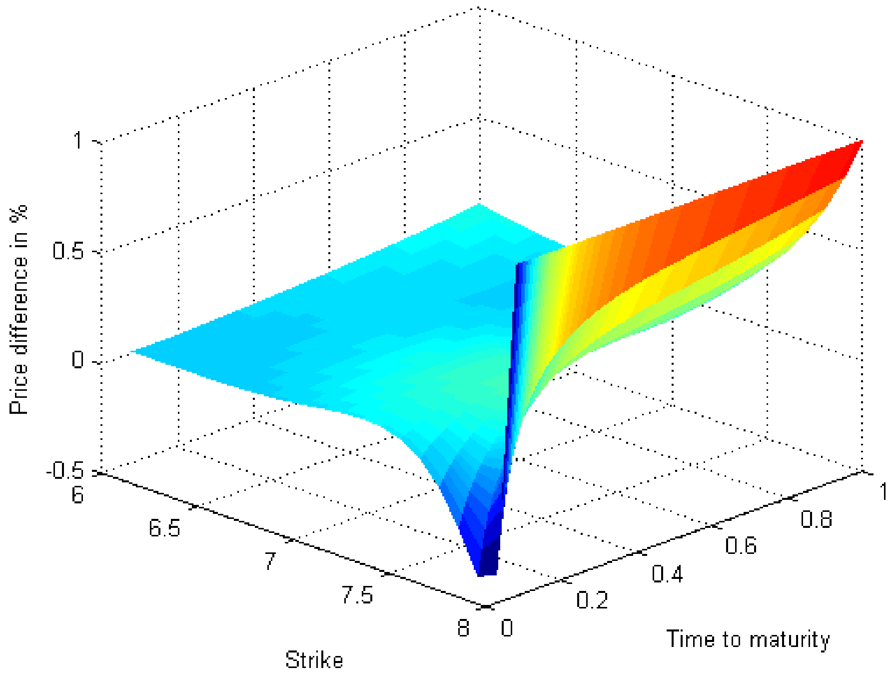


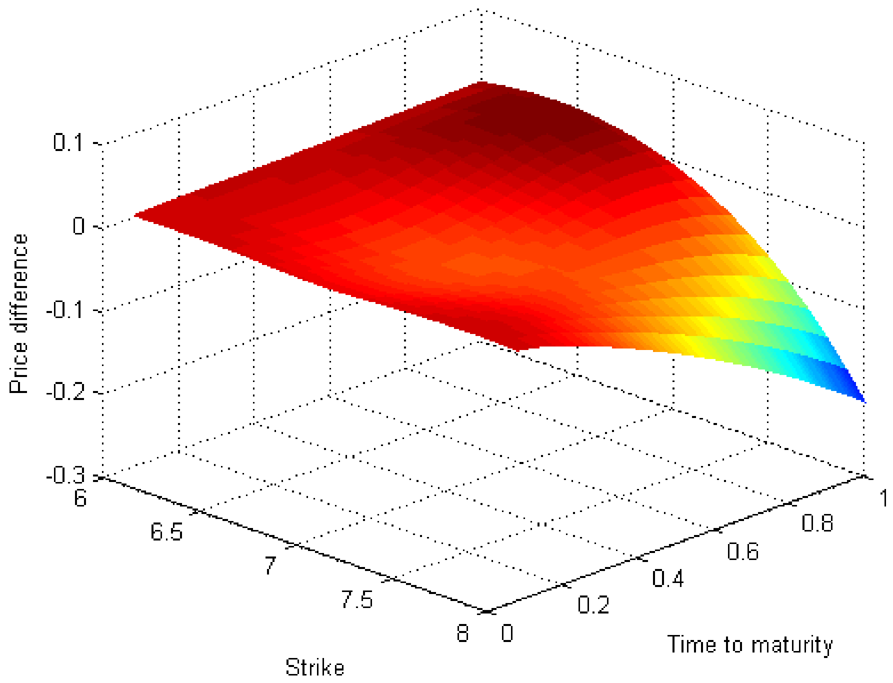
Figure 7 – Price discrepancies (%) between vanilla call and barrier call (Crank-Nicholson)

Finally we plot the discrepancies between the analytically computed barrier call and the Crank-Nicholson method computed barrier call, using

$$Discrepancy = Barrier(numerical) \; - \; Barrier\;(analytical)$$

$$Discrepancy\;in\;\% = \frac{Barrier(numerical) - \; Barrier\;(analytical)}{Barrier(numerical)}$$

The discrepancies are found in figure 8 and figure 9. The overall discrepancies are small, especially for strikes near the barrier. Once again the largest discrepancies were found for large out of the money strikes, where the options have prices near zero.



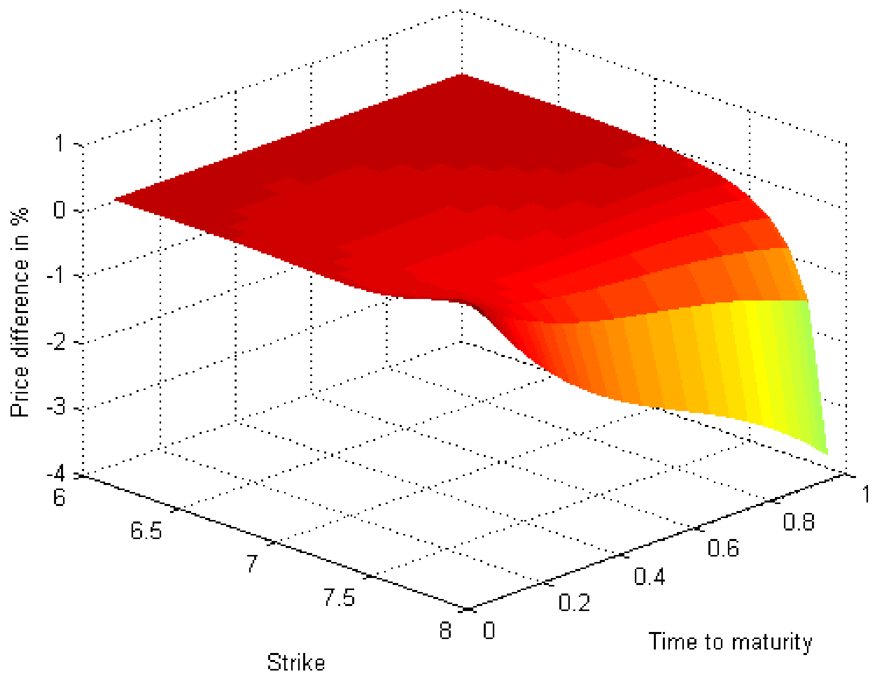


Figure 8 and 9 – Price discrepancies and discrepancies in (%) between barrier call (analytical) and barrier call (Crank-Nicholson).

The implementation of vanilla and barrier calls using the analytical formula in practice was easier and more time efficient than the numerical method by Crank-Nicholson. Before the implementation of Crank-Nicholson, we also tried to compute the prices using the implicit and explicit discretization method, which are both easier to implement. However we experienced severe stability issue with the explicit method and customizations had to be made for boundary conditions, time and asset steps settings. Yet the effect was not satisfying to gain a descent result. The implicit method showed had impaired stability when performing the computation over more steps, but required less time than the Crank-Nicholson method.

Conclusion

This paper has priced barrier options using analytical formula and numerically implementing finite difference method. For the implementation of finite difference method, we needed a volatility that is consistent with the observed market prices of vanilla options and attains completeness of the underlying asset model. This ensures an arbitrage free price and creates foundation for hedging purposes. Dupire (1994) proposed such a model choice, relating the market implied volatility, market observed vanilla price to what is called local volatility. For the implementation of local volatility, a functional form of the implied volatility is presumed. For this, we used the observed discrete implied volatility by the vanilla options for USD/SEK, and fitted the data into a functional form proposed by Duma & Whaley (1998).

The implemented local volatility is then used for the computation of down-and-out barrier calls, using Crank-Nicholson discretization method with finite difference approach. We compared the Crank-Nicholson barrier prices with barrier options calculated analytically, using implied volatility. In addition, we added vanilla option prices into our study.

Overall, the pricing discrepancies between the numerical and the analytical barrier prices are small except for strikes for which we have set as our upper limiting strike, $K = 8$. When comparing with vanilla calls, both the numerically and analytically computed barrier call prices were lower, as expected, except for religions with short maturities, where the barrier call priced similar as a vanilla call, explained by a diminishing barrier feature for the barrier call, reflected by the underlying price has less time for dynamic and hitting the barrier eventually.

We found that the Dupires formula, equation (2.3), were hard to implement in practice, sensitive for the discrete data points and functional model choice. Regarding the pricing implementation using finite difference methods, the stability played a prominent factor. We found that Crank-Nicholson were the most stable one compared to explicit and implicit discretization, although this stability is paid back in computation time. The process from local volatility implementation to option pricing using Crank-Nicholson discretization method was much more complicated than implementing the analytical formula for barrier options. However, it produced a price close to the analytical one, it should be the model to take, as the local volatility produces arbitrage free prices and making static hedging possible, which is very important from a risk management perspective.

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