Risk modeling and pricing of Euribor futures and options using the Ho-Lee model

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Abstract

The main purpose of this master thesis, commissioned by NASDAQ OMX is to study the risk neutral valuation of Fixed Income futures contracts and Fixed Income futures options. More specifically Euribor instruments were studied since they are the most liquid short rate contracts in Europe and hence have the best price picture. Specifying the dynamics of the short rate process to the Ho-Lee model we show the analytical formula for pricing Euribor futures contracts. The valuation method for pricing Euribor futures options is based on restricting the evolution of the short rate process to that of a recombining binomial tree. Also, the historical method for computing Value-at-Risk measure was implemented for portfolios consisting of Euribor futures and Euribor futures options.

Keywords: Euribor futures, Euribor futures options, HJM model, Ho-Lee model, calibration, Value-at-Risk, backtesting.

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Chapter 1

Introduction

As the number of interest derivatives has increased in a dramatic way over the decades, models describing the way interest rates can evolve have become vital for pricing interest derivatives and for pricing the risk associated with interest rate derivatives. The subsequent thesis is dedicated to euro-futures in the HJM framework with constant volatility which is equivalent to Ho-Lee short rate model.

Whereas the original Ho and Lee paper [6] used a binomial setting, Heath, Jarrow, Morton (1990) could describe the limit behavior of the model which implies normally distributed interest rates.

We will show how to implement the Ho-Lee term structure model to price the Euribor futures and Euribor futures options and how to calculate the risk measure of portfolios consisting of these instruments.

Euribor futures options have several features that make them attractive for use in this study. The Euribor futures options market is one of the most attractive option markets for short-term interest rate-sensitive securities. The underlying deliverable asset, a 90-day Euribor time deposit, is essential a pure discount bond and is not complicated by coupon effects. Euribor futures contracts are settled in cash; thus there is no need to determine the cheapest-to-deliver security. Moreover, Euribor futures options contracts and Euribor futures contracts expire concurrently.

Our main task is to determine the Value at Risk (VaR) for portfolios consisting of Euribor futures and Euribor futures options.

How do we estimate portfolios VaR? Most of difficulties are related to pricing. Pricing is the key to VaR estimation; it follows that if we can price positions, then we can also estimate their VaRs.

To estimate the VaR of a fixed-income position, we need to simulate the distribution of possible values of a fixed-income portfolio at the end of the holding period. Our results suggest that if we are dealing with a portfolio of Euribor futures with short maturity, then a terminal term structure provides us with enough information to price the instruments and, hence, value the portfolio at the end of the holding period. The term structure information is sufficient because our findings suggest that for short maturities the futures contracts are equivalent to forward contracts, then we can value this instruments using standard pricing methods.

However, if we are dealing with interest rate options, such as Euribor futures call options, then the information only about the term structure of spot rates will not be enough to value our instruments. To value positions involving options, we need information about terminal volatilities as well. Moreover, in this circumstance, we can no longer price our instruments using simple present-value methods; we need an appropriate option-pricing model.

In the thesis we restrict the evolution of the short-rate to the binomial Ho-Lee model and we present the algorithm for pricing Euribor futures call options. The problem we study then is: how to obtain the terminal volatilities and how we can forecast them.

This paper is organized in the following way. In Chapter 1 we present the basic definition used in the thesis. The short-rate model is described in Chapter 3. The arbitrage free pricing formula of Euribor futures is given in Chapter 4, as well as the analytical formula when the dynamics of the short rate process is specified to the Ho-Lee model. In Chapter 5 we present the results from calibration to market data and the results of pricing Euribor futures options when we calibrate the model to Euribor futures market prices. This thesis also covers the topics of risk measures in Chapter 6 as well as backtesting. Description of backtesting results are provided in Chapter 7. Finally, in Chapter 8, is presented a short summary of the thesis, conclusions and a few suggestions on topic for further research.

Chapter 2

Financial Instruments

2.1 Forwards and Futures

In a forward contract, one party agrees to deliver a specified amount of a specified commodity to the other at a specified date in the future at a specified price. Futures are similar to forwards in all, except two respects:

- Futures are trade on organized commodity exchanges.
- A forward contract involves only one cash flow, at the maturity of the contract, while the futures contracts generally require interim cash flow prior to maturity.

The most important consequence of the restriction of futures contracts to organized exchange is the radical reduction of credit risk by introducing a clearinghouse as the counterparty to each contract and provides a guarantee of performance.

Clearinghouses bring other advantages as well, such as consolidating payment and delivery obligations of participants with positions in many different contracts. In order to preserve these advantages, exchanges offer only a limited number of contract types and maturities. For example, contracts expire on fixed dates that may or may not coincide precisely with the needs of participants.

In order to guarantee performance while limiting risk to exchange members, the clearing house require performance tie from each counterparty. At the initiation of a contract, both counterparties put up initial margin to cover potential default losses. Each day, at the day's closing price, one counterparty will have gained and the other will have lost a precisely offsetting amount. The loser for that day is obliged to increase his margin account and the gainer is permitted to reduce his margin account by an amount, called variation margin, determined by the exchange on the basis of the change in the futures price. Both counterparties earn a short-rate term rate of interest on their margin accounts.

At expiration, counterparties with a short position are obliged to make delivery to exchange, while the exchange is obliged to make delivery to the long.

2.1.1 Euribor Futures and Options

After the introduction, in 1999, of the Euro, European banks establish a new interbank reference rate within the Economic and Monetary Union (EMU) zone. Euribor or Euro Interbank Offer Rate is the rate at which Euro interbank term deposits are offered by one prime bank to another prime bank within EMU zone.

Euribor options and their underlying asset, Euribor futures, are the derivatives of 3-month Euro denominated short-term interest rate traded in LIFFE (London International Financial Futures and Options Exchange) [16]. Euribor futures and options have some important features similar to Eurodollar futures and options.

Definition 1 (Euribor Futures) *A 3-month Euribor futures is a contract to engage in a three months loan deposit of face value* $1,000,000$ *Euro.*

However, all Euribor futures contracts are settled in cash.

Even if the underlying of Euribor futures is an interest rate, the quoted futures prices are expressed in currency units.

If Q is quoted price today for Euribor futures with expiration date T, then the implied Euribor futures interest rate, q is given as

$$q = 100 - Q.$$

The implied futures rate q is quarterly compounded annualized rate expressed in percentage, that can be locked today on a Euribor deposit to be made at time T for a period of 90 days. As a result the contract price is equal to

$$10000 \left[100 - \frac{1}{4} (100 - Q) \right]. \tag{2.1}$$

A minimum price movement of a Euribor futures is 0.5 basis points called a tick size. Thus, futures prices cannot be any positive number, the price must be rounded off to the nearest tick. The value of one tick is the increment in simple interest resulting from a rise of 0.5 basis point

$$1000000 \cdot 0.00005 \cdot \frac{90}{360} = 12.5 \text{ Euro}$$

For example, if the Euribor futures settlement price changes from 95 to 95.50 (a change of 50bp) then the holder of the long position in one futures contract will gain $50 \cdot 12.5 = 625$ Euro.

Euribor futures have a finite life and trades on a quarterly expiration cycle: March, June, September and December (usually denoted by the symbols H, M, U and Z respectively).

Definition 2 (Euribor Futures Option) *Euribor futures option grant the holder the right to purchase or sell a 3-month Euribor futures contract at the strike price prior to or at the option expiration date. So, the options are of American style. One option covers one futures contract and expires at the settlement date of the underlying Euribor futures contract.*

The options on futures are not subject to margining themselves. If the option is exercised, one enter into a futures trade at the price equal to the strike (and no cash is exchanged). The value resulting from exercise comes through the margining of the futures. However the options are still subject to initial margin.

Prices of exchange traded options are expressed in currency units. The tick size is not a major issue with futures prices, but can be quite important for option prices, particularly prices of deep out-of-the-money options with prices close to zero. The price of such an option, if rounded off to the nearest basis point may be zero, close to half, or close to double its true market value. This in turn can violate no-arbitrage conditions on option prices. It can also lead to absurdly high or low, or even undefined, implied volatilities.

2.2 Definitions

In this chapter we present the main definitions that will be used throughout thesis.

Definition 3 (Zero-coupon bond) *A zero coupon bond with maturity date T, is a contract which guarantees the holder one unit of currency to be paid on the date T. The price at time t, $0 \leq t \leq T$ of a bond with maturity T is denoted by $P(t, T)$, and $P(T, T) = 1$.*

The bond price process $P(t, T)$, $T \in [0, T]$ is strictly positive. Now we can define a number of interest rates.

Definition 4 (Yield to maturity) *The continuously-compounded spot interest rate prevailing at time t for the maturity T is a constant rate called yield to maturity such that*

$$P(t, T) = e^{-Y(t,T)(T-t)}, \ \forall t \in [0, T] \,. \tag{2.2}$$

Definition 5 (Instantaneous forward rate) *Let $f(t, T)$ be the forward rate at date t for instantaneous risk-free borrowing or lending at time T. Given $f(t, T)$, the zero-coupon bond prices are then defined by*

$$P(t, T) = e^{-\int_t^T f(t,u)du}, \ \forall t \in [0, T] \,. \tag{2.3}$$

Definition 6 (Short rate) *Short term interest rate is the rate for risk free borrowing or lending at time t over the infinitesimal time interval $[t, t + dt]$.*

$$r(t) = \lim_{T \to \ t} f(t, T) \tag{2.4}$$

2.2.1 Euribor Rates

Definition 7 (Spot Euribor(LIBOR) Rate) *The spot λ-Euribor rate, denoted as $L_\lambda(t)$ is the rate that is offered at time t for a Euro time deposit for a maturity of $360 \cdot \lambda$ days.*

The Euribor rate is defined as the add on yield and are quoted on annual basis. Thus,

- at time t invest one unit
- at time $t + \lambda$ get $1 + \lambda \cdot L_\lambda(t)$

From above argument we get the following equality

$$1 = p(t, t + \lambda)(1 + \lambda L_\lambda(t))$$

which is equivalent to

$$L_\lambda(t) = \frac{1}{\lambda}\left[\frac{1}{P(t, t+\lambda)} - 1\right],\ 0 \le t \le T \tag{2.5}$$

In particular, for a 3-month Euribor, $\lambda = 1/4$ and the above bookkeeping relationship simplifies to

$$L_{1/4}(t) = 4\left[\frac{1}{P(t, t+1/4)} - 1\right],\ 0 \le t \le T$$

Definition 8 (Euribor Forward Rate) *The forward rate $L_\lambda(t, T)$ is the interest rate that is available at time t for a riskless loan that begins at time T and which is paid back at time $T + \lambda$.*

As Euribor spot rates the Euribor forward rates are defined as the add on yield and are quoted on annual basis. So, if we invest one unit at time T we get $1 + \lambda L_\lambda(t, T)$ at time $T + \lambda$. Since, this is a forward contract at time t we do not change any money. So, the present value at time t of this cash flows are equal

$$P(t, T) = P(t, T + \lambda)(1 + \lambda \cdot L_\lambda(t, T))$$

from where we get

$$L_\lambda(t, T) = \frac{1}{\lambda}\left[\frac{P(t, T)}{P(t, T+\lambda)} - 1\right]. \tag{2.6}$$

Chapter 3

Interest Rate Models

3.1 The Heath, Jarrow and Morton Model

The forward rate model of Heath, Jarrow and Morton (HJM) is a general framework to model the evolution of the interest rate curve.

Under the HJM framework is assumed that, for any fixes maturity T the dynamics of the forward rate $f(t,T)$ is given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \tag{3.1}$$

$W = (W_1, W_2, ..., W_N)$ is an N dimensional Brownian motion and $\sigma(t,T) = (\sigma_1(t,T), ..., \sigma_N(t,T))$ is a vector of adapted process and $\alpha(t,T)$ is an adapted process. For simplicity we assume $N = 1$.

Heath, Jarrow and Morton (1992) proved that in order for a unique equivalent measure to exist the drift $\alpha(t,T)$ in the dynamics 3.1 is uniquely determined by the volatility coefficient $\sigma(t,T)$

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds.$$

Then the dynamics of forward rate is given by

$$df(t,T) = (\sigma(t,T)\int_t^T \sigma(t,s)ds)dt + \sigma(t,T)dW_t^Q \tag{3.2}$$

The volatility function $\sigma(t,T)$ completely determine the risk-neutral drift of the forward rate process.

We note that the HJM framework refers to a class of models. A particular HJM model is only specified once $f(0,T)$ and $\sigma(t;s)$ for $0 < t < s$ have been specified.

3.2 The Ho and Lee Model

Under the HJM model the volatility function can be freely specified. We consider volatility function $\sigma(t,T) = \sigma$ as a constant function. Then the HJM equation (3.2) becomes

$$df(t,T) = \sigma \int_t^T \sigma ds + \sigma dW_t^Q$$

or

$$df(t,T) = \sigma^2(T-t)dt + \sigma dW_t^Q \tag{3.3}$$

Integrating stochastically over time the SDE (3.3) we get

$$f(t,T) = f(0,T) + \sigma^2\left(Tt - \frac{1}{2}t^2\right) + \sigma dW_t^Q$$

as $T \to t$ and since $f(t,t) = r(t)$ we get

$$r(t) = f(0,t) + \frac{\sigma^2 t^2}{2} + \sigma W_t^Q \tag{3.4}$$

which gives the SDE

$$dr(t) = \theta(t)dt + \sigma dW_t^Q. \tag{3.5}$$

where

$$\theta(t) = \frac{\partial}{\partial T} f(0,t) + \sigma^2 t$$

The Equation (3.5) is known as the Ho-Lee model in continuous time under risk-neutral measure. One of the reasons for the interest in Ho-Lee model is that it is very tractable analytically.

As is indicated in [12] the dynamics (3.3) implies that the only possible movement of the yield curve in the Ho-Lee model are parallel shifts.

The zero-coupon bond is the basic contract which we want to price, we need the zero-coupon bond dynamics under Q martingale measure. The relationship between zero-coupon bond volatility and forward rate volatility is given by (A.2) in Appendix A. We find that the volatility of zero-coupon bond with maturity T is given by

$$B(t,T) = \int_t^T \sigma ds = \sigma(T-t).$$

Since, under the risk-neutral measure Q the drift of a zero-coupon bond is equal to the short-term interest rate [3, page 184], we obtain the following stochastic differential equation for the dynamics of zero-coupon bonds:

$$dP(t,T) = r(t)P(t,T)dt + \sigma(T-t)P(t,T)dW_t^Q$$

Using expression (3.4) for $r(t)$ we can write

$$dP(t,T) = \left(f(0,t) + \sigma W_t + \frac{\sigma^2 t^2}{2}\right) P(t,T)dt + \sigma(T-t)P(t,T)dW_t^Q \quad (3.6)$$

From above SDE we see that the prices of zero-coupon bonds follow a log-normal distribution. Usinf Formula 2.3 we find the solution of the SDE 3.6

$$\begin{aligned} P(t,T) &= e^{-\int_0^t f(t,s)ds} \\ &= e^{-\int_0^t \left(f(0,s)+\sigma^2(st-\frac{1}{2}t^2)+\sigma W_s^Q\right)ds} \\ &= P(0,T)e^{-\int_0^t \sigma W_s ds} e^{-\frac{1}{2}\sigma^2 Tt(T-t)} \end{aligned} \quad (3.7)$$

This formula can be used to price interest rate derivatives, such as Euribor futures, caps and floors in closed form.

Moreover, the price of zero-coupon bond, as mentioned in [12, page 396] can be expressed as

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-\frac{\sigma^2}{2}Tt(T-t)-\sigma(T-t)W_t^Q}. \quad (3.8)$$

From above equality follows that the bonds of different maturities are perfectly correlated.

3.3 The Binomial Ho-Lee Model

Although, for the Ho-Lee model exists analytical solution for discount bonds and for Euribor futures, in order to price options on Euribor futures we need to represent the term structure model by numerical approximation, since the options are of American type.

Therefore we will present in this section a binomial tree-building procedure for the Ho-Lee short rate model. In the original paper [6] Ho and Lee presented the model as a numerical procedure.

Assume the short term interest rate Ho-Lee model in continuous time evolves as

$$dr(t) = \mu(t)dt + \sigma dW_t^Q \tag{3.9}$$

where $\mu(t)$ is the drift, σ is the instantaneous volatility of the short rate and W_t is a Q Brownian motion.

Since the increments of the Brownian motion are normal distributed with mean 0 and variance dt. We can write

$$dW_t^Q = \sqrt{dt}dz_t$$

where $dz_t \sim N(0,1)$. Hence, the Ho-Lee short-rate model assume normal distributed short rates.

Consider a discrete economy. With $N+1$ trading dates: $t_0, t_1, ..., t_N$, and N trading periods. The trading period is assumed constant $t_k - t_{k-1} = \Delta t$.

A numerical approximation for SDE (3.9) is given by

$$r_{k+1} - r_k = \mu_k \Delta t + \sigma\sqrt{\Delta t}\epsilon_k \tag{3.10}$$

where

$$dz_{t_k} \approx \Delta z_k = \epsilon_k\sqrt{\Delta t}, \epsilon_k \sim N(0,1)$$

and

$$dr \approx r(t_{k+1}) - r(t_k) = r_{k+1} - r_k$$

when

$$dt = \Delta t$$

and r_k is the interest rate from t_{k-1} to t_k, $k = 1, ...N$. Without loss of generality, assume that $\Delta t = 1$ and $t_0 = 0$. Then relation (3.10) becomes

$$r_{k+1} = r_k + \mu_k + \sigma\epsilon_k$$

This yield,
r_1 short rate from t_0 to t_1 is deterministic,
r$_2 = (r_1 + \mu_1) + \sigma\epsilon_1$

$r_3 = r_2 + \mu_2 + \sigma\epsilon_2 = (r_1 + \mu_1 + \mu_2) + \sigma(\epsilon_1 + \epsilon_2)$
$\vdots$
$r_N = (r_1 + \mu_1 + \mu_2 + ... + \mu_{N-1}) + \sigma\,(\epsilon_1 + ... + \epsilon_{N-1})$

So, the short rate process in discrete time under Q martingale measure is

$$r_k = \theta_k + \sigma \sum_{j=2}^{k} \epsilon_j \tag{3.11}$$

with $\theta_1 = r_1$ and

$$\theta_k = (r_1 + \mu_1 + \ldots + \mu_{k-1}), \text{for } k = 2, \ldots, N$$

By approximating normal distributes random variable ϵ_k by a binomial random variable b_k such that

$$b_k = \begin{cases} 1 & \text{with probability } 0.5, \\ -1 & \text{with probability } 0.5 \end{cases}$$

we obtain the binomial Ho-Lee short rate model. Choosing probability of success of binomial random variable 0.5 means that at each node an upstate or downstate is equally likely to occur. Then the formula (3.11) becomes

$$r_k = \theta_k + \sigma \sum_{2}^{k} b_j, k = 1, 2, \ldots, N \tag{3.12}$$

Having specified the dynamics of r_k we can calculate θ_k such that the current term structure is replicated which will assure that the model is arbitrage free.

An evolution of the short rate that precludes arbitrage must satisfy following condition

$$P(0,T) = E^Q\left[e^{-\int_0^T r(s)ds}\right]$$

which in discrete time is equivalent to

$$P(0,T) = E^Q\left[e^{-\sum\limits_{k=1}^{N-1} r_k}\right]$$

The value of zero-coupon bond at time $t = 0$ and maturity t_k is given by (the derivation of the formula is given in [10])

$$P(0,T) = P(0,t_k) = e^{-\sum\limits_{i=1}^{k} \theta_i} \prod_{i=1}^{k-1} \cosh(\sigma i).$$

By taking the quotient of this prices at two consecutive maturity dates t_k and t_{k+1} we obtain

$$\frac{P(0,t_{k+1})}{P(0,t_k)} = e^{-\theta_{k+1}} \cosh(\sigma k)$$

solving for θ_{k+1}

$$\theta_{k+1} = \ln\left(\frac{P(0,t_k)}{P(0,t_{k+1})}\right) + \ln(\cosh(\sigma k)) \tag{3.13}$$

In the Ho-Lee model the initial term structure is taken as given, and is assumed to shift over time so that profitable arbitrage opportunities fail to occur.

Since, today's zero-coupon bond prices come as input in the model, we cannot use the Ho-Lee short rate model to find today price zero-coupon bond prices.

3.4 Calibration

A term structure model has to be calibrated before we can use it for valuation purpose.

The Ho-Lee model takes the current term structure as given, the set $\{P(0,t)\}_{t\geq 0}$ comes as a input in the model, by using (3.13) we determine the first parameter θ.

A second input is the volatility of the short rate, σ which is assumed constant. How we can calculate the volatility parameter?

First approach is to infer short-rate volatility by observing historical short-rates.

The second approach is to estimate short rate volatility based on the observed prices of interest rate derivatives. The volatility calculated using this approach is called *implied volatility*.

Because an implicit estimate contains more information than an historical estimate in assessing the interest rate's volatility expectation, implicit volatility is adopted to prices contingent claims in this thesis.

Thus we choose σ as solution to equation

$$P_{observed}(0,T) = P_{model}(0,T)$$

where $P_{observed}$ is the market price of interest rate derivatives; and P_{model} is the corresponding model price.

The interest rate derivatives used for calibration can be market prices of Euribor futures options or Euribor futures.

It is worth noticing here that when we perform this calibration at the start of the period it will fit the current term structure, but as the time goes on, it will not fit the term structure any more. So then we will need to recalibrate the model. As a result, the model should be recalibrated very often.

3.5 Disadvantages and Advantages of the Model

Ho and Lee's interest rate model retains the distinction of being the first no arbitrage model that can be calibrated to market data.

Advantages:

- It reproduces the current zero coupon yield curve.
- It can be fitted very accurately to the initial term structure and its implementation is relatively straight forward, based on a lattice structure.
- Derivatives prices can be computed quickly. This is very important for risk-management purposes when many securities need to be priced frequently.

Disadvantages:

- It can generate negative short rate.
- It is sensitive to its parameters in the sense; a small change in parameter values may result in a large change in bond price.
- Another disadvantage of the Ho-Lee model follows from the fact that is assumes that the volatility of the short rate is constant in time as well as deterministic.
- Since Ho-Lee model is one factor model, it has the undesirable property that that the yield-curve performs a one-dimensional random motion. In general, yield-curve deformations are more complicated.
- The Ho-Lee model needs to be recalibrated often.

Chapter 4

Pricing Euribor Futures and Options

4.1 Arbitrage Free Euribor Futures Price

In this section we want to answer the question: What is the fair Euribor futures price? And "fair" price we mean in arbitrage free sense.

Let T be the settlement date of the futures contract. The Euribor futures price at maturity T is set to be

$$100 - (\text{3-month Euribor at T})$$

Or equivalently at expiration date T the futures price is

$$F_\lambda(T;T) = 100\left(1 - L^F_\lambda(T,T)\right)$$

where $\lambda = \frac{1}{4}$.
We note that at expiration date T the Euribor futures rate coincide to forward rate and spot rate: $L^F_\lambda(T,T) = L_\lambda(T,T) = L_\lambda(T)$.

At time $t < T$ the futures price is defined by

$$F_\lambda(t;T) = 100\left(1 - L^F_\lambda(t,T)\right) \tag{4.1}$$

where $L^F_\lambda(t,T)$ is the corresponding futures rate and $\lambda = 1/4$.

To simplify the writing we introduce the following notation $E^Q_t[X] = E^Q[X\,|\mathcal{F}_t]$, where $\mathcal{F}_t$ represent all the information available at time t.

Definition 9 (Euribor Futures Rate) *The λ-Euribo futures rate $L_\lambda^F(t,T)$ is given by the martingale representation formula*

$$L_\lambda^F(t,T) = E_t^Q[L_\lambda(T)] \tag{4.2}$$

To motivate the above definition we think as follow: Under risk neutral measure the expected return of all assets should be equal to riskless rate. As a result any zero investment portfolio must provide a zero expected gain (or loss), regardless of the risk it contains, under the risk neutral measure.

Consider initiating a position in a futures contract at any time t and holding it for infinitesimal interval Δt. Since initiating a futures position is always costless, the time t investment is zero. Under the assumption of continuous marketing to market, the expected change in the contract position must be zero under risk neutral measure, or:

$$\begin{aligned} L_\lambda^F(t,T) &= E_t^Q[L_\lambda(T)] \\ &= E_t^Q\left[\frac{1}{\lambda}\left(\frac{1}{P(T,T+\lambda)} - 1\right)\right] \\ &= \frac{1}{\lambda}\left(E_t^Q\left[\frac{1}{P(T,T+\lambda)}\right] - 1\right) \end{aligned}$$

(4.3)

Substituting Equation (4.3) in relation (4.1)we get the price of the Euribor futures contract with maturity T at time t

$$F_\lambda(t;T) = 100\left(1 - \frac{1}{\lambda}\left[E_t^Q\left[\frac{1}{P(T,T+\lambda)}\right] - 1\right]\right) \tag{4.4}$$

or we can write it as

$$F_\lambda(t;T) = 100(1 + \frac{1}{\lambda}) - \frac{100}{\lambda}E_t^Q\left[\frac{1}{P(T,T+\lambda)}\right] \tag{4.5}$$

Thus, the time t risk-neutral expectation of the inverse of the zero coupon bond

$$E_t^Q\left[\frac{1}{P(T,T+\lambda)}\right] \tag{4.6}$$

plays a key role in the valuation of the Euribor futures as well Eurodollar, Euroyen, Short Sterling ect. futures. The result of expectation (4.6) will depend on the interest rate model we are using.

As mention is [15] under the Ho-Lee model the above expectation is determined as

$$E_t^Q\left[\frac{1}{P(T,T+\lambda)}\right] = \frac{P(t,T)}{P(t,T+\lambda)}e^{\frac{\sigma^2}{8}(T-t)(T-t+1/2)}.$$

And the price of Euribor futures contract at time t with maturity T is given by:

$$F_\lambda(t;T) = 100(1+\frac{1}{\lambda}) - \frac{100}{\lambda}\frac{P(t,T)}{P(t,T+\lambda)}e^{\frac{\sigma^2}{8}(T-t)(T-t+1/2)} \tag{4.8}$$

Finally, the American call option value can be obtained by

$$C_t = \max\left\{C_T e^{-\int_t^T r(s)ds}, F_\lambda(t,T) - K, 0\right\} \tag{4.9}$$

where C_t and C_T are the call option prices at time t and T respective; $F_\lambda(t,T)$ is the Euribor futures price observed at time t with maturity at time T; K is the strike price; $r(s)$ is the spot interest rate at date s.

American options give the holder the right to exercise the option at any point of time before the expiration date T. The early exercise feature makes the price of an American option greater than or equal to the price of the corresponding European option. Early exercise opportunities arise when the intrinsic value of the option (i.e., the value of the option exercised immediately) becomes greater than the discounted expected value of holding the option.

4.2 Price of Euribor Futures under the Binomial Ho-Lee Model

According to relation (4.4) in order to compute the Euribor futures prices we need to compute the expectation

$$E_t^Q\left[\frac{1}{P(T,T+\lambda)}\right]$$

The price of zero-coupon bond under the Binomial Ho-Lee model is given by Formula C.2 in Appendix.

$$P(t,T) = P(t_k, t_{k+n}) = e^{-\sum\limits_{i=k+1}^{k+n} \theta_i} e^{-\sigma n \sum\limits_{i=2}^{k+1} b_i} \prod_{i=1}^{n-1} \cosh(\sigma i)$$

We want now to calculate the expected value of the inverse of zero-coupon bond given the information today, let $T = t_k$ and $T + \lambda = t_{k+n}$, then

$$\begin{aligned} E^Q\left[\frac{1}{P(t_k, t_{k+n})}\right] &= e^{\sum\limits_{i=k+1}^{k+n} \theta_i} \left(\prod_{i=1}^{n-1} \cosh(\sigma i)\right)^{-1} E^Q\left[e^{\sigma n \sum\limits_{i=2}^{k+1} b_i}\right] \\ &= e^{\sum\limits_{i=k+1}^{k+n} \theta_i} \left(\prod_{i=1}^{n-1} \cosh(\sigma i)\right)^{-1} (\cosh(n\sigma)) \end{aligned}$$

Then the today price of Euribor futures with maturity at time T is

$$F_\lambda(0;T) = 100\left(1 - \frac{1}{\lambda}\left[e^{\sum_{i=k+1}^{k+n} \theta_i} \left(\prod_{i=1}^{n-1} \cosh(\sigma i)\right)^{-1} (\cosh(n\sigma)) - 1\right]\right).$$

4.3 Pricing Euribor Futures Options using the Binomial Lattice

Suppose that the model is parameterized, and then we can specify the lattice of short rates using relation (3.12).

$$r_n = \theta_n + \sigma \sum_{2}^{n} b_j, n = 1, 2, \ldots, N$$

A movement from time t_n and state i on the lattice to time t_{n+1} and state $i+1$ represents an upward movement in the term structure; and a movement to time t_{n+1} and state i represents a downward movement in the term structure. At each date t_n on the lattice there are $n+1$ states that can occur.

Because the Euribor futures options are American style options, the price of the underlying security, Euribor futures contract, must be determined at each node on the binomial lattice. Euribor futures contracts have a 90 days

Euribor time deposit as their underlying security. In order for the model to price Euribor futures option contracts, we must first find to value the underlying Euribor time deposit and the Euribor futures contract in each state of binomial lattice.

The value of a zero-coupon bond at maturity is equal to its face value in all states, the value of zero-coupon bond at any time t_n and state i prior to its maturity can be determined by discounting the expected value of the bond at time t_{n+1} back one period using the short rate at time t_n and state i of the lattice. Having set binomial probability to 0.5, the value of zero coupon bon at time t_n and state i, $P_{n,i}$ is given as

$$P_{n,i} = \frac{0.5P_{n+1,i+1} + 0.5P_{n+1,i}}{1 + r_{n+1,i}}$$

where $r_{n+1,i}$ is the short rate.

The value of the zero-coupon bond 90 days prior to maturity becomes the maturity value of the Eurodollar futures contract by using Formula (4.4). Once the value of the Euribor futures contract in each state at maturity has been determined, the value of the Euribor futures contract at each time and state prior to maturity can be determined as the expected value of the contract one period into the future. Thus, the value of Euribor futures contract, $F_{n,i}$ at any time t_n and state i prior to maturity, is given as

$$F_{n,i} = 0.5F_{n+1,i+1} + 0.5F_{n+1,i}.$$

In this way the entire lattice of Euribor futures prices can be constructed beginning one period prior to maturity of the futures contract and working backward.

The next step will be to calculate the Euribor futures call option contract prices.

At maturity, the value of a Euribor futures call option, $FO_{m,i}$ is

$$FO_{m,i} = \max(F_{m,i} - K, 0)$$

where m is the maturity of the Euribor futures call option contract and K is the strike price of the option.

Because Euribor futures options can be exercised prior to maturity, the value of the Euribor futures call option at any node on the binomial lattice prior to maturity is equal to the greatest of the value of the option if exercised, the value of the option if held, or zero. Thus, the value of the Euribor futures call option at any time n and state i prior to maturity is given by

$$FO_{m,i} = \max(F_{n,i} - K, 0.5FO_{n+1,i+1} + 0.5FO_{n+1,i}, 0).$$

4.4 Convexity Bias

Convexity bias is defined as the difference between the Euribor futures rate $L^F_\lambda(t,T)$ and Euribor forward rate $L_\lambda(t,T)$

$$Bias_\lambda(t,T) = L^F_\lambda(t,T) - L_\lambda(t,T) \tag{4.10}$$

Using Formula (4.3) and (2.6) for the futures and forward Euribor rate respective we get

$$Bias_\lambda(t,T) = \frac{1}{\lambda}\left(E^Q_t\left[\frac{1}{P(T,T+\lambda)}\right] - \frac{P(t,T)}{P(t,T+\lambda)}\right)$$

The convexity bias can be positive or negative or zero. As mentioned in [7, page 108], for the Ho-Lee model the difference is

$$\frac{1}{2}\sigma^2 T(T+\lambda).$$

Thus when convexity adjustment is zero then the short rate volatility in the Ho-Lee model is zero. We notice that convexity only becomes a meaningful issue in pricing Euribor futures for longer dated futures (in excess of two years) and is greatly affected by interest rate volatility.

Euribor futures give us directly observed futures Euribor rates from actively traded contracts. Both forward and futures Euribor rates are each in essence kinds of proxy for the spot Euribor rate for some future date, and they differ mainly because of the daily 'market to market' conventions that govern futures contracts.

There is a theoretical formula for this bias see Formula B.1 in the Appendix. The formula shows that that the size and sign of convexity bias is determined by the covariance between the Euribor spot rate and a certain discount factor.

Chapter 5

Data and Results

5.1 Calibrate the Model to Market Futures Prices

Our data set consists of the current term structure, the Euribo futures prices and option prices as on February 12, 2010. The current prices of zero coupon bonds are presented in Figure 5.1. All data are extracted from Reuters.

It is worth mentioning here, that the Reuters zero curves are created from the most liquid interest rate instruments that are available: a combination of deposit rates, liquid futures and interest rate swaps.

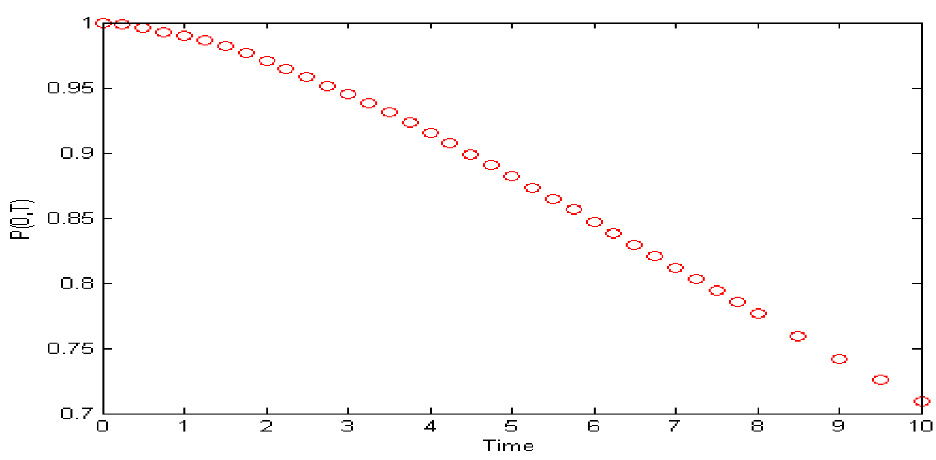

Figure 5.1: *Zero-coupon bond prices.*

Time to Maturity	Days to Expiration	Futures Price	Implied Volatility
March 2010	31	99.325	$0.31830949 \cdot 10^{-7}$
June 2010	122	99.130	$0.50004231 \cdot 10^{-7}$
September 2010	213	98.920	$0.26786242 \cdot 10^{-7}$
December 2010	304	98.710	0.25834754
March 2011	395	98.505	0.12506002
June 2011	486	98.285	$0.80568789 \cdot 10^{-7}$
September 2011	584	98.065	$0.95256770 \cdot 10^{-7}$
December 2011	675	97.840	$0.71703514 \cdot 10^{-7}$

Table 5.1: *Euribor futures market prices as given on February 12, 2010 and respective the short-rate implied volatilities.*

We determine the short-rate volatility by calibrating the model to current futures prices. Having parameterized the model we can use it to price the Euribor futures options.

In Table(5.1) are given the current prices of Euribor futures and the corresponding implied short-rate volatility. As we can notice most of the implied short-rate volatilities are close to zero. Notice, that if $\sigma = 0$, then the futures rates are equal to forward rates.

Further we will consider the call options with the underlying above futures. For each strike price K and maturity T, the market and model option prices are compared. Given the implied short-rate volatility for each Euribor futures, we determine the model prices of Euribor futures options. The gap between model and market prices can thus be found. Figure 5.2 till Figure 5.9 show the plots of the observed call option prices and the model call option prices.

Consider Figure 5.2, it shows the market and the model price of the Euribor futures call option with maturity March 2010. There are 31 days till maturity of the contracts when counting from February 12, 2010. The short-rate volatility which we use in the Binomial Ho-Lee model is given in Table (5.1), $\sigma = 0.31830949 \cdot 10^{-7}$ is very close to zero. It is the volatility which we get when we calibrate the model to Euribor futures with maturity March 2010. We notice that the market and model prices are identical.

If we look at Figure 5.3 to Figure 5.7. The market and the model price of the Euribor futures call option with maturity June 2010, September 2010, December 2010, March 2011 and June 2011 respective. We notice that the

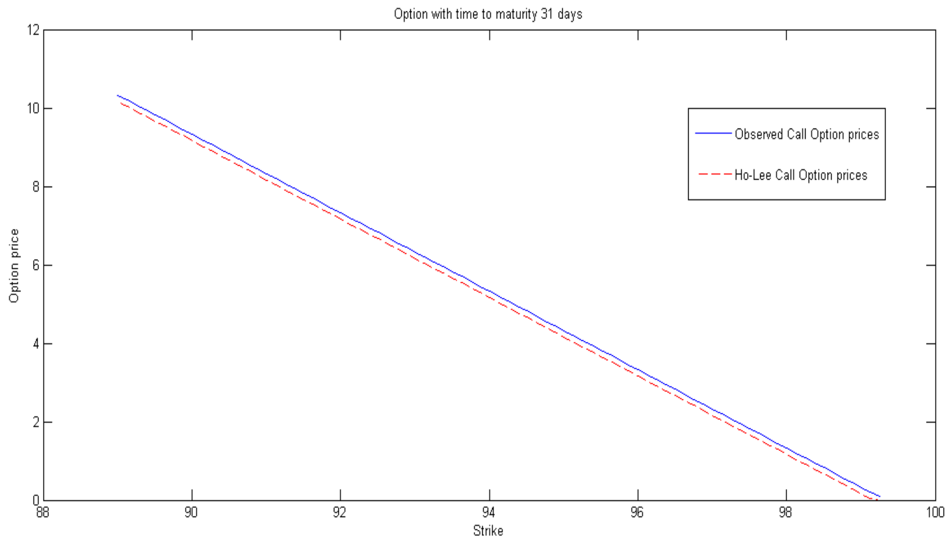

Figure 5.2: *Market options prices versus Ho-Lee model option prices; underling is Euribor futures contract with maturity March 2010.*

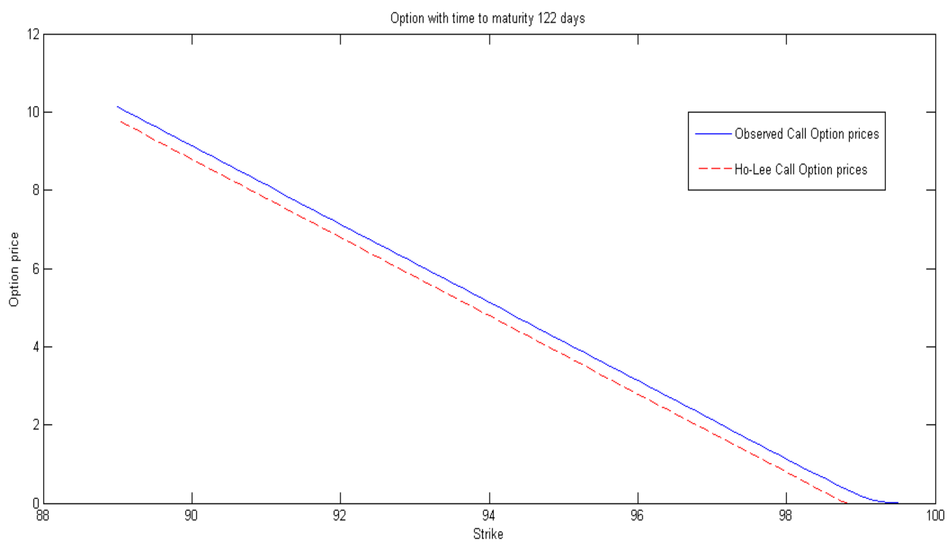

Figure 5.3: *Market options prices versus Ho-Lee model option prices; underling is Euribor futures contract with maturity June 2010.*

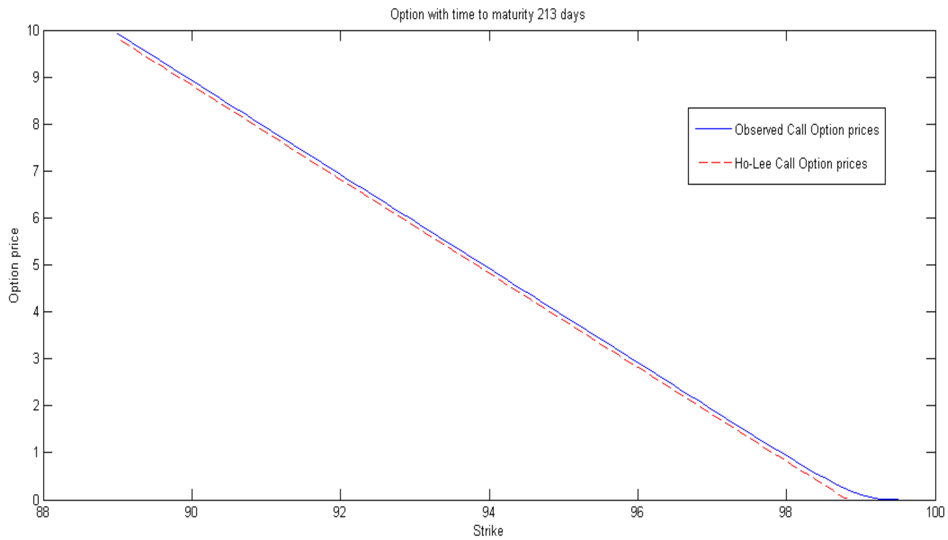

Figure 5.4: *Market options prices versus Ho-Lee model option prices; underling is Euribor futures contract with maturity September 2010.*

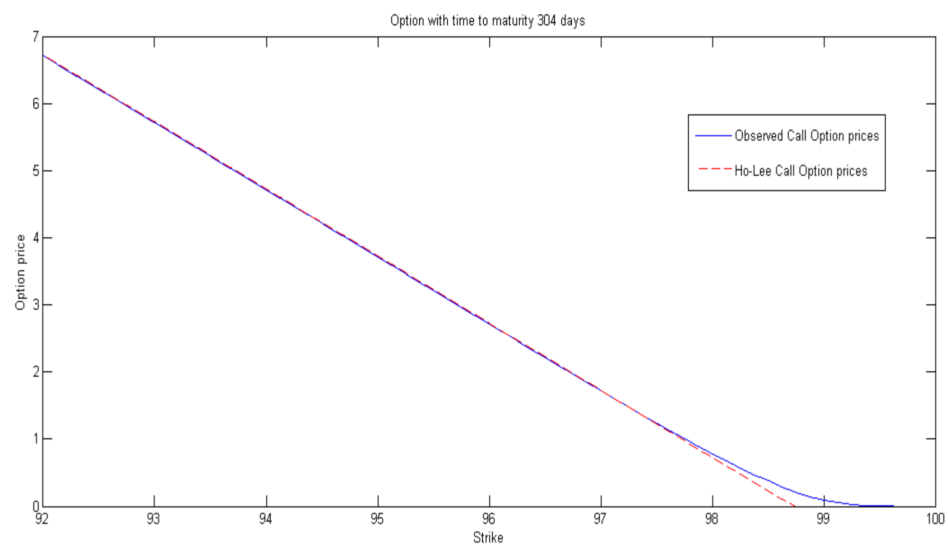

Figure 5.5: *Market options prices versus Ho-Lee model option prices; underling is Euribor futures contract with maturity December 2010.*

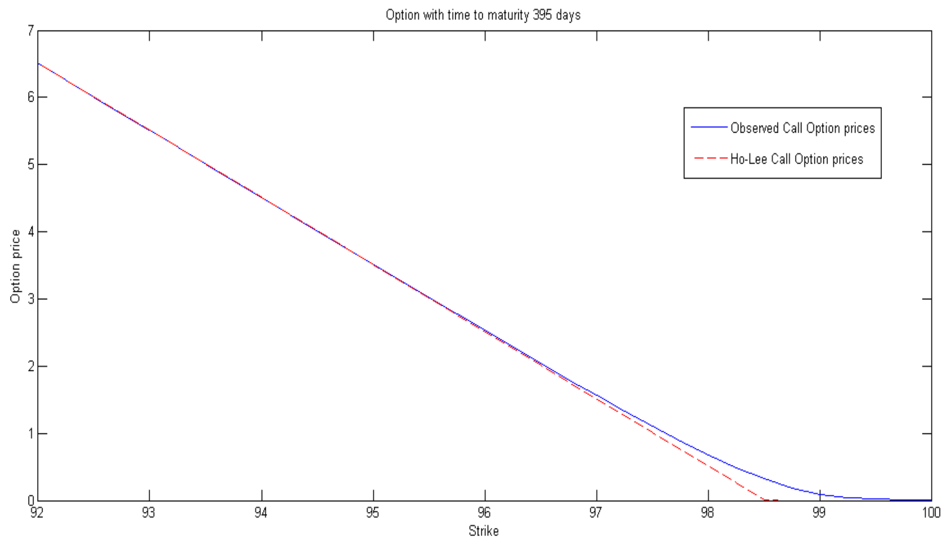

Figure 5.6: *Market options prices versus Ho-Lee model option prices; underling is Euribor futures contract with maturity March 2011.*

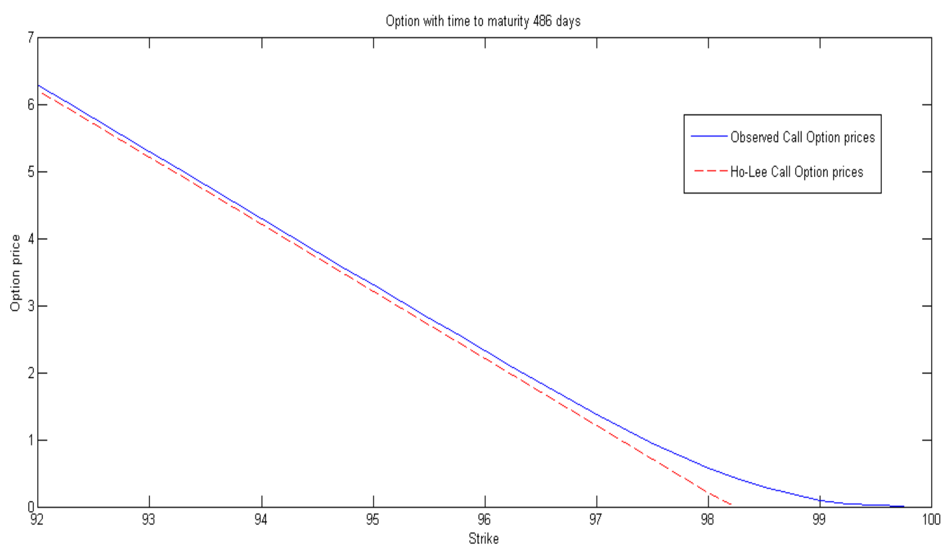

Figure 5.7: *Market options prices versus Ho-Lee model option prices; underling is Euribor futures contract with maturity June 2011.*

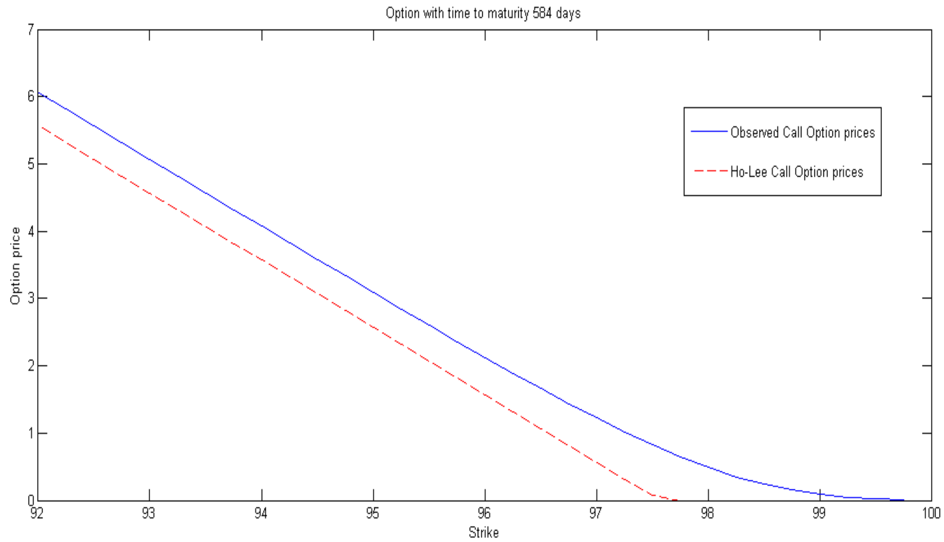

Figure 5.8: *Market options prices versus Ho-Lee model option prices; underling is Euribor futures contract with maturity September 2011.*

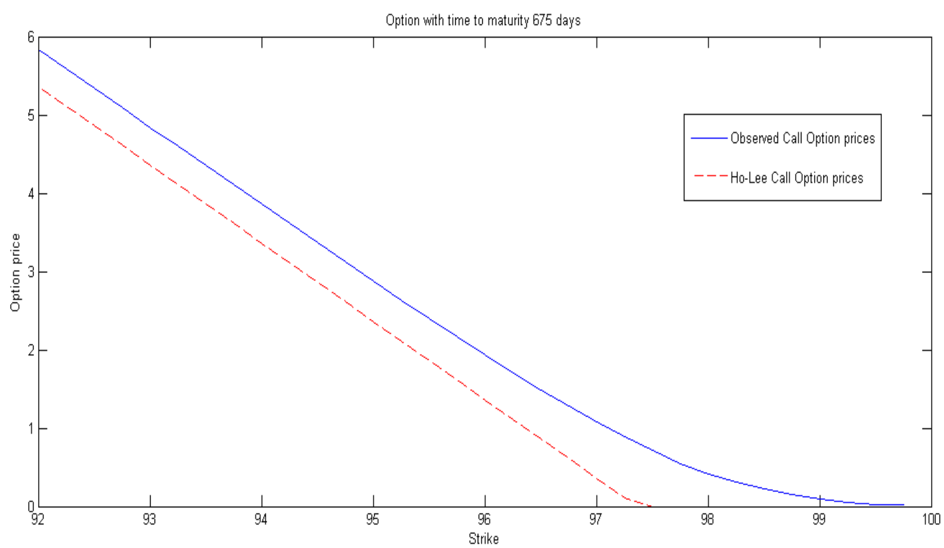

Figure 5.9: *Market options prices versus Ho-Lee model option prices; underling is Euribor futures contract with maturity December 2011.*

market and model prices are very close with few exceptions. In Figure 5.3 we notice a constant spread between prices. In Figure 5.4, Figure 5.6 and Figure 5.7 the difference between market and model prices is increases as the strike approaches Euribor futures price.

Consider Figure 5.8 and Figure 5.9. The market and the model price of the Euribor futures call option with maturity September 2011, December 2011. We see a constant spread between market and model prices.

Conclusions: The above results shows the pricing ability of the Ho-Lee model, when we calibrate the short-rate volatility to market futures prices.

The results indicate that, for short maturities options (up to one year), the pricing of the in-the-money call options generally is better compared with out-of-the-money call option.

One of the explanations can be the lumpiness of the tick size. Recall that for Euribor futures the minimum price movement is 0.5 basis points. Since the prices of deep out-of-the-money options are close to zero. The price of such an option, if rounded off to the nearest basis point may be zero, close to half, or close to double its true market value.

For Euribor futures options with longer maturity, we observe a spread between market and model prices. The results indicate that the short rate volatility is a function of time to maturity and moneyness.

5.2 Calibrate the Model to Market Euribor Futures Call Option Prices

Now we will calibrate the short-rate volatility in Ho-Lee model using observed call options prices as on February 12, 2010.

From above results we notice that the market prices differ mainly from the model prices for out-of-the money call options. As a result we will reduce the number of options strike prices from 65 to 33, since for deep in-the-money call options with short maturity the short-rate volatility implied from Euribor futures call options prices will be equal to short-rate volatility implied from Euribor futures prices.

Here we can make an analogy between the Black-Scholes model for stock prices and the Ho-Lee short-rate model, since both models assume constant volatilities.

However, as everybody agrees the stock volatility is not constant and then it appears that the Black-Scholes model is useless when we want to price options on stocks.

Nevertheless, in practice the Black-Scholes model is not used to price options, instead is used to extract the volatility from observed market data. As a result, traders use the Black-Scholes model to quote option prices in terms of implied volatility. We can recognize the expression *volatility smile*, *buy and sell volatility* and *volatility based trading*.

Theoretically, for options with the same expiration date, we expect the implied volatility to be the same regardless of which strike price we use. However, in reality, the implied volatility we get is different across the various strikes. This disparity is known as the volatility skew.

Analogy when we calibrate the Ho-Lee model to Euribor futures option prices we get the short-rate implied volatility. And we observe in Figure D.3 to Figure D.8, Appendix D how the volatility varies with strike, namely the volatility skew.

The shape of the volatility skew obtained suggests that the implied volatility for call options at the lower strikes is lower than the implied volatility at higher strikes. This suggests that out-of-the-money calls are in greater demand compared to in-the-money calls.

Market prices are determined by traders every day. The models, in particular Ho-Lee model, should be able to match these market prices. We used the calibration methodology to set the parameters in the model so that the models generate market prices. Once the market prices are matched we look at the model and ask what hedge ratios the model tells us. As is the case with the Black-Scholes model, we think that the Ho-Lee model can be used to hedge.

For further research it will be interesting to determine the hedge ratios (delta, gamma) of portfolio consisting of Euribor futures and options contracts when the dynamics of the short-rate is given by the Ho-Lee model.

5.3 Historical Short Rate Volatility

Here we consider the historical implied volatilities.

We pay attention to historical implied volatilities since we want to see if historical data support the statement that: convexity adjustment is zero. Moreover, we want to estimate the Value-at-Risk of portfolios consisting of Euribor futures and of Euribor futures options, we therefore need to simulate the distribution of the possible values of the portfolio. To value position involving futures and options we need information about the short-rate terminal volatility.

5.3.1 Historical Short Rate Volatility Implied from Euribor Futures Prices

Our data set consists of the Euribor futures prices with maturity June 2010, September 2010, December 2010, March 2011, June 2011, September 2011 and December 2011 over the time interval from January 2009 to April 2010. The evolution of the futures prices is shown in Figure 5.10. The evolution of term structure over the considered time period is given in Figure 5.11.

Using the above methodology we determine the implied short-rate volatility when we calibrate the Ho-Lee model to the Euribor futures market prices over given time period. In Figure 5.12 to Figutre 5.18 are given the results.

A popular point of view is that the convexity adjustment defined by (4.10) is close to zero for short maturities Euribor futures rate. And the Euribor futures interest rates can be assumed to be the same as the corresponding forward interest rate.

From time series of implied volatility, we see that as the contract approaches maturity, the frequency of implied volatility being zero increases, see Figure 5.18.

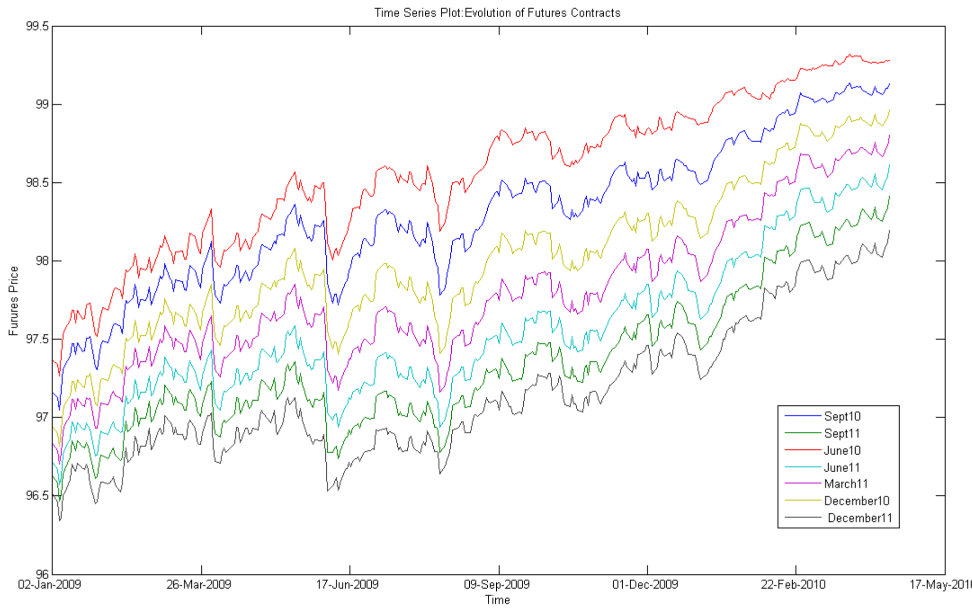

Figure 5.10: *Time series of Euribor futures with different maturities. In the legend is given maturity date of the futures contracts.*

As the results indicate, we conclude that expected value of convexity adjustment is zero and the few exceptions which occur do not depend of maturity of the contracts but rather depend of the uncertainty of the market.

Our results suggest that if we are dealing with a portfolio of Euribor futures (with maturity up to two years) then a terminal term structure provides us with enough information to price the instruments and, hence, value the portfolio at the end of the holding period. The term structure information is sufficient because our findings suggest that for short maturities the futures contracts are equal to forward contracts, then we can value this instruments using standard pricing methods.

The main reasons why we got implied short rate volatility zero is that the Euribor futures are used in construction of zero curve.

We note that Euribor futures should not be used in zero curves to be used for pricing Euribor futures since it effectively becomes a circular reference. They should ideally be replaced with deposits to nine months and swap rates thereafter. However, we suppose that even using this construction will yield

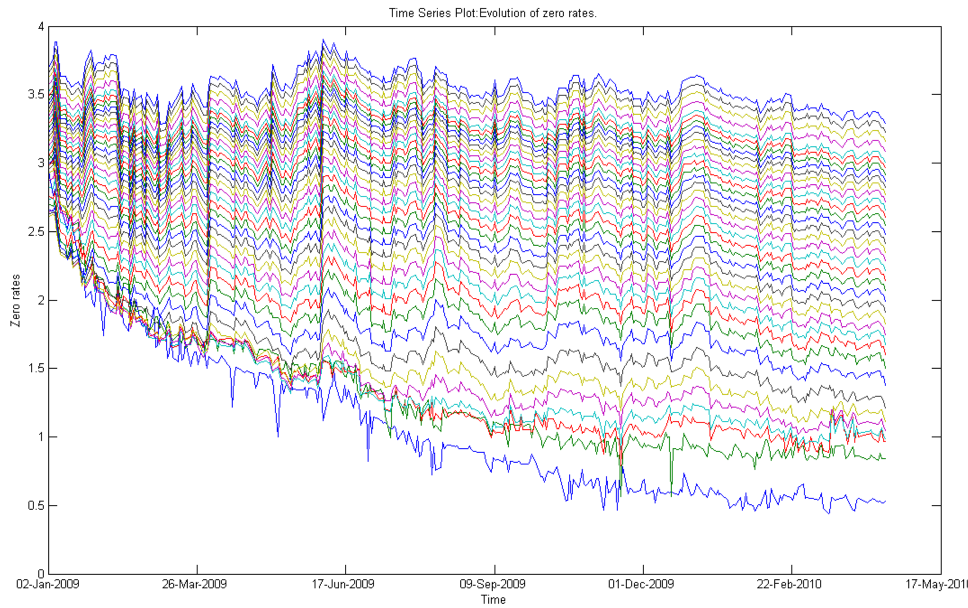

Figure 5.11: *Time series of zero interest rates, from 3 months rate up to 10 years in the future.*

to mixed results for futures with maturity up to two years.

Another possible explanation for this result is that Euribor futures offer an unexplained arbitrage possibility.

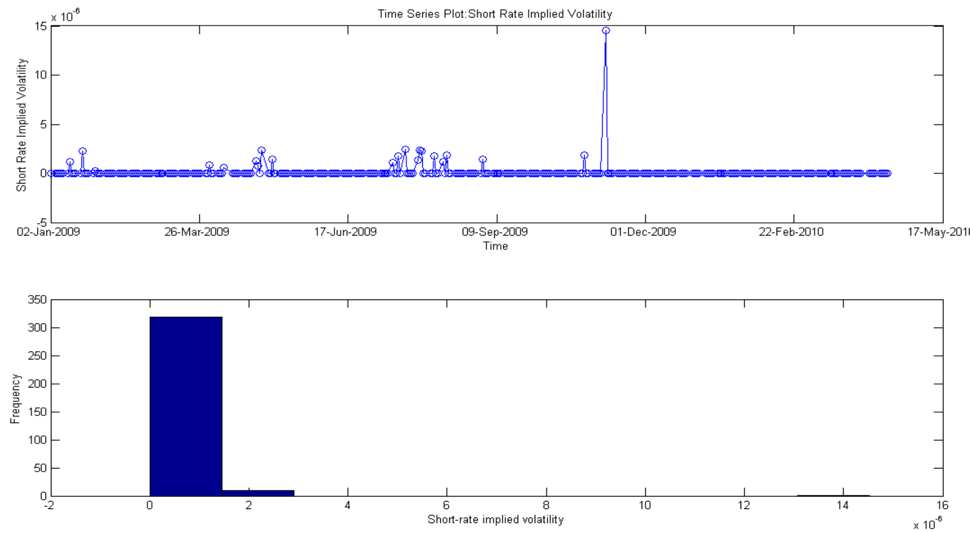

Figure 5.12: *Evolution of short-rate volatility when calibrating the model to market prices of the futures with maturity June 2010 and histogram of the short-rate implied volatility.*

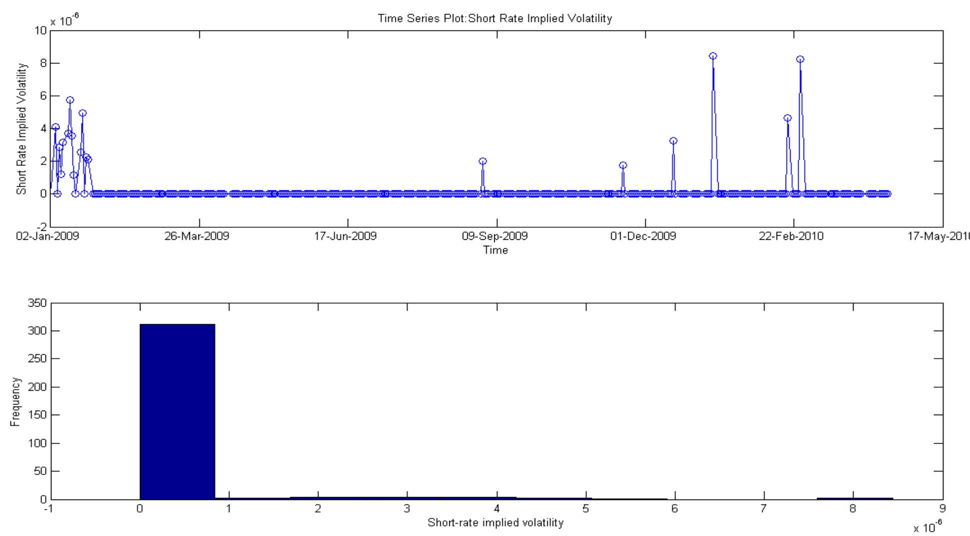

Figure 5.13: *Evolution of short-rate volatility when calibrating the model to market prices of the futures with maturity September 2010 and histogram of the short-rate implied volatility.*

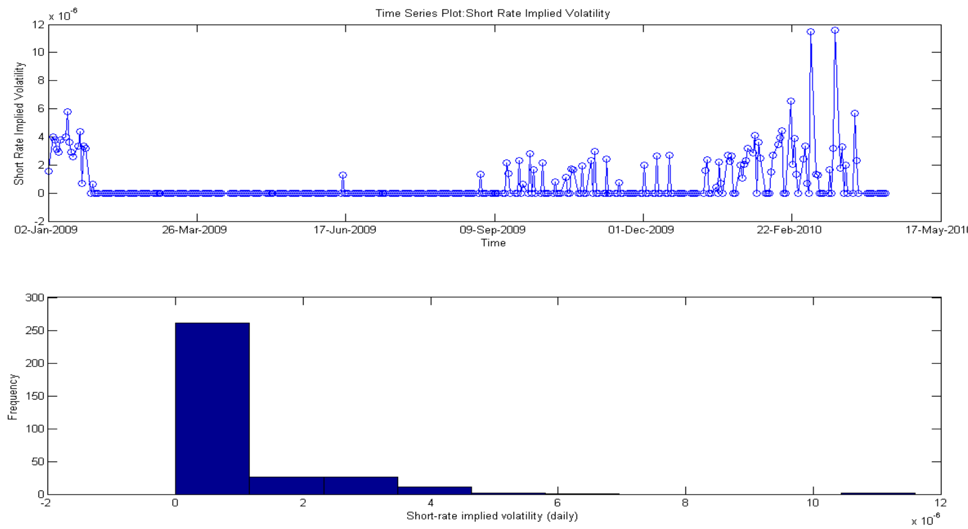

Figure 5.14: *Evolution of short-rate volatility when calibrating the model to market prices of the futures with maturity December 2010 and histogram of the short-rate implied volatility.*

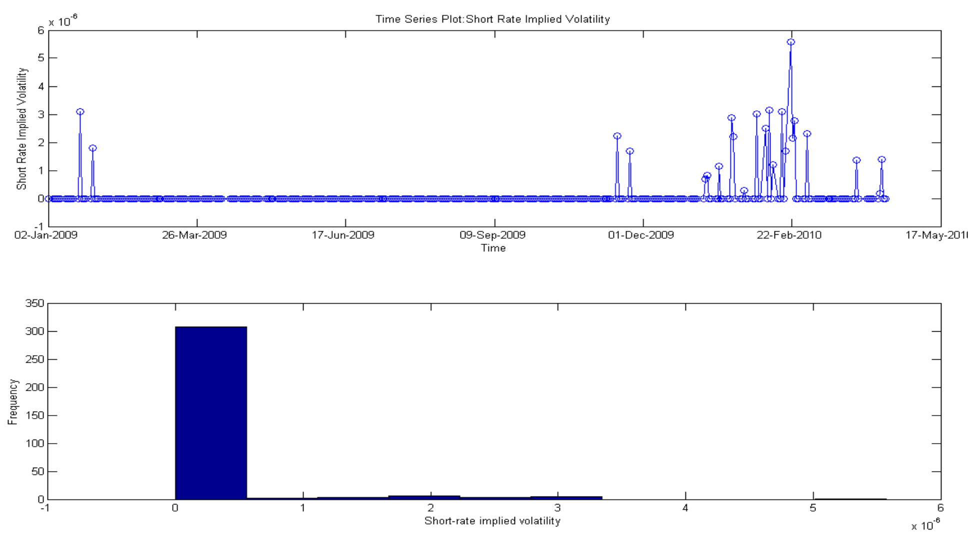

Figure 5.15: *Evolution of short-rate volatility when calibrating the model to market prices of the futures with maturity March 2011 and histogram of the short-rate implied volatility.*

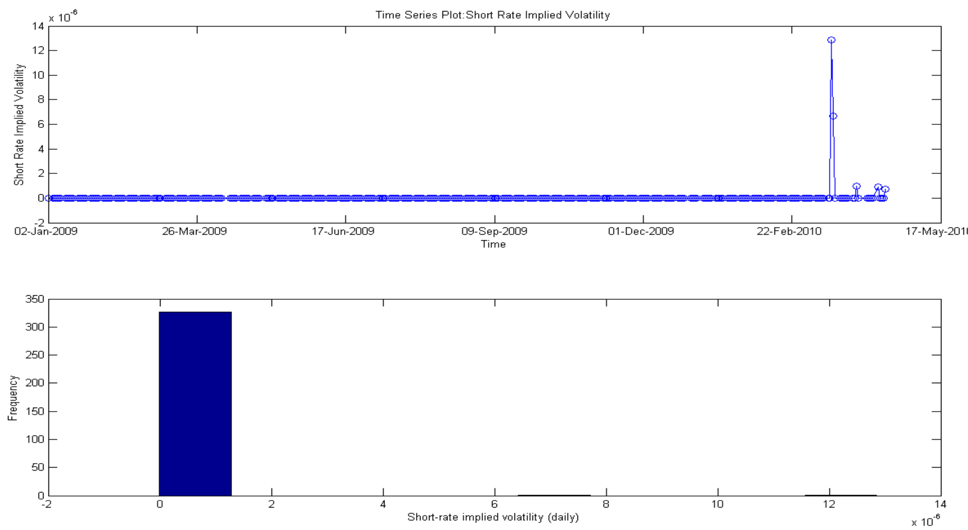

Figure 5.16: *Evolution of short-rate volatility when calibrating the model to market prices of the futures with maturity June 2011 and histogram of the short-rate implied volatility.*

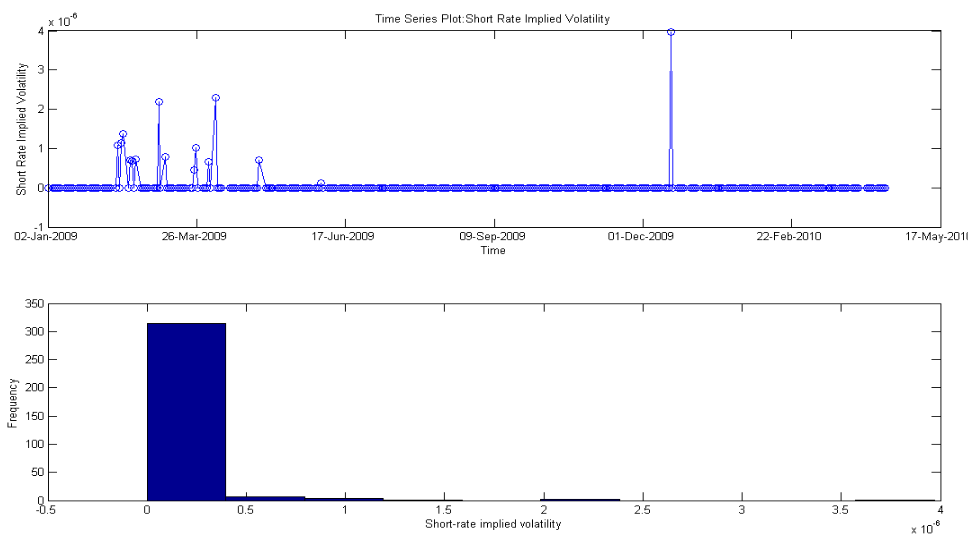

Figure 5.17: *Evolution of short-rate volatility when calibrating the model to market prices of the futures with maturity September 2011 and histogram of the short-rate implied volatility.*

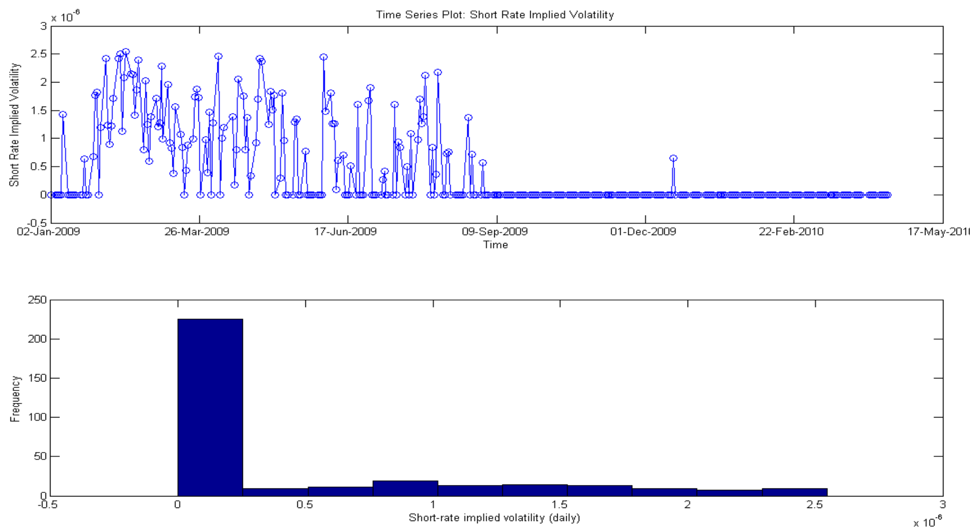

Figure 5.18: *Evolution of short-rate volatility when calibrating the model to market prices of the futures with maturity December 2011 and histogram of the short-rate implied volatility.*

5.3.2 Historical Short Rate Volatility Implied from Euribor Futures Call Options Prices

Now we will calibrate the Ho-Lee model to the historical Euribor futures call option prices. We choose the Euribor futures call option with maturity September 2010. The time interval is from January 2009 to April 2010. The price evolution of the underlying futures over the considered time period is presented in Figure 5.19.

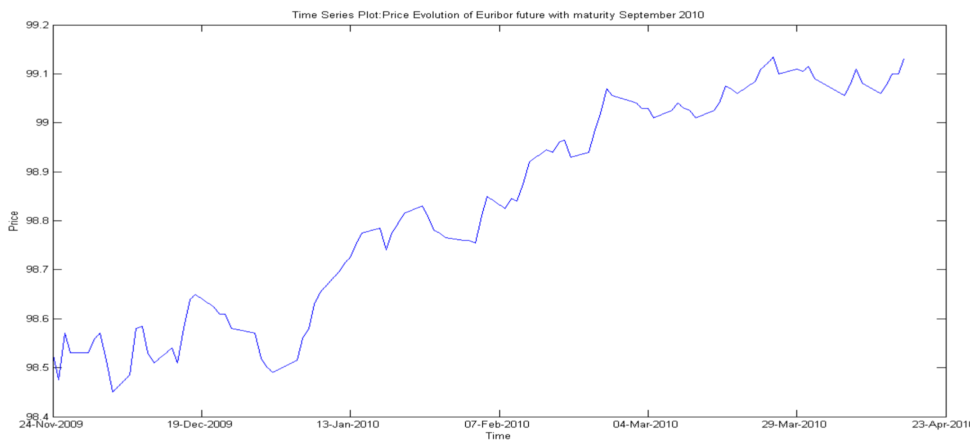

Figure 5.19: *Time series of the Euribor futures with maturity September 2010.*

The results of calibrating the model to Euribor futures call option with maturity September 2010 and different strikes over chosen time period is given in Figure 5.20 to Figure 5.32.

The time series of short-rate volatility implied from deep in-the-money call options, Figure 5.20 and Figure 5.21, show that the implied volatility is rather constant. Thus for deep-in-the-money options, we can forecast terminal short-rate volatilities by assuming that the terminal volatilities are equal to currently prevailing volatilities.

The time-series plots, Figure 5.22 to Figure 5.31 does not support this approach, for forecasting terminal volatilities for this options we need to use GARCH or similar approaches.

In this thesis we forecast terminal short-rate volatilities by assuming that the terminal volatilities are equal to currently prevailing volatilities. In Chapter 6 we use this approach to determine one-day VaR of portfolio consisting of Euribor futures call options.

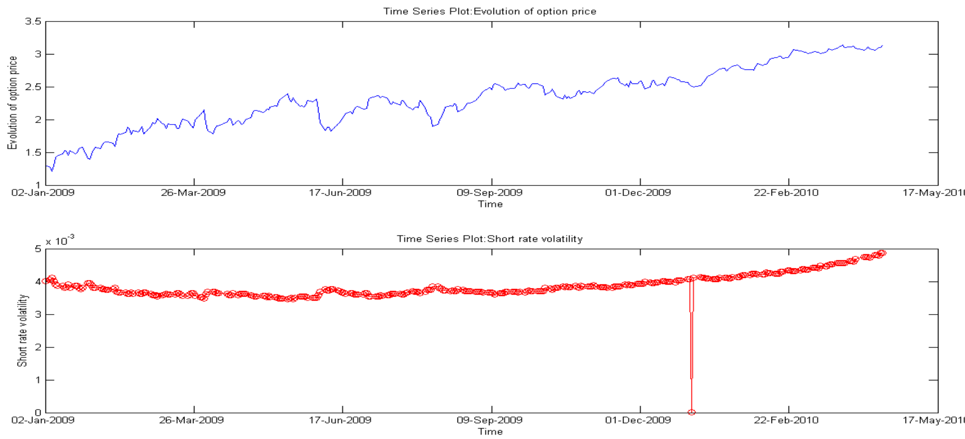

Figure 5.20: *Evolution of the option with Strike* = 96 *and the corresponding historical short-rate implied volatility.*

Figure 5.21: *Evolution of the option with Strike* = 96.50 *and the corresponding historical short-rate implied volatility.*

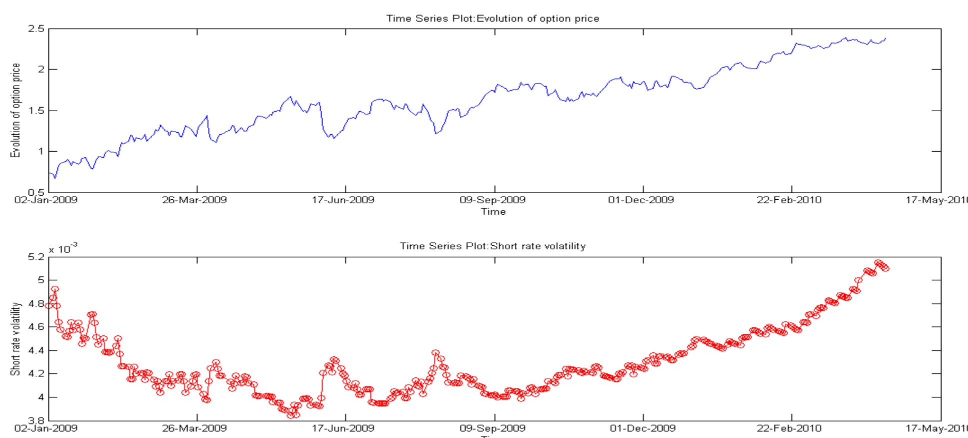

Figure 5.22: *Evolution of the option with Strike* = 96.75 *and the corresponding historical short-rate implied volatility.*

Figure 5.23: *Evolution of the option with Strike* = 97 *and the corresponding historical short-rate implied volatility.*

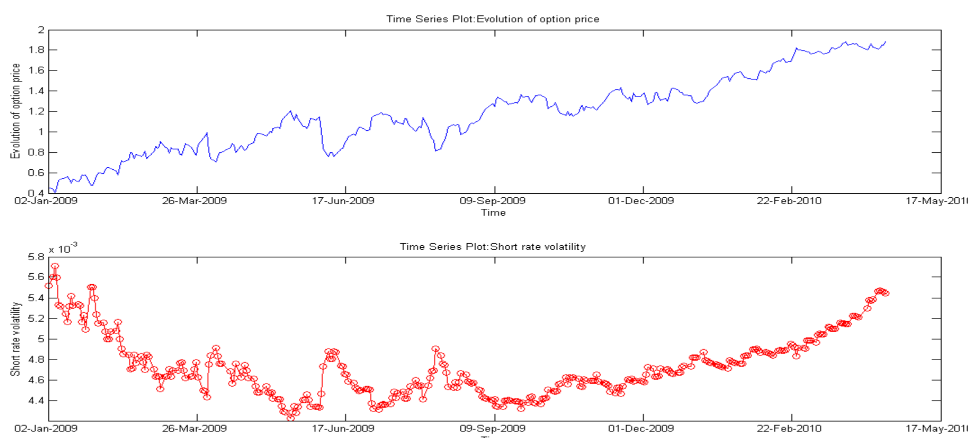

Figure 5.24: *Evolution of the option with Strike* = 97.25 *and the corresponding historical short-rate implied volatility.*

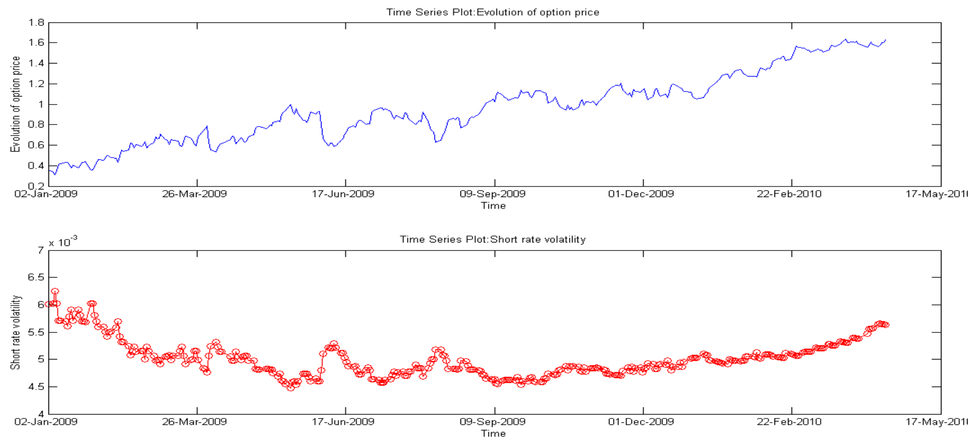

Figure 5.25: *Evolution of the option with Strike* = 97.50 *and the corresponding historical short-rate implied volatility.*

Figure 5.26: *Evolution of the option with Strike* = 97.75 *and the corresponding historical short-rate implied volatility.*

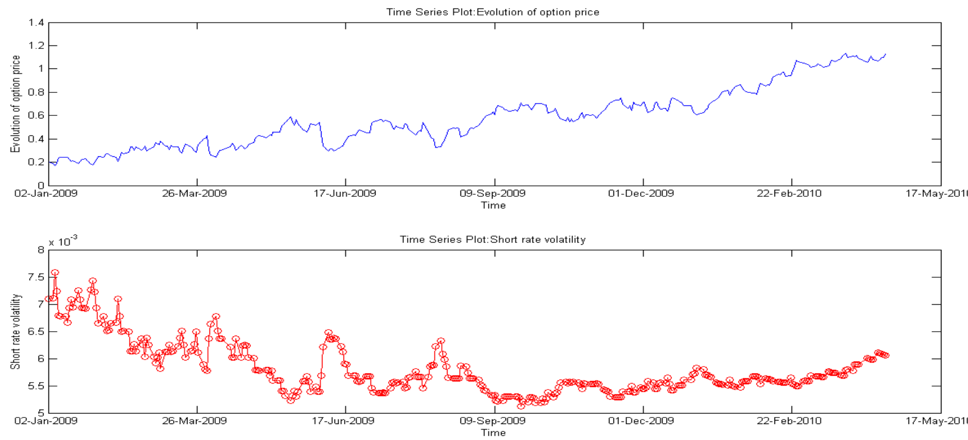

Figure 5.27: *Evolution of the option with Strike* = 98 *and the corresponding historical short-rate implied volatility.*

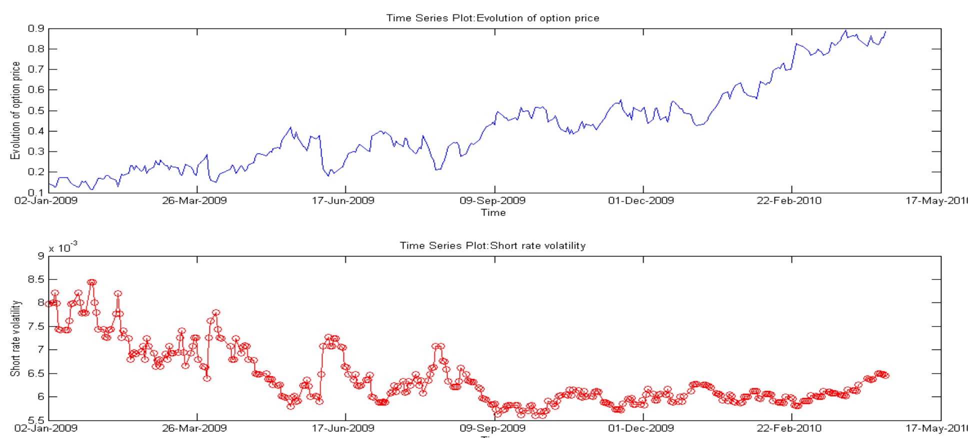

Figure 5.28: *Evolution of the option with Strike* = 98.25 *and the corresponding historical short-rate implied volatility.*

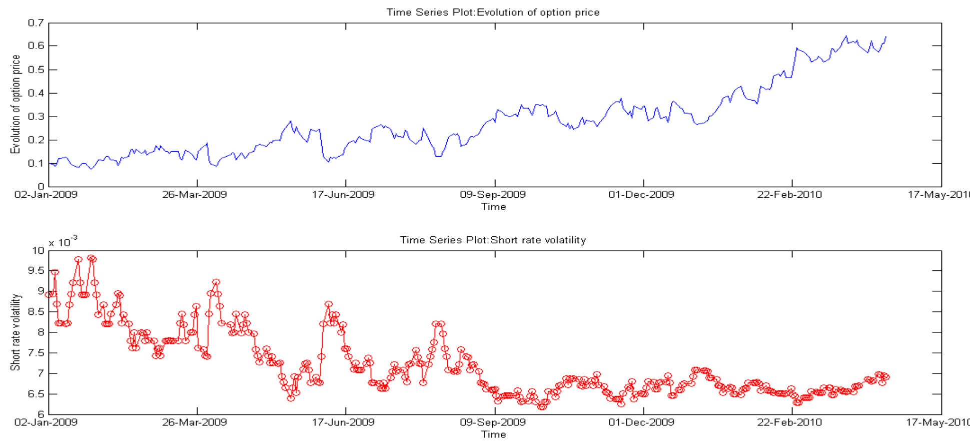

Figure 5.29: *Evolution of the option with Strike* $= 98.50$ *and the corresponding historical short-rate implied volatility.*

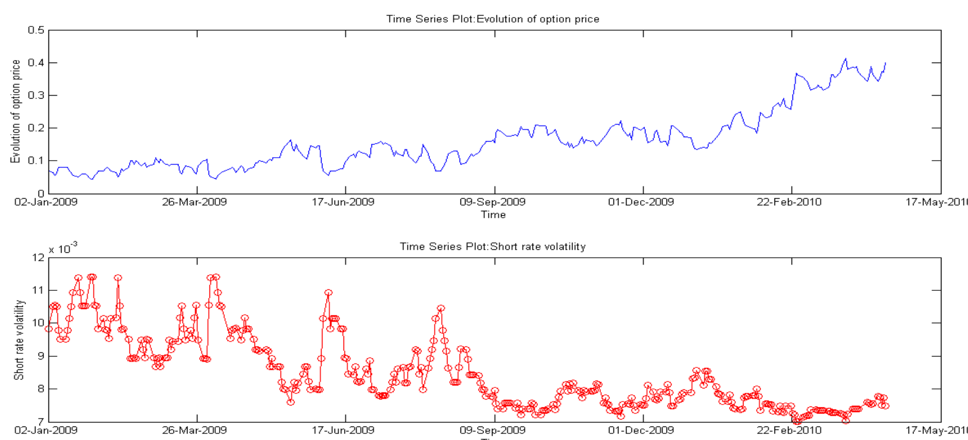

Figure 5.30: *Evolution of the option with Strike* $= 98.75$ *and the corresponding historical short-rate implied volatility.*

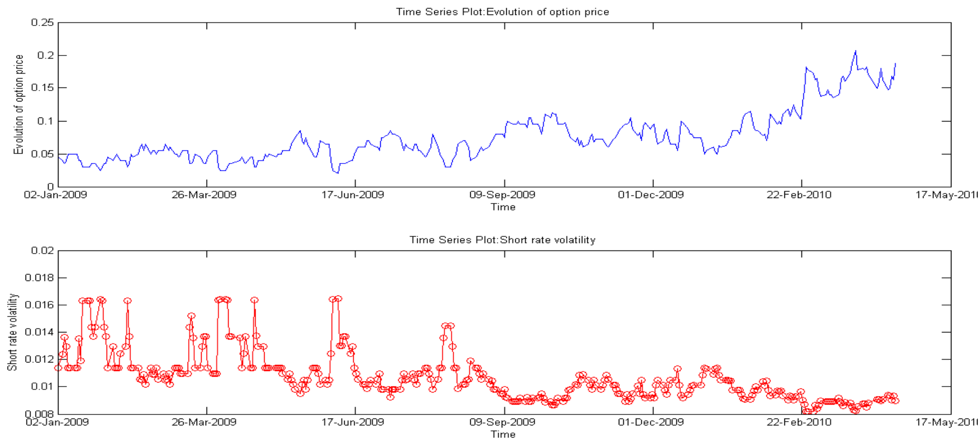

Figure 5.31: *Evolution of the option with Strike* = 99 *and the corresponding historical short-rate implied volatility.*

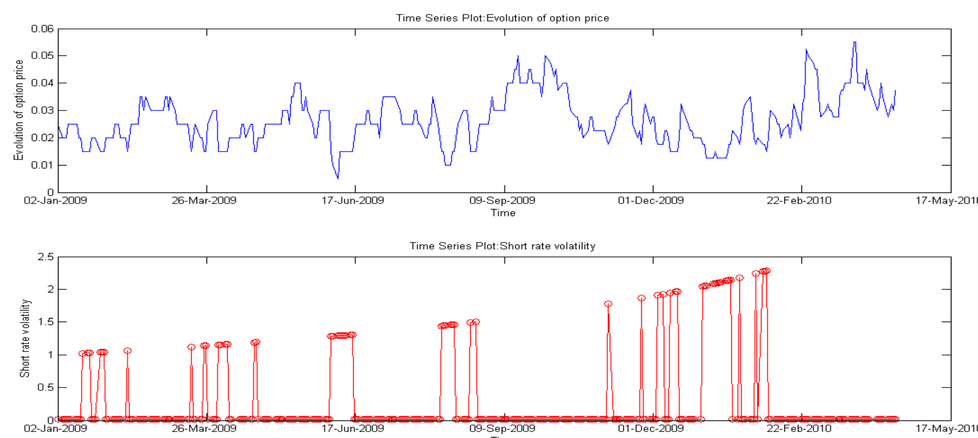

Figure 5.32: *Evolution of the option with Strike* = 99.25 *and the corresponding historical short-rate implied volatility.*

Chapter 6

Pricing Risk

In this chapter we define market risk and Value-at-Risk (VaR) measure to price risk. The historical simulation approach to determine VaR is described. Moreover, the valuation of VaR model using backtesting procedure is explained.

6.1 Market Risk

Financial risks include market risk, credit risk, and operational risk.

Market risk is defined as the risk of losses due to movements in financial market prices or volatilities.

Credit risk is defined as the risk of losses due to the fact that counterparties may be unwilling or unable to fulfill their contractual obligations.

Operational risk is defined as the risk of loss resulting from failed or inadequate internal processes, systems and people, or from external events.

Here our focus is on market risk. According to [5], market risks can be subdivided into four classes:

1. interest rate risks
2. equity price risks
3. exchange rate risks
4. commodity price risks

Since, our portfolio consists of Euribor futures and options; our main concern is the interest rate risk.

6.2 Value-at-Risk

Value-at-Risk (VaR) measures the worst expected loss under normal market conditions over a specific time interval at a given confidence level. Or as is state in [9] "VaR answers the question: how much can I lose with $x\%$ probability over a pre-set horizon".

To define Value-at-Risk, we use the notations given in [8].

Let V_0 be the value of the portfolio at time $t = 0$ and V_1 value of the portfolio at time $t = 1$. Thus, V_1 is a random variable as seen from $t = 0$. The final net worth of the investment is

$$X = V_1 - V_0(1 + r_f)$$

with r_f is risk free rate.
The portfolio loss (L) as measured in money at time $t = 0$ is defined as

$$L = \frac{-X}{1 + r_f}.$$

With above notations the VaR can be defined as:

Definition 10 (Value-at-Risk) *Value-at-Risk at time $t = 1$ at level α is defined as*

$$VaR_\alpha(X) = min\{m : P(L \leq m) \geq 1 - \alpha\}$$

Another way of expressing the above definition is that VaR is the lowest quantile of the potential losses that can occur with a given portfolio during a specified time period.

$$VaR_\alpha(X) = F_L^{-1}(1 - \alpha) \tag{6.1}$$

with $F_L^{-1}(\cdot)$ the inverse function of the cumulative distribution function F. Different interpretations of VaR definition can be found in [8].

6.2.1 Historical Simulation

For computing Value-at-Risk in practice, three approaches are proposed:

- parametric approach
- analytical or variance-covariance approach

- historical and Monte Carlo simulation.

The method we choose to implement in this thesis is *historical simulation*. The basic principle of this method is to assume that the near future will be sufficiently like recent past. So then we can use the data from the recent past to forecast the future. However, this assumption may only be valid on a relatively short time interval and is very sensitive to the quality of the data, [5].

The main interest in using historical simulation approach to determine VaR, is that no assumption is made on the distribution of profit & loss (P/L), we let the P/L data to speak for themselves. As a result we use empirical distribution to estimate VaR.

The first step of historical simulation is to identify the instruments in the portfolio and to obtain time series for these instruments over some defined historical period. Second step is to use the weights in the current portfolio to simulate hypothetical returns that would have realized assuming that the current portfolio had been held over the observation period. VaR estimates can then be read off from histogram of the portfolio returns using Formula 6.1. Since we use empirical distribution of P/L, Formula6.1 is simplified to:

$$\widehat{\mathrm{VaR}}_\alpha(X) = L_{[n\alpha]+1,n} \tag{6.2}$$

where $L_{1,n} \geq \ldots \geq L_{n,n}$ is ordered empirical loss distribution and $[x]$ is the integer part of x.

Consequently we calculate VaR using these steps:

1. Calculate the present value of the portfolio.

2. Generate a set of historical changes in our risk factors.

3. Calculate the loss in portfolio value for each historical change, this give the sample $L_{1,n}, \ldots, L_{n,n}$.

4. From the empirical loss distribution L_n estimate empirical $\widehat{\mathrm{VaR}}_\alpha$.

6.2.2 Advantages and Disadvantages of Historical Simulation Approach

Some of the advantages of historical simulation are:

- No statistical estimation of the multivariate distribution of risk-factor changes is necessary and no assumptions about the dependence structure of risk-factors changes are made.
- Can account for fat tails of the return distributions.
- Easy to implement and conceptually simple.
- Uses data that is readily available.

and disadvantages:

- Total dependence of data set.
- Assumes that history will repeat itself.
- There may be potential risks that are not captured by the historical data set.

The historical simulation method is usually very time consuming, but its main advantage is that it catches all recent market crashes. This feature is very important for risk measurement.

6.3 Backtesting

Value-at-Risk has become one of the most popular risk measurement techniques in finance. However, VaR model are useful only if it predict future risks accurately. In order to evaluate the quality of the VaR estimates, the model should always be backtested with appropriate methods.

Definition 11 (Backtesting) *Backtesting is a statistical procedure where actual profits and losses are systematically compared to corresponding VaR estimates.*

As explained in [13, page 55] the backtesting is performed as follow. We choose a fixed portfolio and use this portfolio to assess the accuracy of VaR model. We assume that the weights of the portfolio are fixed for the entire backtest. Moreover, we assume that the risk factor changes are constant throughout the backtest.

With above assumption we proceed as follow: At time t we estimate $\widehat{\mathrm{VaR}}_\alpha^t$ for one day. At time $t+1$ we can compare our one period estimate $\widehat{\mathrm{VaR}}_\alpha^t$, with what actually loss and continue so for t running throughout chosen backtesting period.

Let I_{t+1} be indicator notation for violation of the VaR estimate.

$$I_{t+1} = \begin{cases} 1, & \text{when } L_{t+1} > \widehat{VaR}^t, \\ 0, & \text{otherwise.} \end{cases}$$

If the VaR model is accurate then this indicators follows an i.i.d. Bernoulli process. Thus the expected number of successes in a test sample with n observation is $n\alpha$.

For example, if the confidence level used for calculating daily VaR is 99% which in our notations will mean that $\alpha = 0.01$, we expect an exception to occur once in every 100 days on average.

If the number of exceptions is less than the selected confidence level would indicate, the system overestimates risk. On the contrary, too many exceptions signal underestimation of risk. Naturally, it is rarely the case that we observe the exact amount of exceptions suggested by the confidence level. It therefore comes down to statistical analysis to study whether the number of exceptions is reasonable or not, i.e. will the model be accepted or rejected.

Chapter 7

VaR Calculation and Backtesting Process

The purpose of this chapter is not only to present the backtesting process and results in detail, but also to analyze the outcome and the factors that may have affected the outcome. First, we will consider a portfolio of Euribor futures options and we will give an analytical and numerical example of VaR calculation. After that, we will consider a portfolio of Euribor futures options and determine its VaR. Finally, the backtest of the portfolio will be conducted and the results are interpreted at portfolio level.

7.1 Portfolio Setup

This study utilizes the Euribor futures prices, Euribor futures option prices, and interest rate over the period January 2009 through March 2010 to perform the backtest of portfolio consisting of these derivatives. The pricing of the securities are done using the binomial Ho-Lee model, and one day binomial tree period was employed. To create a full spectrum of maturities for the Euribor rate, cubic spline interpolation was employed. In this manner, zero coupon bond prices could be obtained for each time date on the binomial lattice.

For backtesting purpose we choose two portfolios. The first one, consist one Euribor futures with maturity September 2010. The second portfolio consists of the Euribor futures option with maturity September 2010 and strike equal to $K = 98$. The reason for choosing just one futures contract and respective one option is that the futures contracts with different maturities are highly correlated see Figure 5.10 as well as options with different maturities and strikes.

Historical data are from January 2009 to November 2009, in total 229 trading days. The distribution of the risk factor changes is presented in Appendix E.

7.2 VaR Calculation

The choice of parameters in VaR calculations is not arbitrary whenever backtesting is involved. To construct a solid view on the validity of the model, relatively low confidence levels should be used. According to Jorion (2001), a confidence level of 95% suits well for backtesting purposes. With this approach, it is possible to observe enough VaR violations within the one year time period. However, we estimate VaR under different confidence levels 95% and 99%. Having more than one level of confidence in the backtesting process makes the testing more effective.

7.2.1 VaR Calculation for Portfolio of Euribor Futures Contracts

We consider a portfolio consisting of Euribor futures options. And we want to determine one-day VaR of this portfolio.

Recall Formula 4.8

$$F_\lambda(t;T) = 100(1+\frac{1}{\lambda}) - \frac{100}{\lambda}\frac{P(t,T)}{P(t,T+\lambda)}e^{\frac{\sigma^2}{8}(T-t)(T-t+1/2)}$$

The cost of initial portfolio consisting of h Euribor futures contracts with maturity T at time $t_0 = 0$ is

$$V_0 = hF_\lambda(0,T)$$

The value of the portfolio at time t_1 is $V_1 = hF_\lambda(t_1,T)$. Then the future net worth X of the portfolio is given by

$$X = V_1 - V_0 = -h\frac{100}{\lambda}\left(\frac{P(t_1,T)}{P(t_1,T+\lambda)}e^{\frac{\sigma^2}{8}(T-t_1)(T-t_1+1/2)} - \frac{P(0,T)}{P(0,T+\lambda)}e^{\frac{\sigma^2}{8}T(T+1/2)}\right)$$

where $P(t_1, T)$ and $P(t_1, T+\lambda)$ are random variables as seen from time $t_0 = 0$. To predict the T-zero rate at time t_1 we stress the current term structure using historical changes in zero-rates

$$Y(t_1, T) = y(0, T) + \Delta_{Y(T)} = y(T) + \Delta_{y(T)}$$

Using Formula 2.2 we get

$$P(t_1, T) = e^{-(y(T)+\Delta_{y(T)})(T-t_1)}$$

and

$$P(t_1, T + \lambda) = e^{(y(T+\lambda)+\Delta_{Y(T+\lambda)})(T+\lambda-t_1)}$$

Historical zero rates give us the empirical distributions of random variables $\Delta_{y(T)}$ and $\Delta_{y(T+\lambda)}$. Using empirical distributions we can determine Value-at-Risk at level α as

$$VaR_\alpha(X) = \widehat{F_L^{-1}}(1 - \alpha) \tag{7.1}$$

with $L = -X$ and $\widehat{F_z^{-1}}(\cdot)$ empirical distribution.

7.2.2 Empirical Example

Our portfolio consists of one Euribor futures with maturity September 2010. We want to determine one day VaR_α where $\alpha = 0.01$ and 0.05.

The current date is November 23, 2009. And the current value of the portfolio is $V_0 = 98.55$. Using historical changes of zero rates presented in Appendix E we stress the current term structure. Figure 7.1 shows the simulated term structure.

As results from section 5.3.1 suggests we choose the short-rate volatility equal to zero in the Ho-Lee model.

The portfolio loss distribution is given in Figure 7.2. Then we can determine empirical VaR using Formula (6.2), we obtain $VaR_{0.01} = 0.1711$ and $VaR_{0.05} = 0.1110$.

However, since 3-month Euribor futures is a contract to engage in a three month loan deposit of face value 1000000 Euro. We can express determined VaR in terms of the contract itself by using Formula (2.1), we get

$$\begin{aligned} VaR_{0.01} &= \frac{0.1711}{100}\frac{1}{4}1000000 \\ &= 427.7500 \text{ Euro} \end{aligned}$$

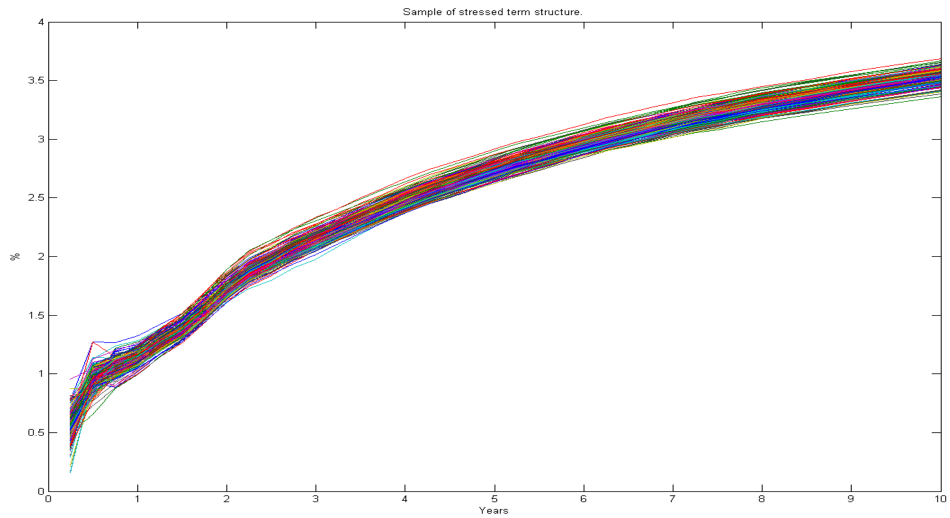

Figure 7.1: *Historical simulated term structure.*

and

$$
\begin{aligned}
VaR_{0.05} &= \frac{0.1110}{100}\frac{1}{4}1000000 \\
&= 277.5000 \text{ Euro.}
\end{aligned}
$$

We can compare this numbers with the actual initial margin of one Euribor futures contract 475 Euro. Using our data the actual initial margin will correspond to 99.35% confidence level VaR.

We want to mention here that Eurex calculates the initial margin requiems quarterly using a method called SPAN. This stands for Standard Portfolio ANalysis of Risk developed by the Chicago Mercantile Exchange in 1988. More information about SPAN can be found in [17].

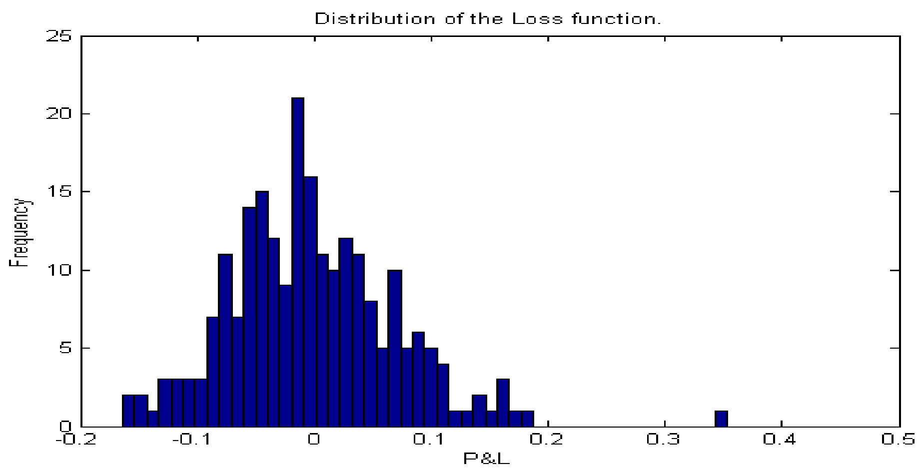

Figure 7.2: *Distribution of the loss function.*

7.2.3 VaR Calculation for Portfolio of Euribor Futures Options

Now we pay attention to a portfolio consisting of Euribor futures options. As in the previous section we want to determine VaR of this portfolio.

For portfolios containing Euribor futures options the value of the portfolio at time $t = 1$ is determined using algorithm described in Section 4.3.

Recall that the current term structure will determine the drift coefficient in the Ho-Lee short rate model. So when we calculate the value of the portfolio at time $t = 1$ the stressed term structure will determine the drift θ in the Ho-Lee model.

The results from Section 5.3.2 suggest, to choose the implied volatility at time $t = 0$ as input in the Ho-Lee model for calculating the option prices at time $t = 1$. Thus we don't stress the volatility.

However, for the purpose of comparison, and following the results from Section 5.1 and Section 5.3.1 we will determine the VaR of the portfolio of options by consider the short-rate volatility equal to zero and we will stress just the zero rates.

By assuming that the short-volatility is zero, we mean that there is not randomness in the behavior of the short-rates. We suggest that only the current term structure itself tells us everything about Euribor futures options prices.

7.2.4 Empirical Example

Our portfolio consists of hundred Euribor futures call options with maturity September 2010. We want to determine one day VaR_α with $\alpha = 0.01$ and 0.05.

The current date is November 23, 2009. The current value of the portfolio is $V_0 = 66.25$. Figure 7.1 shows the historical simulated term structure. We determine the VaR when the short-rate volatility in Ho-Lee model first is set to zero, $\sigma_1 = 0$; and in a second case the short-rate volatility is set equal to implied volatility determined at current time, $\sigma_2 = 0.0054$ (daily volatility since the step in the binomial Ho-Lee model is one day). One day Value-at-Risk of the portfolio at different confidence levels are presented in Table 7.2

Value-at-risk of the portfolio when short-rate volatility in Ho-Lee model is σ_2 are much lower compare with VaR of the portfolio when short-rate

VaR	Volatility zero	Implied volatility
$\mathrm{VaR}_{0.01}$	33.39	1.16
$\mathrm{VaR}_{0.05}$	27.33	0.99

Table 7.1: *One day VaR for portfolio of Euribor futures option.*

volatility is zero. To see which of VaR estimates are reliable we will perform backtest of the portfolio.

7.3 Backtesting Result

The backtesting results are obtained by computing daily VaR for a time period ranging from November 23, 2009 to April 16, 2010. The number of trading days (observations) totals 100. A useful way of displaying backtesting data is the backtesting graph. The P&L is shifted in time relative to risk so that the risk figure for a particular day is compared with the P&L for the following trading day.

Throughout the backtesting process daily trading outcomes are compared to daily VaR estimates. If we let $x_{t,t+1}$ denote profit or loss of the portfolio over one day time interval. Corresponding VaR estimate is then defined as VaR_t, which is calculated at the beginning of the period, using the closing prices of day t.

For example, the first VaR estimate is calculated with the closing prices on November 23, 2009. This estimate is then compared to the trading outcome (profit or loss) that is realized at the end of November 24, 2009.

Figures 7.4, 7.5 and 7.6 presents the backtesting graphs. Daily returns are displayed with VaR estimates at two different confidence levels over the 100 days time period. The graphs show not only the how many exceptions there where, but also their timing and magnitude. Figures 7.7 and 7.8 show the P&L histograms of our portfolios. Figure 7.7 shows the evolution of zero rates during backtesting period.

First backtesting graph, Figure 7.4 refers to the portfolio of Euribor futures. The graph shows one exception for both confidence levels. The result is consistent with statistical expectation for $VaR_{0.01}$ but not for $VaR_{0.05}$. However, apart from this, the risk measure is high compared to actual changes in portfolio value, see Figure 7.7. The result indicate that by considering

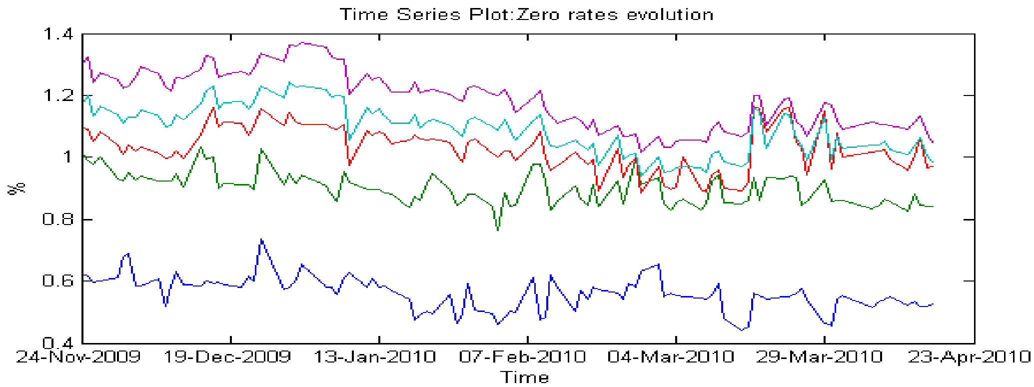

Figure 7.3: *Evolution of 3 months, 6 month, 9 months, 1 year, 1year and 3 months Zero rates over backtesting period.*

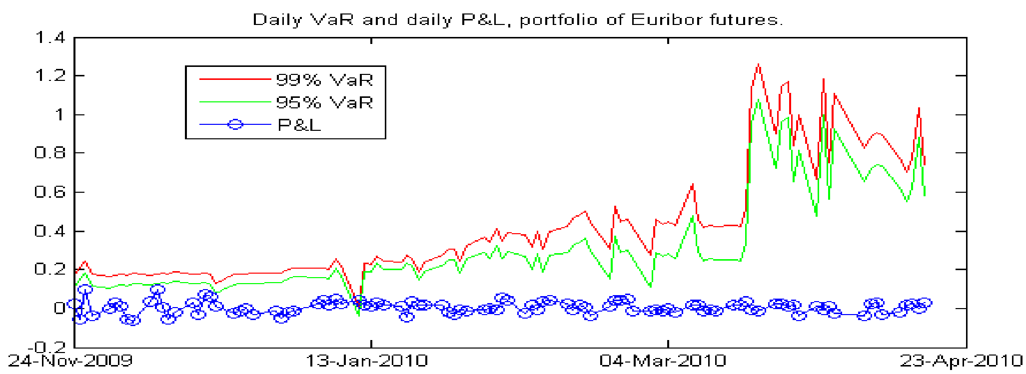

Figure 7.4: *Daily P&L and 95%, 99% daily VaR for portfolio consisting of Euribor futures contract.*

the price of Euribor futures contracts is equal to the price of the respective forward contract is a conservative method to aggregate risk.

Second backtesting graph, Figure 7.5 refers to the portfolio of Euribor futures option when short-rate volatility is zero. The graph shows no exceptions, thus the risk figure measured is overestimating the volatility of earnings, compare with histogram in Figure 7.8.

As expected, when short rate volatility is zero, the correlation between VaR_α, 9 months zero-rate and one year zero-rate is high during the backtesting period, see Figures 7.7, 7.4 and 7.5.

Third backtesting graph, Figure 7.6 refers to the same portfolio of Euribor

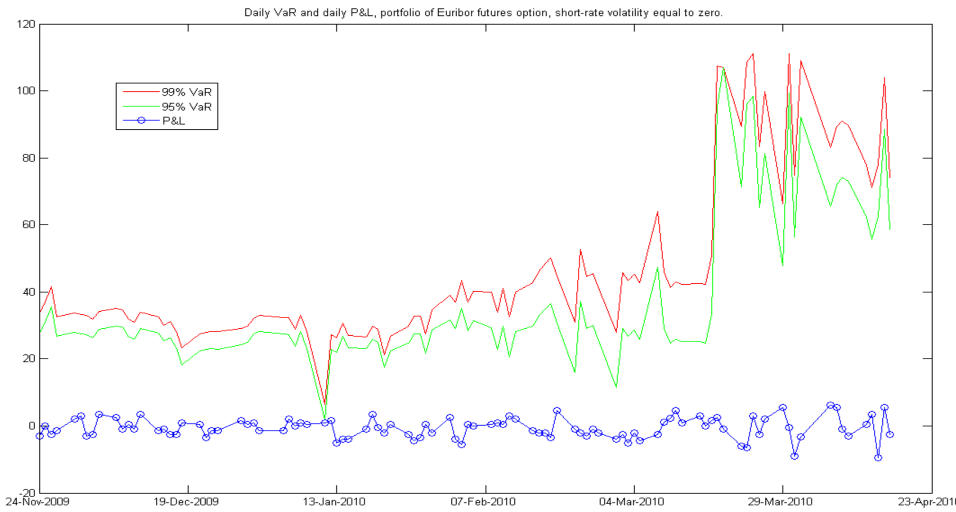

Figure 7.5: *Daily P&L and 95%, 99% daily VaR for portfolio consisting of Option on Euribor futures contract.*

Confidence Level	Exceptions/Observations futures portfolio	Exceptions/Observations option portfolio $\sigma = 0$	Exceptions/Observations option portfolio σ = implied volatility
99%	1/100	0/100	28/100
95%	1/100	0/100	31/100

Table 7.2: *Backtesting results.*

futures option when terminal short-rate volatility is current implied volatility. The graph shows too many exceptions for both confidence levels, the risk being measured is too low. We believe that this result can be improved by using longer historical data or by choosing another method to compute VaR.

Before rejecting Ho-Lee short-rate model for pricing Euribor futures and options we will suggest additional testing of VaR of portfolio of Euribor futures options with a new set of data. We may use a longer historical time period; or we can use other method for computing VaR. A powerful method for stressing the term structure is to use Principal Component Analysis (PCA) and stress scenarios.

Moreover, we believe that this result can be improved by using different

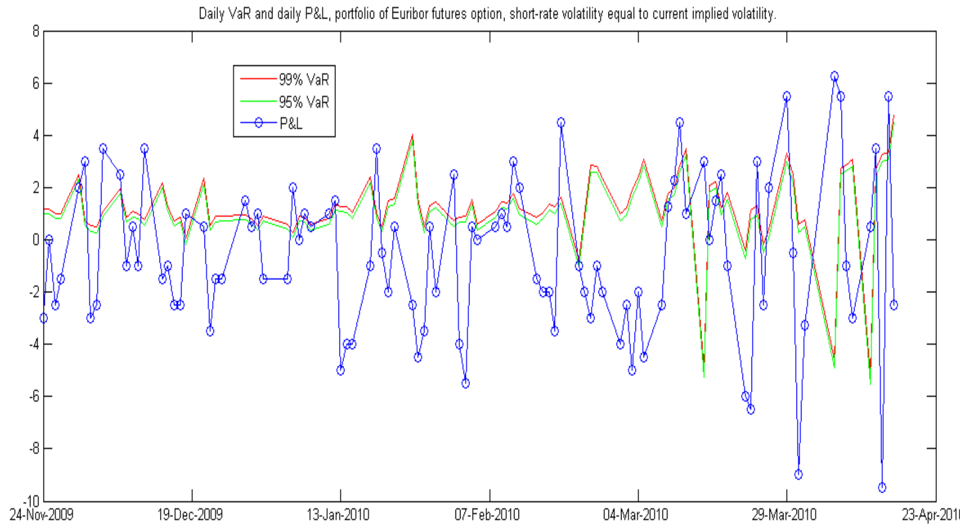

Figure 7.6: *Daily P&L and 95%, 99% daily VaR for portfolio consisting of Option on Euribor futures contract.*

approach to forecast terminal volatility, i.e. GARCH or similar approaches.

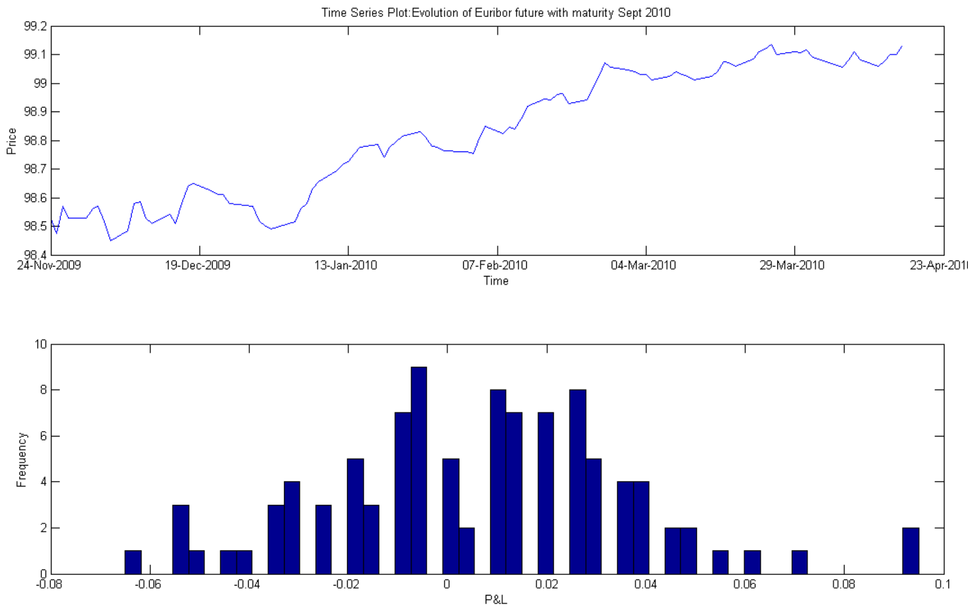

Figure 7.7: *Evolution of the Euribor futures and the loss distribution histogram.*

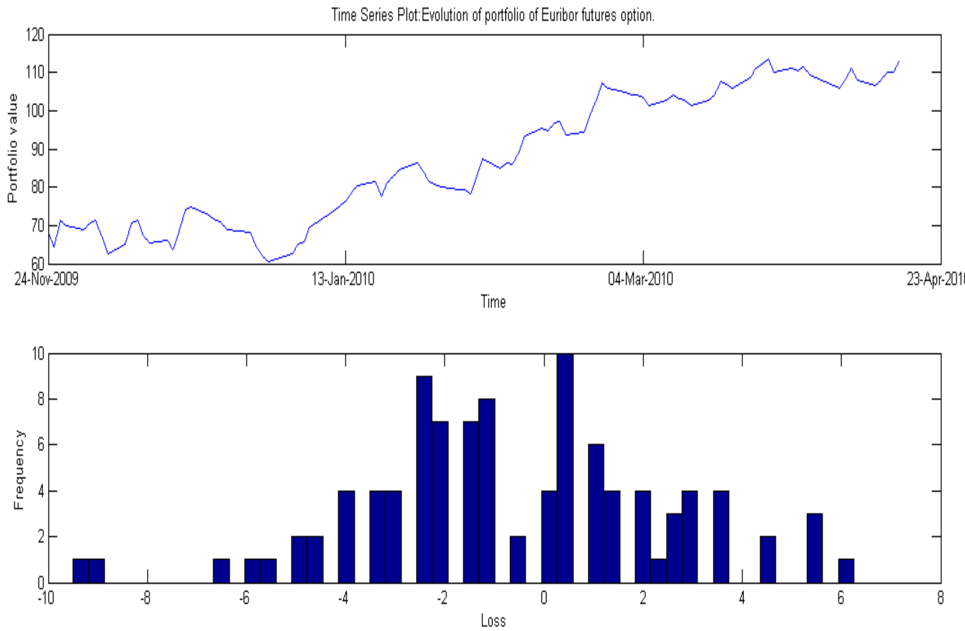

Figure 7.8: *Evolution of the Option on Euribor futures and the loss distribution histogram.*

Chapter 8

Concluding Remarks

This section summarizes the findings and provides suggestions for further research.

In general single-factor models are believed impossible to fit the market data, no matter how you change the combination of the parameters [4].

Although Ho-Lee model takes the current term structure as input it does not fit the volatility curve. To solve this problem we can add some flexibility to the model so that the fitting curves becomes no problem.

- Under the HJM model the volatility function can be freely specified. A series of volatility functions can be used to fit the volatility curve. These volatility functions can be nested into a general form as follows [2]

  $$\sigma(t, T, f(t, T)) = [\sigma_0 + \sigma_1(T - t)]\exp[-\lambda(t - t)]f(t, T)^{\gamma}$$

  For example by choosing constant volatility we get the Ho-Lee model; if we choose exponential volatility function we get the Hull-White short rate model.

- We can allow more than one state variable for the term structure. We can consider two- or three- factor HJM models.

Moreover, no matter which approach is adopted in modeling the term structure, it is important to understand its hedge. The next natural step is too study delta-gamma hedging of Euribor futures and Euribor futures options.

Our results indicate that the convexity adjustment is close to zero for Euribor futures rate with short maturities. The main cause for this result is the fact that the Euribor futures are used in construction of zero curves, after that we use the zero curve as an input to the Ho-Lee model. It is recommended to replace the futures with deposits to nine months and swap rates thereafter when building the zero curves. However, apart from this, even the formula for convexity adjustment determined from the Ho-Lee model, suggest that convexity adjustment is close to zero for short maturities futures rates.

Another result is that the short-rate volatility implied from deep in-the-money Euribor futures call options with short maturity (one year) is equal to short-rate volatility implied from underling futures contract.

From time series of implied volatilities we find out that there is not recommended to stress the short-rate volatility when we simulate the prices of Euribor futures call options. On the other hand, backtesting results are very poor for portfolio of Euribor futures option when we choose the terminal volatility equal to the current implied volatility.

For further research we recommend to perform the backtesting of VaR using a longer historical time or to use scenario simulation in order to simulate the movements of the term structure. In addition we recommend trying using different approach to forecast terminal volatilities, i.e. GARCH or similar approaches.

Appendix A

Relation between Zero Coupon Bonds and Forward Rates

The forward rate $f(t,T)$ dynamics is given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t^Q.$$

Relation between zero-coupon bond and forward rates is given by formula [3, page 305]

$$P(t,T) = e^{-\int_t^T f(t,s)ds}$$

we can prove that $P(t,T)$ satisfies (the proof can be found [3, page 307])

$$dP(t,T) = p(t,T)\left\{r(t) - A(t,T) + \frac{1}{2}B^2(t,T)\right\}dt - p(t,T)B(t,T)dW_t^Q \tag{A.1}$$

where

$$A(t,T) = \int_t^T \alpha(t,s)ds$$

and

$$B(t,T) = \int_t^T \sigma(t,s)ds \tag{A.2}$$

where $B(t,T)$ can be interpreted as volatility of zero-coupon bond with maturity T.

Appendix B

Relation between Forward and Futures Prices

We use the notations and the Theorem in [10, page 3] to prove the relationship between futures and forward prices.

Let $P_0^{(T)}[X]$ be the present value today of a contract that delivers the random value X at time T; $G_0^{(T)}[X]$ denote the forward price today of the value X delivered at time T; $F_t^{(T)}[X]$ is the futures price as of time t of the value X delivered at time T; Z_T discount factor and $R(0,T)$ is the money market account.

Theorem 1 *The following relations hold:*

- $P_0^{(T)}[X] = Z_T G_0^{(T)}[X]$
- $P_0^{(T)}[Xe^{R(0,T)}] = F_0^{(T)}[X]$
- $P_0^{(T)}[X] = F_0^{(T)}[Xe^{-R(0,T)}]$

Using the fact that the futures prices are martingale under futures distribution

$$F_t^{(T)}[X] = E_t^Q[X]$$

and following general formula

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y].$$

We can determine the relation between forward and futures prices as follow:

$$\begin{aligned} G_0^{(T)}[X] &= Z_T^{-1} P_0^{(T)}[X] \\ &= Z_T^{-1} F_0^{(T)}[Xe^{-R(0,T)}] \\ &= Z_T^{-1} E[Xe^{-R(0,T)}] \\ &= Z_T^{-1}\mathrm{cov}(X, e^{-R(0,T)}) + Z_T^{-1} E[X]E[e^{-R(0,T)}] \\ &= Z_T^{-1}\mathrm{cov}(X, e^{-R(0,T)}) + Z_T^{-1} F_0^{(T)}[X] F_0^{(T)}[e^{-R(0,T)}] \\ &= Z_T^{-1}\mathrm{cov}(X, e^{-R(0,T)}) + F_0^{(T)}[X] \end{aligned}$$

(B.1)

To prove the last equality we used the relation

$$\begin{aligned} Z_T^{-1} F_0^{(T)}[e^{-R(0,T)}] &= Z_T^{-1} P_0^{(T)}[e^{-R(0,T)} e^{R(0,T)}] \\ &= Z_T^{-1} P_0^{(T)}[1] \\ &= Z_T^{-1} Z_T = 1 \end{aligned} \tag{B.2}$$

Appendix C

Pricing Zero Coupon Bonds using the Binomial Ho-Lee Model

In following chapter, we provide a derivation of the valuation of zero coupon bond prices when specifying the dynamics of the short rate to the binomial Ho-Lee model.

We use the notations and assumptions introduced in Section 3.3. Using risk neutral valuation, the value of zero coupon bond at time $t = t_k$ with maturity $T = t_{k+n}$, $P(t,T)$ should satisfy the relation

$$\begin{aligned} P(t,T) &= E_t^Q\left[1 \cdot e^{-(r_{k+1}+r_{k+2}+\ldots+r_{k+n})}\right] \\ &= E_t^Q\left[e^{-\sum\limits_{i=k+1}^{k+n}\theta_i - \sigma\left(\sum\limits_{i=2}^{k+1} b_i + \sum\limits_{i=2}^{k+2} b_i + \ldots + \sum\limits_{i=2}^{k+n} b_i\right)}\right] \\ &= e^{-\sum\limits_{i=k+1}^{k+n}\theta_i - \sigma n \sum\limits_{i=2}^{k+1} b_i} E_t^Q\left[e^{-\sigma\left(\sum\limits_{i=k+2}^{k+2} b_i + \sum\limits_{i=k+2}^{k+3} b_i + \ldots + \sum\limits_{i=k+2}^{k+n} b_i\right)}\right] \end{aligned} \tag{C.1}$$

We continue with calculation by focusing only on the expectation:

$$\begin{aligned} E_t^Q\left[e^{-\sigma\left(\sum\limits_{i=k+2}^{k+2} b_i + \sum\limits_{i=k+2}^{k+3} b_i + \ldots + \sum\limits_{i=k+2}^{k+n} b_i\right)}\right] &= E_t^Q\left[e^{-\sigma((n-1)b_{k+2}+(n-2)b_{k+3}+\ldots+b_{k+n})}\right] \\ &= \cosh(\sigma(n-1)) \cdot \cosh(\sigma(n-2)) \cdot \ldots \cdot \cosh(\sigma) \end{aligned}$$

Finally the price of the zero coupon bond at time t with maturity T is determined as

$$P(t,T) = e^{-\sum\limits_{i=k+1}^{k+n} \theta_i - \sigma n \sum\limits_{i=2}^{k+1} b_i} \prod_{i=2}^{n-1} \cosh(\sigma i) \tag{C.2}$$

Appendix D

Short Rate Implied Volatility Smile

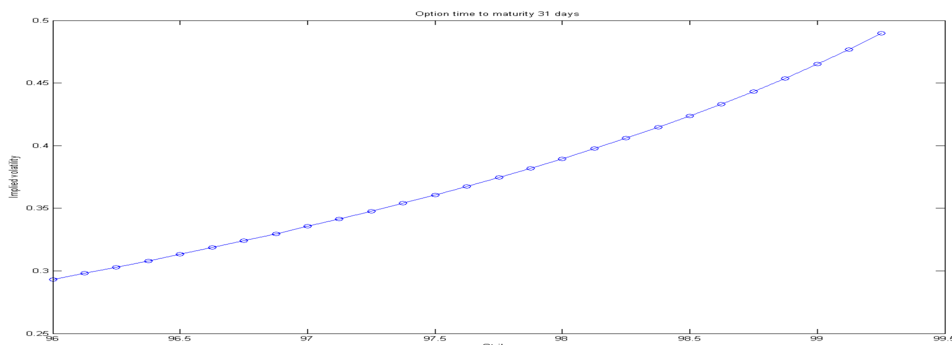

Figure D.1: *Short-rate volatility implied from option with maturity March 2010.*

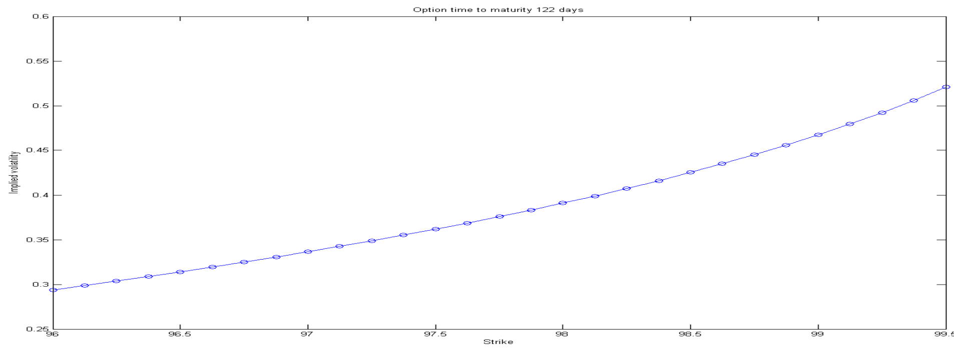

Figure D.2: *Short-rate volatility implied from option with maturity June 2010.*

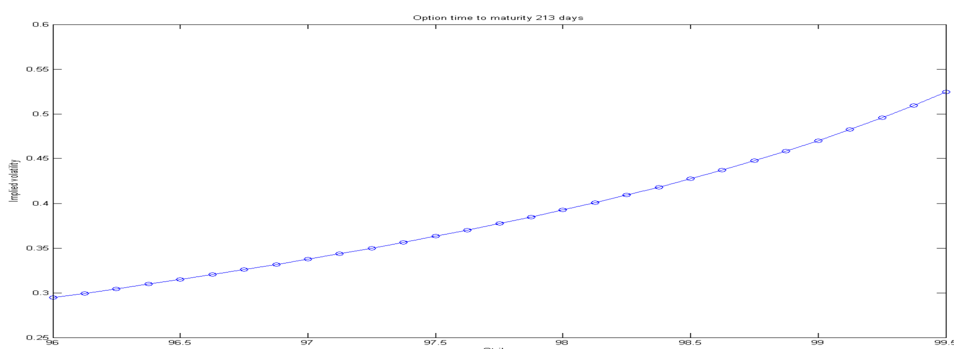

Figure D.3: *Short-rate volatility implied from option with maturity September 2010.*

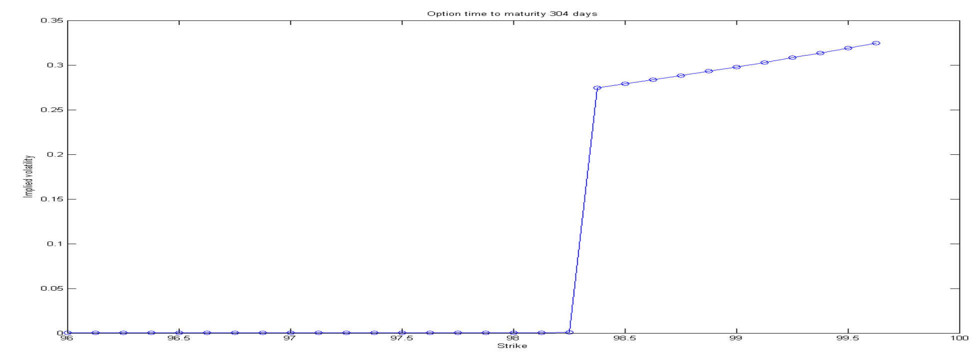

Figure D.4: *Short-rate volatility implied from option with maturity December 2010.*

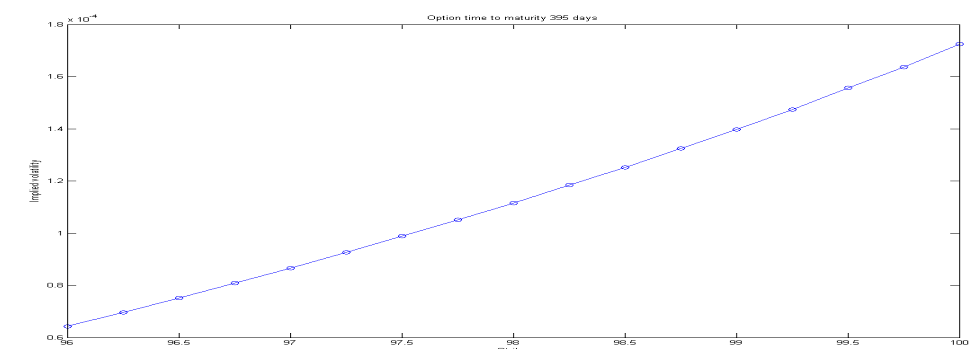

Figure D.5: *Short-rate volatility implied from option with maturity March 2011.*

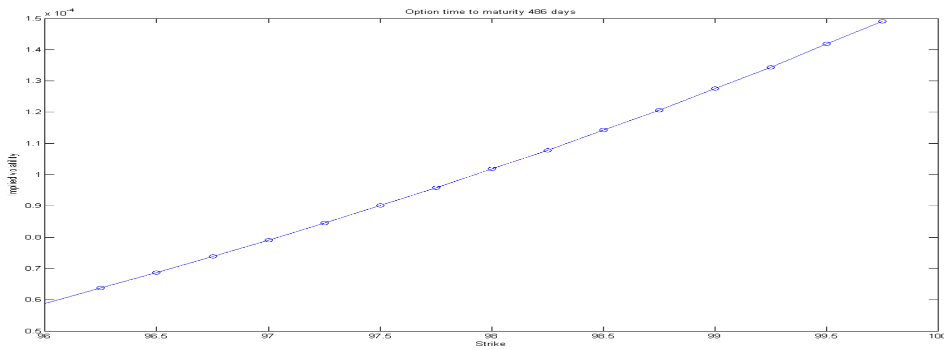

Figure D.6: *Short-rate volatility implied from option with maturity June 2011.*

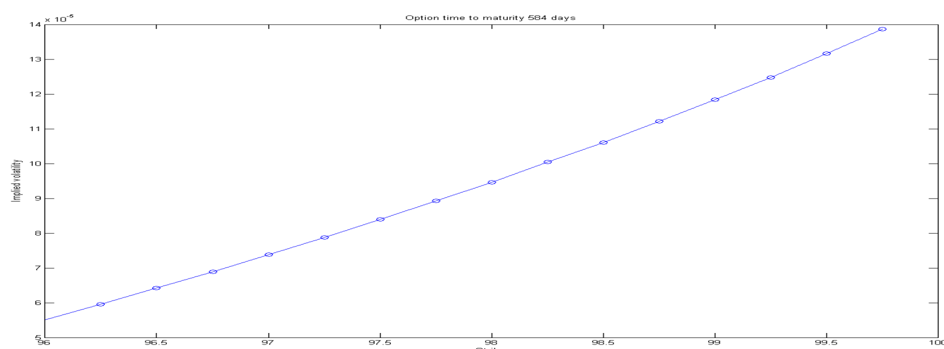

Figure D.7: *Short-rate volatility implied from option with maturity September 2011.*

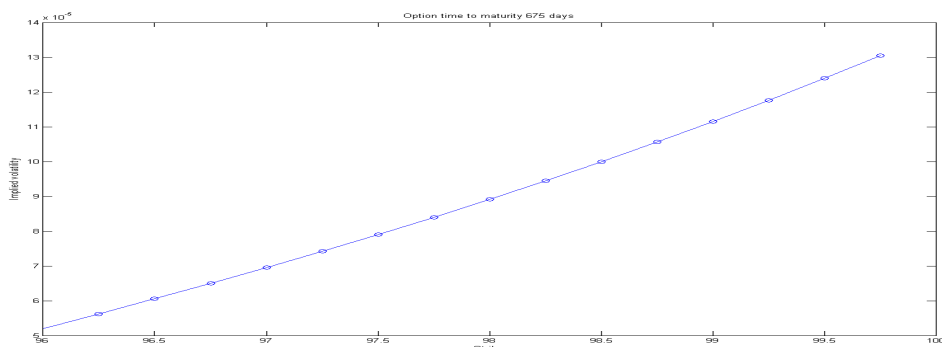

Figure D.8: *Short-rate volatility implied from option with maturity December 2011.*

Appendix E

Risk Factors Changes

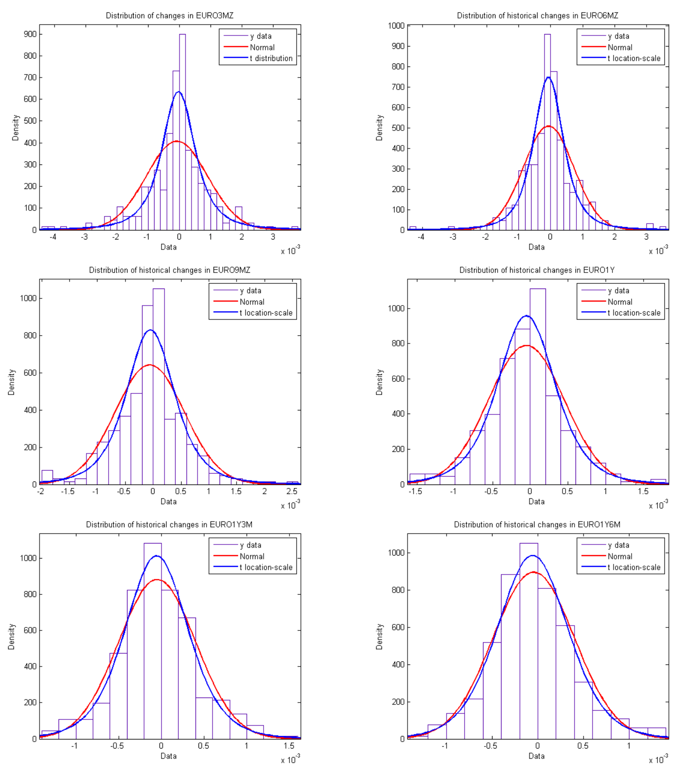

Figure E.1: *Distribution of historical chances in zero interest rates with maturity 3 months, 6 months, 9 months, 1 year, 1 year and 3 months, 1 year and 6 months.*

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