ON IMPLEMENTING EURO-BUND FUTURES

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Abstract

One of the exciting developments in finance over the last 25 years has been the growth of the derivatives markets. In many situations, both hedgers and speculators find it more attractive to trade a derivative on an asset than to trade the asset itself.

In this thesis, futures on bonds or so called Cheapest-to-Deliver (CTD), are studied. Bond futures contracts are futures contracts that allow investor to buy in the future a theoretical government notional bond at a given price at a specific date in a given quantity. Compared to other futures, bond futures are slightly more complicated as the underlying bond of the futures contract is not a physical bond but rather a theoretical notional bond determined by the basket of available deliverable government bonds issued in the market. At delivery of the futures contract, the holder will want to deliver the cheapest bond, hence bond futures are often called Cheapest-to-Deliver (CTD). The main objective of this thesis will be to, if possible develop a method to determine which bond will be the cheapest to deliver at expiration of the future.

The term structure of the interest rates is assumed to be stochastic and in particular the Ho-Lee model with a binomial short rate lattice is employed for the term structure modelling. Further, it is observed what effects changes in the terms structure have on which bond is the cheapest-to-deliver, since when interest rate change it is natural to presume that another bond will become the cheapest-to-deliver.

The following questions will be discussed

- How well does the Ho-Lee model fit the prices of bonds and futures?
- How many steps are needed in the binomial tree to get a good result?
- How does changes in the term structure affect which bond is the cheapest to deliver?
- Is it possible to predict which bond will be the cheapest to deliver?
- How sensitive is the futures price to changes in the zero curve?
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Chapter 1

Introduction

Growth in the derivatives markets has brought with it an ever-increasing volume and range of interest rate dependent derivative products. To allow profitable, efficient trading in these products, accurate and mathematically sound valuation techniques are required to make pricing, hedging and risk management of the resulting positions possible.

The importance of managing interest rate risk cannot be overstated and bond futures are widely used to hedge interest rate risk on long maturities, especially by swap dealers that need to cover their risk against various points of the interest rate curve. Bond futures bear an additional risk often referred to as the basis risk compared to swaps.

The Eurex Euro-Bund futures contract is a debt instrument issued by the Federal Republic of Germany and is the exchange’s most liquid contract, with volume of 338 million contracts. It is based on long-term debt with a term of 8.5 to 10.5 years, and is considered the benchmark for long-term Euro-denominated government debt. A futures contract is an agreement between two parties to trade an asset at some future date for a fixed price agreed upon today. Due to features such as standardization, marking to market, and so forth, futures contract are more liquid and easier to trade than forward contracts.

The aim of this paper is to provide an insight in the pricing and dynamics of bond futures and determining which bond will be the cheapest to deliver. Much effort has been put into keeping this paper as understandable and readable as possible. We will proceed in a stepwise manner and mostly build from basic principles.

1.1 Outline of the Thesis

Chapter 2 presents an overview of the most common term structure models and the basics of short rate modelling. The Ho-Lee model is studied in more detail and it is shown how the short rate and bond prices can be computed using the continuous time Ho-Lee model and how to build a short rate tree with the discrete time Ho-Lee model. Further, we make a comparison of the two methods in Section (2.2) and a numerical test in Section (2.2.1).

In the introduction of Chapter 3, the basics of bond pricing is explained and Section (3.2) gives an overview of bond futures pricing. Further, we explain the conversion...
factor methodology and the idea behind it. The following section analyses the characteristics of the Euro-Bunds and the Euro-Bund futures and the specific FGBL contract and its underlying studied in this thesis. Section (3.3) and (3.4) thoroughly examines the relevant mathematics that will be used in order to determine the Cheapest-To-Deliver (CTD) and the elements of bond pricing that will be employed to price the Euro-Bund futures in this thesis. We also take a look at how to build a zero curve using a series of coupon-bearing bonds, and how they can be interpolated. To conclude this chapter, we describe how a short rate tree is built from the zero curve obtained in the previous section.

Chapter 4 discusses the results of the model using real market data. We start by investigating how many steps are needed to get a good fit and how well the futures and bond prices fit considering the assumption of a constant volatility. In Section (4.2) we study which bond will be the cheapest to deliver and what change in the short rate makes the CTD change from one bond to another. Finally, Section (4.3) analyses how sensitive the futures price is to changes in the bond prices.

Chapter 5 summarises and discusses the material covered in the thesis and also presents suggestions for further research and study in the area.
Chapter 2

Term structure models

In order to study interest rate derivatives, or other situations where interest rates are assumed to be random and the underlying asset depends on the interest rate, we need to study the dynamics of the short rate of interest. The pricing of interest rate contingent claims has two parts. Firstly, a finite number of relevant economic fundamentals are used to price all 'default-free' zero coupon bonds of varying maturities. This gives rise to an interest rate term structure, which describes the relation between the pricing of zero coupon bonds and various maturities. Secondly, taking these zero coupon bond prices as given, all interest rate derivatives may be priced. As with asset prices, the movement of interest rates is assumed to be determined by a finite number of random shocks, which is fed into the model through stochastic processes. Assuming continuous time and hence also continuous interest rates, these sources of randomness are modelled by Brownian motions (Wiener processes). The theory of interest rate dynamics relies on the assumption that the assets are default-free and available in a continuum of maturities.

The short rate that we look at in this section applies to an infinitesimally short period of time at time $t$ and is referred to as the instantaneous short rate. All processes for $r$ that are studied will be processes in the risk-neutral world. In the risk-neutral world we assume that investors earn on average $r(t) \cdot \Delta(t)$ in the very short time period between $t$ and $t + \Delta(t)$.

Term structure models can more or less be divided into two categories; the equilibrium models and the no-arbitrage models. There are a large number of proposals on how to specify the short rate dynamics in the risk-neutral world. The following are a couple of the most popular models.

Equilibrium models

- the Vasicek model, $dr(t) = a(\theta - r(t))dt + \sigma dW_t$
- the Dothan model, $dr(t) = ar(t)dt + \sigma r(t)dW_t$
- the Cox-Ingersoll-Ross model, $dr(t) = a(\theta - r(t))dt + \sigma \sqrt{r(t)}dW_t$
The no-arbitrage models

- the Black-Derman-Toy model, $dr(t) = \theta(t) r(t) dt + \sigma(t) r dW_t$
- the Ho-Lee model, $dr(t) = \theta(t) dt + \sigma dW_t$
- the Hull-White model, $dr(t) = (\theta(t) - ar(t)) dt + \sigma dW_t$

The equilibrium models have a deterministic drift and therefore do not automatically fit today’s term structure of interest rates. They can only provide an approximate fit to many of the term structures that are encountered in practice. This can be of great disadvantage since the model cannot price the underlying bond correctly which can lead to large errors in the pricing of interest rate derivatives. The no-arbitrage models are designed to be exactly consistent with today’s term structure of interest rates. The main difference between an equilibrium and a no-arbitrage model is therefore as follows. In an equilibrium model, today’s term structure of interest rates is an output and the drift of the short rate is not usually a function of time. In a no-arbitrage model, today’s term structure of interest rates is an input and the drift is in general dependent on time. This is because the shape of the initial zero curve governs the average path taken by the short rate in the future. If the zero curve is steeply upward-sloping for maturities between $t_1$ and $t_2$, then $r$ has a positive drift between these times and vice versa.

2.1 The Ho-Lee model

In this thesis we will use the Ho-Lee model for bond pricing. The model has many advantages. Bond prices, forward and zero rates are explicitly computable from the model, it is an exogenous term structure model and very well suited for building recombining lattices. The model is relatively simple and can be calibrated so as to fit the current term structure perfectly. However, there are two features with this model that are a disadvantage. The first is that it does not incorporate mean reversion which means that all shocks to the short rate are permanent and do not wear off with time. On the basis of economic theory, there are compelling arguments for the mean-reversion of interest rates. When rates are high, the economy tends to slow down and investments will decline. This implies that there is less demand for money and rates will tend to decline. Vice versa, when rates are low, it is relatively cheap to invest, hence rates will tend to rise. The second weakness of the Ho-Lee model is that interest rates are assumed to be normally distributed, which implies that the interest rate can become negative with positive probability. And far into the future the short rate will eventually take negative values with a probability that approaches one half.

Thomas S. Y. Ho and Sang Bin Lee proposed the first no-arbitrage model of the term structure in 1986. Instead of modelling the short-term interest rate, Ho-Lee developed a discrete time model of the evolution of the whole yield curve. The model was presented in the form of a binomial tree of bond prices with two parameters, namely the short-rate standard deviation and the market price of risk of the short rate. The continuous-time limit of the model is defined as

$$dr = \theta(t) dt + \sigma dW_t$$

where $\sigma$ is the deterministic instantaneous standard deviation and $\theta(t)$ is defined as the average direction that $r$ moves at time $t$. $\theta(t)$ is a function of time and should be chosen to ensure that the model fits the current term structure, this can be done analytically.
2.1.1 Continuous time Ho-Lee model

The \( \theta(t) \) variable in the continuous Ho-Lee model can be determined analytically. By integrating \( dr = \theta(t) \, dt + \sigma \, dW_t \), we obtain

\[
r(u) = r(t) + \int_t^u \theta(s) \, ds + \sigma (W_u - W_t)
\]

The short rate \( r(u) \) conditional on \( F_t \) is normally distributed with mean

\[
E[r(u)|F_t] = r(t) + \int_t^u \theta(s) \, ds
\]

and variance

\[
Var(r(u)|F_t) = \sigma^2 (W_u - W_t^2)|F_t| = \sigma^2 (u - t)
\]

The bond price at time \( t \) with maturity \( T \) is

\[
P(t, T) = E^Q[e^{-\int_t^T r(u) \, du}|F_t] = E^Q[e^{-\int_t^T \theta(s) \, ds + \sigma (W_u - W_t)}|F_t]
\]

The integral \( Z = -\sigma \int_t^T (W_u - W_t) \, du \) is normally distributed with mean zero and variance

\[
Var(-\sigma \int_t^T (W_u - W_t) \, du|F_t) = \sigma^2 Var(\int_0^{T-t} W_u \, du) = \sigma^2 Var((T-t) W_{T-t} - \int_0^{T-t} u \, dW_u)
\]

\[
= \sigma^2 E\left[\left(\int_0^{T-t} u \, dW_u\right)^2\right]
\]

\[
= \sigma^2 \int_0^{T-t} (T-t - u)^2 \, du
\]

\[
= \frac{1}{3} \sigma^2 (T-t)^3,
\]

where the second equality follows from Ito’s formula.

Since \( Z \) is a normally distributed variable with variance \( \frac{1}{3} \sigma^2 (T-t)^3 \), we can conclude that

\[
E^Q[e^Z|F_t] = e^{\frac{1}{6} \sigma^2 (T-t)^3}
\]

Putting this into the bond price equation

\[
P(t, T) = e^{-(T-t)r(t) - \int_t^T \int_s^T \theta(s) \, ds \, du + \frac{1}{6} \sigma^2 (T-t)^3}.
\]

The Ho-Lee model can be fitted to the initial term-structure of interest rate by solving for \( \theta(t) \). We know that the instantaneous forward rate can be expressed as

\[
f(t, T) = -\frac{\partial}{\partial T} \ln P(t, T)
\]

hence

\[
f(0, T) = -\frac{\partial}{\partial T} (-Tr(0) - \int_0^T \int_0^T \theta(s) \, ds \, du + \frac{1}{6} \sigma^2 T^3)
\]

\[
= r(0) + \int_0^T \theta(s) \, ds - \frac{1}{2} \sigma^2 T^2
\]

Taking logarithms and differentiating twice with respect to \( T \) yields

\[
\theta(T) = -\frac{\partial}{\partial T} f(0, T) + \sigma^2 T.
\]

We now turn the pricing of bonds with the Ho-Lee model. The bond price at time \( t \) for maturity \( T \) is

\[
P(t, T) = e^{-(T-t)r(t) - \int_t^T \int_s^T \theta(s) \, ds \, du + \frac{1}{6} \sigma^2 (T-t)^3}.
\]
We also know that the model can be fitted to the initial term structure by choosing
\[ \theta(t) = -\frac{\partial}{\partial t} f(0, t) + \sigma^2 t. \]

Employing this into the integral \( \int_t^T f(s, T) ds \) that occurs in the bond pricing equation yields
\[
\int_t^T f(s, T) ds = \int_t^T \int_s^T \left( \frac{\partial}{\partial s} f(0, s) + \sigma^2 s \right) ds du \\
= \int_t^T \int_s^T \left( f(0, u) - f(0, t) + \frac{1}{2} \sigma^2 (u^2 - t^2) \right) du \\
= -(T - t) f(0, t) - \frac{1}{2} \sigma^2 (T - t) t^2 + \int_t^T \left( f(0, u) + \frac{1}{2} \sigma^2 u^2 \right) du \\
= \text{reintroducing}(f(0, u) = -\frac{\partial}{\partial u} \ln P(0, u)) \\
= -(T - t) f(0, t) - \frac{1}{2} \sigma^2 (T - t) t^2 + \int_t^T (-\frac{\partial}{\partial u} \ln P(0, u) + \frac{1}{2} \sigma^2 u^2) du \\
= -(T - t) f(0, t) - \frac{1}{2} \sigma^2 (T - t) t^2 - \ln P(0, T) + \ln P(0, t) + \frac{1}{2} \sigma^2 (T^3 - t^3).
\]

Hence,
\[
P(t, T) = e^{-(T-t)\hat{r}(t)-\int_t^T \int_s^T \theta(s) ds du + \frac{1}{2} \sigma^2 (T-t)^2} \\
= e^{-(T-t)\hat{r}(t)+\int_0^T f(0, t) + \frac{1}{2} \sigma^2 (T-t)^2 + \ln P(0, T) - \ln P(0, t) - \frac{1}{6} \sigma^2 (T^3 - t^3) - \frac{1}{2} \sigma^2 (T-t)^2}.
\]

But,
\[
\frac{1}{2} \sigma^2 (T-t)^2 - \frac{1}{6} \sigma^2 (T^3 - t^3) + \frac{1}{6} \sigma^2 (T-t)^3 = -\frac{1}{2} \sigma^2 (T-t)^2.
\]

Thus, the bond price can be expressed as
\[
P(t, T) = \frac{P(0, T)}{P(0, t)} e^{-(T-t)(\hat{r}(t) - f(0, t)) - \frac{1}{2} \sigma^2 (T-t)^2}. \tag{2.1}
\]

### 2.1.2 Discrete time Ho-Lee model

In almost all cases when the whole price path has to be taken into account, it is necessary to use some numerical procedure to calculate the price of the derivative. By using a binomial lattice the Ho-Lee model can be applied in a discrete time frame. We can also see that in accordance with theory, the discrete time model always converges to the continuous model with smaller time intervals and increasing number of intervals \( n \to \infty, \Delta t \to 0 \). Since we are looking at the discrete time frame, we will use the discrete short rate.

We set up a lattice with a time span between the nodes equal to the time period we want to use to represent the term structure. At each node we assign a short rate, which is the one-period interest rate for the next period. The nodes in the lattice are indexed by \((k, s)\), where \( k \) is the time variable, \( k = 0, \ldots, T \), for maturity \( T \) and \( s \) represents the state, \( s = 0, \ldots, k \). There are \( M \) steps in the tree which gives us the length of the time intervals \( \Delta t = \frac{T}{M} \).

Figure (2.1) illustrates binomial tree where each column represents an instant of time, and moving from one instant to the next means either an up or down movement. Under the forward distribution the probabilities of moving up and down are both \( \frac{1}{2} \). The short rate at node \((k, s)\) is given by \( \hat{r}(k, s) \geq 0 \). Given the short rate, the one-period discount rate can be calculated by
\[
d(k, s) = e^{-\Delta t \hat{r}(k, s)}. \tag{2.2}
\]
Figure 2.1: Binomial tree with nodes \((k, s)\), where \(k = 0, \ldots, T\) and \(s = 0, \ldots, k\)

When the process is at any arbitrary node, the value of any interest rate security depends only on that node.

For a large number of single-node evaluations it is more convenient to apply the forward recursion method than working backwards through the lattice. The forward process requires only a single recursion whereas the backward process can become extremely computationally extensive. The forward recursion is based on calculating elementary prices, \(P_0(k, s)\) which is the price at time zero of a security that pays one unit at time \(k\) and state \(s\), and pays nothing at any other time or state. The prices \(P_0(k, s)\) are called elementary prices because they are the prices of elementary securities that have payoff at only one node.

If we suppose that elementary prices have been found for all nodes in the lattice for time from 0 through \(k\). Looking at node \((k+1, s)\) for \(s \neq 0\) and \(s \neq k + 1\), we see that at the previous time \(k\), there are two nodes leading to this state, nodes \((k, s)\) and \((k, s - 1)\), see Figure (2.2).

\[
\begin{align*}
(k, s) & \quad \rightarrow \quad (k+1, s) \\
(k, s-1) & \quad \rightarrow \quad (k, s)
\end{align*}
\]

Figure 2.2: One timestep from \(k\) to \(k + 1\)
Assuming a bond that only has the payment one unit at node \((k + 1, s)\), we know that by going backwards the bond would have the value \(\frac{1}{2}d(k, s - 1)\) at node \((k, s - 1)\) and \(\frac{1}{2}d(k, s)\) at node \((k, s)\). By the definition of the elementary prices, the values at time zero are

\[
\frac{1}{2}d(k, s - 1)P_0(k, s - 1)
\]

and

\[
\frac{1}{2}d(k, s)P_0(k, s).
\]

The total value at time zero is the sum of these two, and this is the elementary price at \((k + 1, s)\). Thus

\[
P_0(k + 1, s) = \frac{1}{2}d(k, s - 1)P_0(k, s - 1) + \frac{1}{2}d(k, s)P_0(k, s)
\]

This is a forward recursion since the value at time \(k + 1\) is expressed in terms of values at time \(k\). In the same manner, we can compute the elementary prices at every node in the lattice, except at the top and bottom nodes that only depend on one preceding node. Overall, we obtain the three forms of the forward equation, depending on whether the node is in the middle, at the bottom or at the top of the lattice.

\[
P_0(k + 1, k + 1) = \frac{1}{2}d(k, k)P_0(k, k), \quad s = k + 1
\]

(2.3)

\[
P_0(k + 1, s) = \frac{1}{2}[d(k, s - 1)P_0(k, s - 1) + d(k, s)P_0(k, s)], \quad 0 < s < k + 1
\]

(2.4)

\[
P_0(k + 1, 0) = \frac{1}{2}d(k, 0)P_0(k, 0), \quad s = 0
\]

(2.5)

Once the elementary prices are known, the price of any interest rate security can be found by multiplying the payoff at any node \((k, s)\) by the price \(P_0(k, s)\) and sum the results over all nodes that have payoffs. For example, summing all the elementary prices at time \(k\), which are the elements in column \(k\) of the lattice, gives the price of a zero-coupon bond with maturity \(k\)

\[
P(0, k) = \sum_{s=0}^{k} P_0(k, s).
\]

The price of a bond that pays 1 unit at time \(k = 0\) and state \(s = 0\), is 1, so the first elementary price \(P_0(0, 0)\) equals 1. Knowing this, it is possible to determine all other prices by forward recursion. The prices are strictly positive, since at each node the price is multiplied by \(\frac{1}{2}\) and by the strictly positive discount factors and are eventually summed up. If the elementary prices would not be strictly positive, it would mean that at some node \((k, s)\) the payout would be 1 unit by definition of the elementary price, although the price of the contract is zero or negative, which leads to an arbitrage opportunity. Conversely, since this is not true in our model, we know that there is no arbitrage. Hence the probabilities \(\frac{1}{2}\) are risk-neutral. We use the probability \(\frac{1}{2}\) because we want the short rate to have the same probability to go up as to go down.
In the Ho-Lee model, the short rate at node \((k, s)\) is represented as

\[
\hat{r}(k, s) = a(k) + b(k) \cdot s
\]

(2.6)

where the parameter \(a(k)\) is a measure of the aggregate drift from period 0 to \(k\) and \(b(k)\) is a volatility parameter. From node \((k, s)\), the short rate goes to node \((k + 1, s)\) with the risk-neutral probability \(\frac{1}{2}\) and to node \((k + 1, s + 1)\) with probability \(\frac{1}{2}\) with corresponding values

\[
\hat{r}(k + 1, s) = a(k + 1) + b(k + 1) \cdot s
\]

\[
\hat{r}(k + 1, s + 1) = a(k + 1) + b(k + 1) \cdot (s + 1)
\]

By matching the zero rates implied by the lattice method, with the known zero rates from the market data, one can adjust the parameter \(a(k)\) to make the term structure perfectly fit the market. The next section compares the parameters \(a(k)\) and \(b(k)\) with the parameters \(\theta(t)\) and \(\sigma\) to ensure that the discrete model is a good approximation of the continuous case.

![Figure 2.3: Ho-Lee short rate tree](image)

### 2.2 Comparing the continuous and discrete time Ho-Lee model

In this section we will compare the continuous time and discrete time Ho-Lee model and see if the discrete time model converges to the continuous model with very small \(\Delta t\). This is done by comparing the two models expectations and variances.

It should be noted that the discrete time short rate process and the ordinary (continuous time) short rate should be distinguished. Let us denote the discrete time short rate \(\bar{r}\) and the short rate as usual by \(\hat{r}\), then from Equation (2.6) it follows

\[
\bar{r}(k + 1) - \hat{r}(k) = a(k + 1) - a(k) + (b(k + 1) - b(k))s,
\]

with probability \(\frac{1}{2}\) and

\[
\bar{r}(k + 1) - \hat{r}(k) = a(k + 1) - a(k) + (b(k + 1) - b(k))s + b(k + 1),
\]

with probability \(\frac{1}{2}\).
The conditional expectation and variation in the discrete time frame equal

\[
E[\bar{r}(k+1) - \bar{r}(k) \mid \bar{r}(k) = \bar{r}(k,s)] = \frac{1}{2}[a(k+1) - a(k) + (b(k+1) - b(k))s] + \frac{1}{2}[a(k+1) - a(k) + (b(k+1) - b(k))s + b(k+1)]
\]

\[
= a(k+1) - a(k) + (b(k+1) - b(k))s + \frac{b(k+1)}{2}
\]

\[
Var(\bar{r}(k+1) - \bar{r}(k) \mid \bar{r}(k) = \bar{r}(k,s)) = \frac{(b(k+1))^2}{4}
\]

The expectation and variation in the continuous time frame equal

\[
E[r(t + \Delta t) - r(t) \mid F_t] = \theta(t)\Delta t + o(\Delta t)
\]

\[
Var(r(t + \Delta t) - r(t) \mid F_t) = \sigma^2\Delta t + o(\Delta t)
\]

We know that the discrete model converges to the continuous model when the first and second moment of the discrete model equals the first and second moment of the continuous model. Which means that the following must hold

\[
a(k+1) - a(k) + (b(k+1) - b(k))s + \frac{b(k+1)}{2} = \theta(t)\Delta t \quad (2.7)
\]

and

\[
\frac{(b(k+1))^2}{4} = \sigma^2\Delta t \Rightarrow \frac{b(k+1)}{2} = \sigma\sqrt{\Delta t}
\]

The Ho-Lee model assumes a constant volatility which implies that \( \frac{b(k+1)}{2} \) is independent of \( k \) and equal to \( \sigma\sqrt{\Delta t} \). Therefore \( b(k+1) - b(k) = 0 \). Hence Equation (2.7) equals

\[
a(k+1) - a(k) = \theta(t)\Delta t - \sigma\sqrt{\Delta t}. \quad (2.8)
\]

Since the discrete model should converge to the continuous model as \( \Delta t \to 0 \), the parameter \( a(k) \) should increase with \( \theta(t)\Delta t - \sigma\sqrt{\Delta t} \). The next section shows a numerical test of this approximation.
2.2.1 Numerical test

We test the approximations for a couple of different scenarios.

If the interest rates are constant, \( \theta(t) \), \( \sigma \) and the increments of \( a \) are zero. Which means that all \( a \)'s are equal.

When \( \theta(t) \) equals some constant \( \alpha \)

\[
a(k + 1) - a(k) = \alpha \Delta t - \sigma \sqrt{\Delta t}
\]

From the equation we see that \( a \) is a linear function, it grows with constant value since \( \Delta t \) and \( \sigma \) are constants. This can be seen in Figure (2.4)

![Figure 2.4: Plot of a(k) with \( \theta = 0.5, \sigma = 0.0001, \Delta t = \frac{1}{100} \), and \( r = 0.05 \)](image)

If we consider \( \theta(t) \) to be linear, \( \theta(t) = \alpha t \) then

\[
a(k + 1) - a(k) = \alpha t \Delta t - \sigma \sqrt{\Delta t}
\]

where \( \Delta t \) and \( \sigma \) are constants. We can see from the equation that \( a \) increases with \( \alpha t \) which means that \( a \) should be a quadratic function. This can indeed be seen in Figure (2.5).
When equating the bond prices from the discrete model with the bond prices from the continuous model, we find that Equation (2.8) holds for very small $\Delta t$, where $\Delta t = \frac{T}{m}$. Therefore we can say that this discrete time model gives a good approximation of the continuous time Ho-Lee model.
Chapter 3

Bond and bond futures pricing

3.1 Bond pricing

When trading with bonds and derivatives on bonds, it is important to know about the factors that have bearing on the bond price. Bonds can be priced at premium, discount or at par. If the bond’s price is higher than its par value, it will sell at a premium, which means that its interest rate is lower than current prevailing interest rates. Usually, the required yield on a bond is equal to or greater than the current interest rate in order to offer a decent rate of return to encourage investors. Some bonds trade at a premium because they offer attractive particularities such as that the coupon can be stripped easily from the bond etc. Fundamentally, however, the price of a bond is the sum of the present values of all expected coupon payments plus the present value of the par value at maturity. If \( B(c_i, t_i, t) \) denotes the coupon-bearing bond price with coupons \( c_i \) paid at time \( t_i \) and final payment at maturity \( L_N \) we have the following formula:

\[
B(c_i, t_i, t) = \sum_{i=1}^{N} c_i P(t, t_i) + L_N P(t, t_N)
\]

where \( P(t, t_i) \) is the price of a bond at time \( t \) that pays one at time \( t_i \).

A bond’s yield \( y \) is defined as the interest rate at which the present value of the stream of cashflows equals the bond’s market price.

\[
B(c_i, t_i, t) = \sum_{i=1}^{N} c_i e^{-(t_i-t)y} + L_N e^{-(t_N-t)y}.
\]

This equation is usually solved by using an iterative method.

To price a bond between payment periods an appropriate day-count convention needs to be applied. The day count is a way of measuring the appropriate interest rate for a specific period of time. In this thesis, an actual/actual day count will be applied since this is the type of day count used for Euro-Bund futures.

Accrued interest is the fraction of the coupon payment that the bond seller earns for holding the bond for a period of time between coupon payments. Hence, the accrued interest should be added to the price of the bond, this is called the dirty price. In
newspapers, the bond prices quoted are often clean prices. The accrued interest is calculated as follows

\[ AI = \frac{\text{Interest in the reference period} \times \text{Days between settlement and the last coupon}}{\text{Total days in period}} \]  
(3.1)

### 3.2 Bond futures pricing

A future on a bond is a contract that obliges the holder to buy or sell a bond at maturity. Often, this future consists of a basket of bonds. The basket of bonds to choose from consists of several bonds with different maturities and coupon payments. A conversion factor is needed to be able to compare them. The idea behind the conversion factor is to make the various bonds equal provided that the interest rate curve is flat with a given yield. In case of delivery, the short decides which bond to deliver among the basket of potential bonds. Obviously, the short would choose the Cheapest-to-Deliver bond, referred to as the CTD and it will be the one that maximizes the profit for the short position \(i = \text{ArgMax}_i (F_T \times CF^i - B^i)\).

The design of the bond futures purposely avoids a single underlying security. One reason for this is that if the underlying bond should lose liquidity, perhaps because it has been accumulated over time by buy and hold investors and institutions, then the futures contract would lose its liquidity as well. If we assume that there is only one bond deliverable in the futures contract, a trader may profit by simultaneously purchasing a large fraction of that bond issue and a large number of contracts. As the short party of the contract scrambles to buy that bond to deliver or buy back the contract she has sold, the trader can sell the holding of both bonds and contracts at prices well above their fair values. However, by making shorts hesitant to take positions, the threat of a squeeze can prevent a contract from attracting volume and liquidity. The existence of a basket of securities effectively avoids the problems of a single deliverable only if the cost of delivering the next to CTD is not that much higher than the cost of delivering the actual CTD.

#### 3.2.1 The conversion factor

The conversion factors represent the set of prices that would prevail in the cash market if all the bonds were trading at a yield equivalent to the contract’s notional coupon. The FGBL contract that we are studying in this thesis, has a notional coupon of 6%. It is assumed that the cash flows from the bonds are discounted at six percent and the notional of the bond to be delivered equals one. This means that when a bond has a yield of six percent, the conversion factor is equal to one. Further, if the bond has yield larger than six percent, the conversion factor is larger than one but the shorter the maturity, the closer the conversion factor comes to one. Likewise, when the yield of a bond is less than six percent the conversion factor is smaller than one, but with longer maturity the conversion factor converges to one. Generally the conversion factor for a bond with coupons \(c_i\) paid at time \(t_i\) is given by the following formula.

\[
CF = \frac{\sum_{i=1}^{N} c_i}{(1+0.06)^t} + \frac{LN}{(1+0.06)^N} = \frac{\sum_{i=1}^{N} c_i e^{-(t_i-t)0.06}}{LN e^{-(t_N-t)0.06}} + \frac{LN}{(1+0.06)^N}
\]
3.2.2 Features of the Euro-Bund futures contract - FGBL

The Euro-Bund futures contract is traded on the Eurex (www.eurexchange.com), the leading clearing house in Europe and one of the world’s largest derivatives exchanges which also operates the most liquid fixed income markets in the world. Eurex is jointly operated by Deutsche Börse AG and SIX Swiss Exchange. The trading volume at Eurex exceeds 1.5 billion contracts a year and connects market participants from 700 locations worldwide.

The underlying of the Euro-Bund futures contract are three bonds with an 8.5 to 10-year term and a 6% coupon. The par value is 100,000 euro and the price quotation is in the usual percent of the par value. The minimum price change is one basis point (0.01 percent) which corresponds to 10 euros. The delivery takes place every tenth calendar day of the respective quarterly month, if that day happens to be a non-trading day; it is the exchange day immediately succeeding that day. The last trading day is two exchange days prior to the Delivery Day of the relevant maturity month. It is on this last trading day that the clearing members with open short positions of the maturing futures must notify Eurex which debt instrument they will deliver (the CTD). The Final Settlement Price is established by Eurex on the Final Settlement Day at 12.30 CET based on the volume-weighted average price of all trades during the final minute of trading provided that more than ten trades occurred during this minute; otherwise the volume-weighted average price of the last ten trades of the day, provided that these are not older than 30 minutes. If such a price cannot be determined, or does not reasonably reflect the prevailing market conditions, Eurex will establish the Final Settlement Price.

In this thesis we will study the FGBL contract with expiry March 2010, which means that the delivery day is March 10, 2010. As all FGBL contracts, the March 2010 contract has three bonds as underlying assets

<table>
<thead>
<tr>
<th>Bond (ISIN)</th>
<th>Settle</th>
<th>Coupon</th>
<th>First Coupon</th>
<th>Maturity</th>
<th>Conversion Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE0001135374</td>
<td>2008-11-14</td>
<td>3.75</td>
<td>2010-01-04</td>
<td>2019-01-04</td>
<td>0.849118</td>
</tr>
<tr>
<td>DE0001135382</td>
<td>2009-05-22</td>
<td>3.50</td>
<td>2010-07-04</td>
<td>2019-07-04</td>
<td>0.825135</td>
</tr>
<tr>
<td>DE0001135390</td>
<td>2009-11-13</td>
<td>3.25</td>
<td>2011-01-04</td>
<td>2020-01-04</td>
<td>0.799913</td>
</tr>
</tbody>
</table>

Table 3.1: Deliverable bonds for the March 2010 FGBL contract

For simplicity, we rename the bonds to

<table>
<thead>
<tr>
<th>Bund 1</th>
<th>DE0001135374</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bund 2</td>
<td>DE0001135382</td>
</tr>
<tr>
<td>Bund 3</td>
<td>DE0001135390</td>
</tr>
</tbody>
</table>

Table 3.2: Bund 1, 2 and 3

Following is a graph of the price of Bund 1, 2 and 3 from 25 January - 8 March 2010.
3.3 Determining the CTD

The short position of the bond futures can choose which bond to deliver among a basket of bonds. Naturally, she will choose the bond that is the cheapest to deliver and that bond becomes the underlying asset of the contract. This brings us to two questions:

- Which bond is the Cheapest-to-Deliver?
- Given the price of this bond, what should be the quoted futures price?

Looking at the March contract, the Cheapest-to-Deliver bond must have been determined by the last trading day March 8, 2010. Although the bond will not be delivered until March 10, 2010 it must be determined which of the underlying will be the CTD on March 8, 2010. There are many parameters that influence the CTD. When yields are higher than the assumed yield for the conversion factor, the conversion methodology tend to favour low coupon bonds with long maturity. Similarly, when yields are lower, the CTD bonds are often high coupon bonds with short maturities. Further, when the yield curve is upward-sloping, there is a tendency for bonds with a long time to maturity to be favored whereas when it is downward-sloping, there is a tendency for bonds with short time to maturity to be delivered. To determine which bond is the Cheapest-to-Deliver, one needs to look at the existing cash flows. The short position receives

\[(\text{Settlement price} \times \text{Conversion Factor}) + \text{Accrued Interest}\]

by selling the futures contract. In addition to that, the short would need to buy the bond upon delivery to the long party and hence needs to pay the price of the bond;

\[
\text{Quoted clean price} + \text{Accrued Interest}.
\]

The CTD is therefore the bond that minimizes the following

\[
\text{Dirty bond price} - (\text{Settlement price} \times \text{Conversion Factor}) + \text{Accrued Interest} \Rightarrow
\Rightarrow \text{Quoted bond price} - (\text{Settlement Price} \times \text{Conversion Factor})
\]

where settlement price refers to the price of the future.
The cost of delivering each of the three underlying bonds is

Bund 1: 105.266 – 123.971 × 0.849118 = 0
Bund 2: 103.123 – 123.971 × 0.825135 = 0.835
Bund 3: 100.799 – 123.971 × 0.799913 = 1.634

We see that it costs nothing to deliver Bund 1 and to deliver any of the other bonds costs more. The basis always equals zero at maturity. Thus we can change the formula

\[ \text{Futures price} = \min_{i=1,2,3} \left( \frac{(\text{Clean Price of Bond})_i}{(\text{Conversion Factor of Bond})_i} \right) \]  \hspace{1cm} (3.2)

The following a graph of the price of the March 2010 futures contract from today, 25 January 2010 until maturity of the contract

![Figure 3.2: March 2010, Euro-Bund futures price from today until maturity](image)

We want to calculate the prices of the bonds at delivery of the future. One way to do this, is by making a Ho-Lee short rate lattice from today until maturity of the bonds. Since we are only interested in the bond prices between today and delivery, we could price the bonds by making a short rate tree from today until delivery of the future, and calculate the bond prices at delivery analytically with Equation (3.3) and Equation (2.1). When a bond has a series of coupons \(c_{ij}\) that are paid at \(t_{ij} = 1, \ldots, N\), where \(t_N\) is the maturity of the bond, then the price of this bond \(j = 1, 2, 3\) at time \(t\) is

\[ V(c_{ij}, t_{ij}, t) = \sum_{i=1}^{N} c_{ij} P(t, t_{ij}) + P(t, t_{ij}) \] \hspace{1cm} (3.3)

The total price of a coupon-bearing bond is the sum of the discounted coupon payments and the discounted notional. \(P(t, t_{ij})\) is the price of a bond at time \(t\) that pays one at time \(t_{ij}\). Hence a coupon \(c_{ij}\) that is paid at time \(t_{ij}\) must be discounted with \(P(t, t_{ij})\) to find the value of the coupon at time \(t\).

We want to find the price of the bonds at delivery \(t = t_d\), at the end nodes of a binominal tree with \(M\) steps. Hence we want to know the prices of the three underlying bonds at the nodes \((M, s)\), where \(s = 1, \ldots, M\). To simplify calculations we introduce two
new notations, the first is the discount factor at node \((M, s)\) with maturity \(t\), which is denoted by \(H((M, s), t)\) and the second is the price at node \((M, s)\) of a bond with coupons \(c^j\) at times \(t^j\) which we define as \(\bar{V}(c^j, t^j, (M, s))\). With these two new notation the bond pricing formula above can be expressed as

\[
\bar{V}(c^j, t^j, (M, s)) = \sum_{i=1}^{N} c_i^j H((M, s), t_i^j) + H((M, s), t_N^j). \tag{3.4}
\]

When all the bond prices are known at delivery, at the nodes \((M, s)\), we can calculate the bond prices at the nodes prior to delivery. For example, starting at node \((M - 1, s)\), one step later we are either at node \((M, s)\) or at \((M, s + 1)\), with probability \(\frac{1}{2}\). The price of a bond at node \((M - 1, s)\) is therefore

\[
\bar{V}(c^j, t^j, (M - 1, s)) = \frac{1}{2} d_{M-1,s} [\bar{V}(c^j, t^j, (M, s)) + \bar{V}(c^j, t^j, (M, s + 1))]
\]

where \(d_{M-1,s}\) is the discount factor at state \(s\), for the period \(M - 1\) to \(M\).

To find the price of the FGBL contract at delivery at the final nodes of the tree \((M, s)\) for \(s = 1, \ldots, M\) we start by finding the price one time step before at \(M - 1\).

\[
F((M - 1, s), t) = \frac{1}{2} \left[ F((M, s), t) + F((M, s + 1), t) \right]
\]

since from node \((M - 1, s)\) we move to node \((M, s)\) with probability \(\frac{1}{2}\) and to node \((M, s + 1)\) with probability \(\frac{1}{2}\). Futures prices are settled daily, which means that we do not have to add a discount factor. In other words, the futures price is the average of the next two prices using the risk-neutral probabilities without discounting. By going backwards through the lattice and computing the averages in the same manner, we can find today’s price of the future with delivery \(t\) at the initial node \((0, 0)\).

3.4 The elements of bond pricing

We have seen that the bond prices can be computed by Equation (3.4),

\[
\bar{V}(c^j, t^j, (M, s)) = \sum_{i=1}^{N} c_i^j H((M, s), t_i^j) + H((M, s), t_N^j).
\]

To calculate the discount factor \(H((M, s), t_i^j)\) for bond \(j\) we make use of Equation (2.1)

\[
P(t_i^j) = \frac{P(0, t_i^j)}{P(0, t)} e^{-(t_i^j - t)(t_i^j - f(0, t)) - \frac{1}{2} \sigma_i^2 (t_i^j - t)^2}
\]

which means that at node \((M, s)\) the discount factor \(H((M, s), t_i^j)\) equals

\[
H((M, s), t_i^j) = \frac{P(0, t_i^j)}{P(0, t)} e^{-(t_i^j - t_d)(t_i^j - f(0, t_d)) - \frac{1}{2} \sigma_d^2 (t_i^j - t_d)^2}. \tag{3.5}
\]

The parameters in Equation (3.5) need to be determined

- To find \(P(0, t_i^j)\) and \(P(0, t_d)\) we interpolate the logarithm of the discount factors, which follow from the zero curve.
• The short rates $\hat{r}(M, s)$ follow from the binomial tree that is constructed according to the Ho-Lee method, see section (2.1.2).

• The forward rate is determined by

$$f(0, t_d) = -\frac{\partial}{\partial t} \ln P(0, t)|_{t=t_d}$$

We know that this derivative must be a constant since we interpolate linearly over the logarithm of the discount factors.

• To match the spot rate curve to the Ho-Lee model we need to make some assumption concerning the volatility. We will assume that the volatility is $0.01$ per year which means that the short rate is likely to fluctuate about $1$ percentage point during the year.

Further details on how to calculate a zero curve and a Ho-Lee short rate tree can be found in the following sections.

### 3.4.1 Zero Curve

Bootstrapping is a method to calculate the zero curve from a series of coupon-bearing bonds. With the zero rates at time $t$ with maturity $t_i$ for $i = 1, \ldots, N$ denoted by $z(t, t_i)$ we have the price of a coupon-bearing bond with coupons $c$ at times $t_c$

$$V(c, t_c, t) = \sum_{i=1}^{N} c_t e^{-(t_i-t)z(t, t_i)} + e^{-(t_N-t)z(t, t_N)}.$$  \hfill(3.6)

In order to get reliable rates, the bonds that are used for bootstrapping must be liquid. If not, it is better to exclude them from the bootstrapping.

We want to solve for the zero rate, hence we rewrite Equation (3.6)

$$z(t, t_N) = -\frac{1}{t_N-t} \ln \left( \frac{V(c, t_c, t) - \sum_{i=1}^{N-1} c_t e^{-(t_i-t)z(t, t_i)}}{1 + c_t} \right).$$

As can be seen, if we know the bond price $V(c, t_c, t)$ and the zero rates $z(t, t_i)$ for $i = 1, \ldots, N-1$ we can solve for the next zero-rate $z(t, t_N)$. Which means that if we know the zero rates up to a certain time, the next zero rate can easily be found by discounting all earlier cash flows with the known zero rates. This can be done for every maturity $t_N$ since the above equation is an iterative solution algorithm.

The bootstrapping method is closely related to a method called interpolation. This is because it is unlikely that there are always bonds available that expire on the exact times $t = 1, \ldots, N-1$, which means that the earlier rates are not always known. This can be solved by using the interpolation method. Interpolation is a technique that calculates the intermediate (unknown) zero rates when the zero rates are only known for a couple of time points. Since the zero curve cannot be determined uniquely by bootstrapping it is often necessary to complement with an interpolation scheme.

There are a lot of different ways to interpolate. The following is a quality criteria for choosing an interpolation method.
The forward rates must be positive to avoid arbitrage.

Continuous forward rates

How local is the interpolation method? If an input is changed, does the interpolation function only change nearby or also for the rest of the rates?

Stable forward rates. If an input is changed with one basis point up or down, what is the change in the forward rate?

The best interpolation method takes all the points above into account. However, implementing such an interpolation method can be very complex, thus we will keep to the raw interpolation or linear interpolation on the log of discount factors. In this method, the forward rates are not continuous but piecewise continuous. Naturally, the forward rates are positive to avoid arbitrage.

For \( t_i < t < t_{i+1} \) the continuously compounded risk-free rate for maturity \( t \) is

\[
z(t) = \frac{t - t_i}{t_{i+1} - t_i} z(t_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} z(t_i)
\]

Taking the exponential

\[
e^{z(t)} = e^{\frac{t - t_i}{t_{i+1} - t_i} z(t_{i+1})} \cdot e^{\frac{t_{i+1} - t}{t_{i+1} - t_i} z(t_i)}.
\]

We take the \( t \)-th power to get the continuously compounded zero rates

\[
(e^{z(t)})^t = (e^{\frac{t - t_i}{t_{i+1} - t_i} z(t_{i+1})})^t \cdot (e^{\frac{t_{i+1} - t}{t_{i+1} - t_i} z(t_i)})^t
\]

\[
\Rightarrow e^{t z(t)} = e^{\frac{t - t_i}{t_{i+1} - t_i} t z(t_{i+1})} \cdot e^{\frac{t_{i+1} - t}{t_{i+1} - t_i} t z(t_i)}
\]

\[
= (e^{-t_i z(t_{i+1})})^{\frac{t - t_i}{t_{i+1} - t_i}} \cdot (e^{-t_i z(t_i)})^{\frac{t_{i+1} - t}{t_{i+1} - t_i}}
\]

Substituting \( P(0, t) = e^{-t z(t)} \) we get

\[
P(0, t) = P(0, t_{i+1})^{\frac{t - t_i}{t_{i+1} - t_i}} \cdot P(0, t_i)^{\frac{t_{i+1} - t}{t_{i+1} - t_i}}
\]

(3.7)

Hence, by linearly interpolating \( \log(P) \), and then reverting it to \( P \) by taking the exponential, we can find all the intermediate discount factors. The corresponding spot rates are given by \( z(t) = -\frac{1}{t} \ln P(0, t) \) and the forward rates equal \( f(0, t) = -\frac{\partial}{\partial t} \ln P(0, t) \).

Following is a general explanation of how to interpolate.

From the bond prices that are known, the zero rates can be computed. The spot rates are given by

\[
z(1), z(2), \ldots, z(T)
\]

where \( T \) is the maturity. We want the tree to have \( M \) steps, so the time steps in the interpolated tree are \( \Delta t = \frac{T}{M} \). The new interpolated spots rates are thus

\[
z(\Delta t), z(2\Delta t), \ldots, z((M - 1)\Delta t), z(M\Delta t).
\]
In Bloomberg and Reuters, the prices of coupon-bearing bonds can be found, that can be used for bootstrapping and interpolation. In this thesis we will use bonds with maturities of two until ten years. Since these bonds are coupon-bearing, we first need to strip them by applying the bootstrapping method.

The first bond we use for the bootstrapping is a 2 year bond, which means that we need to find the zero curve for the periods shorter than two years. This is done by utilizing the Eonia swap, which is a type of plain vanilla interest rate swap. The Eonia swap gives the zero rates from today until one week, two weeks, three weeks, one month, \ldots, twelve months. Once the zero rates have been found we can calculate the discount factors by applying the following equation

\[ P(t, T) = e^{-(T-t)z(t, T)}. \]

There are bonds available for maturities three and six months as well as one year, however, they are not considered to be liquid. Which means that they are not traded much, this in turn would result in a zero curve with extreme fluctuations. The Eonia swap rates on the contrary are very liquid and give more reliable rates. It is possible for us to combine the Eonia rate with the zero curve from the government bonds since in both cases the credit risk is considered very low. To make the zero curve as accurate as possible for the pricing of bonds 1, 2 and 3, we have included the bonds in the bootstrapping. In this way, we are sure that the bonds are priced well.

The following table illustrates the bonds that were used to create the zero curve.

<table>
<thead>
<tr>
<th>Bond</th>
<th>Settle</th>
<th>Next Coupon</th>
<th>Maturity</th>
<th>Coupon Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 2</td>
<td>2002-07-05</td>
<td>2010-07-04</td>
<td>2012-07-04</td>
<td>5.00</td>
</tr>
<tr>
<td>Bond 3</td>
<td>2003-07-04</td>
<td>2010-07-04</td>
<td>2013-07-04</td>
<td>3.75</td>
</tr>
<tr>
<td>Bond 5</td>
<td>2005-03-20</td>
<td>2010-07-04</td>
<td>2015-07-04</td>
<td>3.25</td>
</tr>
<tr>
<td>Bond 6</td>
<td>2006-11-17</td>
<td>2011-01-04</td>
<td>2017-01-04</td>
<td>3.75</td>
</tr>
<tr>
<td>Bond 7</td>
<td>2007-05-25</td>
<td>2010-07-04</td>
<td>2017-07-04</td>
<td>4.25</td>
</tr>
<tr>
<td>Bond 8</td>
<td>2008-05-30</td>
<td>2010-07-04</td>
<td>2018-07-04</td>
<td>4.25</td>
</tr>
<tr>
<td>Bond 8.5</td>
<td>2008-11-14</td>
<td>2011-01-04</td>
<td>2019-01-04</td>
<td>3.75</td>
</tr>
<tr>
<td>Bond 9.5</td>
<td>2009-11-13</td>
<td>2011-01-04</td>
<td>2020-01-04</td>
<td>3.25</td>
</tr>
<tr>
<td>Bond 14</td>
<td>1994-01-04</td>
<td>2011-01-04</td>
<td>2024-01-04</td>
<td>6.25</td>
</tr>
<tr>
<td>Bond 17</td>
<td>1997-07-04</td>
<td>2010-07-04</td>
<td>2027-07-04</td>
<td>6.50</td>
</tr>
</tbody>
</table>

Table 3.3: Data of the bonds used for bootstrapping the zero curve
Figure 3.3: Zero curve on 25 January 2010

Figure 3.4: Interpolated zero curve on 25 January 2010
3.4.2 Short Rate Tree

We want to build a short rate tree with M steps, from today $t=0$ until March 8, 2010, $t = t_d$. According to the Ho-Lee model, we need to know the zero curve and the volatility $\sigma$ on $[0, t_d]$. As previously mentioned, the volatility is assumed to be $\sigma = 0.01$ and the zero curve is derived through bootstrapping and interpolation of Euro bonds in addition to the Eonia swap rates. To build a complete short rate tree we need to determine the parameters $a(k)$ and $b(k)$ in Equation (2.6). $b(k)$ follows from the comparison of the continuous and discrete case of the Ho-Lee model and equals

$$\frac{b(k + 1)}{2} = \sigma \sqrt{\Delta t} \Rightarrow b(k + 1) = 2 \cdot \sigma \sqrt{\Delta t}.$$ 

The parameter $a(k)$ is found through the forward rates which are extracted from the zero curve,

$$a(k) = f(k) - b(k) \cdot \Gamma(k)$$

where $f(k)$ is the forward rate for the period $k$ and $\Gamma(k) = \sum_{i=1}^{k+1} \gamma_i \cdot (i - 1)$ with $\gamma_i = Bin(i, k, \frac{1}{2})$.

Once the short rate tree has been determined we can find the bond prices $V(c, t_c, (M, s))$ through Equation (3.4) and (3.5). Finally, with Equation (3.2) we can calculated the futures prices $F((M, s), t_d)$ at the nodes $(M, s)$.

To determine the bond prices at the earlier nodes, we need to create a discount factor tree which is done by Equation (2.2) applied to the nodes in the short rate tree.
Chapter 4

Analysis

In this chapter, the results of the model is studied using real market data. We calculate the bond prices and the futures prices according to Equations (3.5), (3.4) and (3.2) at time $t = t_d$ of the short rate tree. In the first section of the analysis, we take a look at how many steps are needed in the Ho-Lee model to obtain reliable results. Further, it is shown how well the futures and bond prices are fitted when assuming a constant volatility. In section (4.2) we analyse which bond is the cheapest to deliver and at what short rate the CTD switches from one bond to another. Concluding, we take a look at the sensitivity of the futures prices when the bond prices change.

4.1 Increasing the number of steps in the tree

We want to find out how many steps are needed to obtain reliable values for the bond and futures prices. We expect that the bond and futures prices converge to some constant value, because when increasing the number of steps, the more the discrete time model should converge to the continuous time model.

![Figure 4.1: Difference in theoretical and market prices of Bunds](image)
In figure(4.1) we can see that the differences between the Ho-Lee and the market bund prices decrease when the numbers of steps increase. As we assumed, when taking more steps, the discrete time model converges to the continuous time model and the errors become smaller. The reason why they do not converge to zero might be due to the assumption regarding a constant volatility and how the short rate tree constructed according to the Ho-Lee model. Also we should consider the difference in term structure of interest rate that the market utilizes. In the zero curve constructed in section (3.4.1), we applied the Eonia swap rates, which are interbank interest rates. We made the decision that the Eonia swap rates would be a good replacement for the short term government rates, since they have very low credit risk, just like the government bonds. However, usually there is a slight difference in these two rates, which might have caused the gap between the market and theoretical bond prices. Further, there will always be a minor error due to the discretization of the continuous Ho-Lee model. We see that from approximately 300 steps and on, the bond and futures prices do not improve significantly anymore.

### 4.2 Determining which bond is the cheapest to deliver

By calculating the bond and futures prices at every end node, we can find out which bond is the cheapest to deliver at each node. We construct a table of each day from 25th February to maturity, the highest and lowest short rate at that date with the corresponding cheapest to deliver. We can see for which values of the short rate Bund 1 and Bund 3 switch from being the cheapest to deliver. In Table (4.2) we can see that between the lowest short rate and the lower switching value, Bund 1 is the CTD and between the higher switching rate and the highest short rate Bund 3 is the CTD. The table also gives an overview for which value the CTD changes from Bund 1 to Bund 3. The short rates can be negative, as was already mentioned in Section 2.1. This can be seen clearly in Figure (4.2)

![Figure 4.2: Highest, lowest short rate levels and Bund 1 and 3 switching levels](image)
Table 4.1: Data of short rates at which the CTD changes from Bund 1 to Bund 3

<table>
<thead>
<tr>
<th>Date</th>
<th>Lowest rate</th>
<th>Bund 1 switch</th>
<th>Bund 3 switch</th>
<th>Highest rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>25 February 2010</td>
<td>-3.6</td>
<td>5.09</td>
<td>5.12</td>
<td>9.43</td>
</tr>
<tr>
<td>26 February 2010</td>
<td>-3.77</td>
<td>4.82</td>
<td>4.90</td>
<td>9.72</td>
</tr>
<tr>
<td>27 February 2010</td>
<td>-4.30</td>
<td>4.69</td>
<td>4.74</td>
<td>9.89</td>
</tr>
<tr>
<td>28 February 2010</td>
<td>-4.55</td>
<td>4.57</td>
<td>4.61</td>
<td>10.02</td>
</tr>
<tr>
<td>1 March 2010</td>
<td>-4.10</td>
<td>4.23</td>
<td>4.30</td>
<td>9.70</td>
</tr>
<tr>
<td>2 March 2010</td>
<td>-3.61</td>
<td>4.55</td>
<td>4.58</td>
<td>9.13</td>
</tr>
<tr>
<td>3 March 2010</td>
<td>-3.44</td>
<td>4.81</td>
<td>4.85</td>
<td>9.24</td>
</tr>
<tr>
<td>4 March 2010</td>
<td>-3.17</td>
<td>4.73</td>
<td>4.78</td>
<td>9.05</td>
</tr>
<tr>
<td>5 March 2010</td>
<td>-2.12</td>
<td>4.56</td>
<td>4.59</td>
<td>7.36</td>
</tr>
<tr>
<td>6 March 2010</td>
<td>-1.42</td>
<td>4.52</td>
<td>4.56</td>
<td>6.95</td>
</tr>
<tr>
<td>7 March 2010</td>
<td>-1.17</td>
<td>4.49</td>
<td>4.52</td>
<td>6.88</td>
</tr>
<tr>
<td>8 March 2010</td>
<td>-1.10</td>
<td>4.70</td>
<td>4.73</td>
<td>7.06</td>
</tr>
</tbody>
</table>

We have seen which bond is the cheapest to deliver at the delivery of the future. Knowing which bond is the cheapest to deliver at maturity, we can calculate the probability that Bund 1, Bund 2 or Bund 3 will be the cheapest at maturity in the nodes prior to delivery. To do this we construct a tree with the same number of nodes as the short rate tree, at each node we know the probabilities going backwards from maturity by weighing the CTD at the end node by $\frac{1}{2}$, which is the probability of going up or down in the tree. Doing this for every node we can create a graph to summarize the total probabilities. Naturally, since Bund 2 is the cheapest to deliver in none of the end nodes of the tree, the probability that Bund 2 becomes cheapest in any node prior to delivery is zero in the whole tree.

The lower right triangle in Figure (4.3) shows the probability tree, where the light blue area is where Bund 1 is cheapest and the red area illustrates the probability of Bund 3 being the cheapest. In the multi-coloured area in between there is a probability that Bund 1 as well as Bund 3 will become the cheapest at delivery.

Figure 4.3: Lower right triangle; probability tree
4.3 Sensitivity of the futures price

In this section we will look at how the bond prices influence the futures price. We assume that the bond prices are not correlated and look at what happens to the theoretical futures price when one of the three bond prices increases or decreases by one.

To measure the sensitivity we will study the ratio

$$\frac{\Delta F_j}{\Delta Bond_j} = \frac{\text{Original futures price} - \text{futures price after changing the bond price}}{\text{Original bond price} - \text{bond price after change}},$$

where $j = 1, 2, 3$. To see how bond prices affects the futures price we will start by changing one of the three bond prices by one, which is approximately a one percent change. Changing the bond price means that the zero curve, the short rate tree as well as the futures prices will change.

We know that at delivery, the futures price equals the minimum of the three bonds divided by their conversion factors, see Equation (3.2). This means that when changing the bond prices, it is possible that another bond becomes the cheapest to deliver. Naturally, price changes of the bonds that are not the cheapest to deliver do not influence the futures as much as prices changes of the CTD.

How much influence each bond has on the futures price can be seen in Figure (4.4), (4.5) and (4.6). Starting with Bund 1, we see that if it were to decrease by one, the futures price decreases with approximately 1.17 at every time point, see Figure (4.4). Whereas changing Bund 2 and 3 by one has less affect on the futures price. The reason for this is quite intuitive, the futures price depends solely on the price of the CTD and since Bund 1 is always cheaper than the other bonds it has the most bearing on the futures price.

![Futures Sensitivity, Bund 1](image)

Figure 4.4: Bund 1’s influence on the futures price
Dividing the price change of Bund 1 by the conversion factor, we get

\[
\frac{1}{CF_1} = \frac{1}{0.849118} \approx 1.18.
\]

However, if Bund 1 increases by one the influence on the futures price is smaller. This follows from the change of the cheapest to deliver bond. After increasing Bund 1 by one it is no longer the cheapest at all time points. For example, looking at date 11 February 2010 we see that Bund 2 is the cheaper and hence determines the futures price.

\[
\frac{\text{Bund 1} + 1}{CF_1} = \frac{105.77}{0.849118} \approx 124.56
\]
and
\[
\frac{\text{Bund 2}}{CF_2} = \frac{102.50}{0.825135} \approx 124.22.
\]

Studying the price changes of Bund 2, we see that when decreasing the price by one while keeping the other bond prices fixed, Bund 2 also becomes cheaper than Bund 1 at some time points. We analyze the break even point and the influence on the future price when decreasing the price of Bund 2 by looking at the value of

\[
\frac{\text{Bund 1}}{CF_1} - \frac{\text{Bund 2-1}}{CF_2}.
\]

(4.1)

The following is a diagram Equation (4.1) plotted against changes in the futures price.

![Figure 4.7: Difference in futures price when decreasing Bund 2 by one](image)

From the diagram we can see that when Equation (4.1) is negative, which means that Bund 1 is cheaper than Bund 2, Bund 2 barely influence the futures price. And the opposite holds when Bund 2 is the cheapest, it has a big effect on the futures price.
Chapter 5

Summary and conclusions

In this thesis we have discussed the following questions:

- How well does the Ho-Lee model fit the prices of bonds and futures?
- How many steps are needed in the binomial tree to get a good result?
- How does changes in the term structure affect which bond is the cheapest to deliver?
- Is it possible to predict which bond will be the cheapest to deliver?
- How sensitive is the futures price to changes in the zero curve?

To answer these questions, we firstly determined an appropriate term structure model, namely the Ho-Lee model. The reason behind the choice of model was based on the qualities of the model. The Ho-Lee model is an exogenous term structure model and is very suitable for building recombining trees. Further, the Ho-Lee model has a decent method for computing the bond prices analytically. In Section (2.2) we compare the discrete and continuous time Ho-Lee models and a numerical approximation of this comparison was made.

In Chapter 3, the Euro-Bunds are introduced with details and specifications of the contracts. Further, we take a look at how the bonds and futures prices are calculated as well as how the cheapest to deliver bond can be determined from the bond prices and their conversion factors. In the subsequent sections, the necessary elements of bond pricing is studied and the method of constructing a Ho-Lee short rate tree is explained.

Concluding, in Chapter 4 we make use of market data to find the answers to the questions stated above.

- **How many steps are needed in the binomial tree to get a reliable result?**
  We made calculations for different numbers of time steps in Section (4.1) and found that from 300-400 steps and onwards, the bond and futures prices do not improve significantly. Hence we concluded that at least 300 steps are needed in the binomial tree to make the results reliable.

- **How does changes in the term structure affect which bond is the cheapest to deliver?**
  As studied in Section (4.2), it could be seen that the most probable cheapest to
deliver bond is Bund 1. Further, we could conclude that Bund 3 only becomes cheapest when the short rate is very high.

- **Is it possible to predict beforehand which bond will be the cheapest to deliver?**  
  Obviously, it is impossible to give an exact prediction of which bond will be the CTD on March 8 2010, since the term structure is stochastic. However, we can calculate the probabilities of the different bonds to be the CTD. To make a prediction we create a tree with probabilities at each node that Bund 1, 2 or 3 will become the CTD. Looking at the March 2010 contract, we concluded that Bund 1 is the cheapest at most of the nodes in the tree which means that this is the most probable scenario at maturity. The probability tree, Figure (4.3) shows that Bund 3 can become the cheapest at some points. The most probable scenario is that Bund 1 is the cheapest, since only when the short rate increases enormously the CTD changes from Bund 1 to Bund 3. The probability that Bund 2 would become cheapest is zero at the end node and hence also at all other nodes in the three. However, the next question discusses scenarios when Bund 2 becomes the cheapest to deliver.

- **How sensitive is the futures price to changes in the zero curve?**  
  The futures price is very sensitive to changes in the term structure. When Bund 1, that is the CTD decreased by one, the futures price decreased by approximately 1.17 at all time points. Increasing Bund 1 by one changed the futures price by 0.5 – 1.17 (see Figure (4.4) ) because at the majority of the dates, Bund 1 remained the CTD. Bund 2 and 3 also effects the futures price, however with much less than the CTD. If the price of Bund 2 decreases by one, the change in futures price is at most 0.72 and small changes in the price of Bund 3 barely effects the futures prices.  
  It should be noted that bond prices are usually highly correlated. This means that when changing the price of one of the bonds, it is unlikely that the prices of the other bonds stay fixed. There is a small probability that any of the other bonds will become the CTD, since Bund 1 has the most influence on the futures price even when it decreases or increases by one. We previously mentioned that Bund 2 could become the CTD at some dates, but this conclusion was based on the fact that the other bonds were kept fixed. If taking into account that the bonds are correlated, then the probable scenario would be that Bund 1 as well as Bund 3 will decrease as Bund 2 decreases, which leaves very little chance that Bund 2 would indeed become the CTD.

- **Is the Ho-Lee model a satisfactory model to price bonds and futures?**  
  From the analysis made in Chapter 4 we can conclude that the Ho-Lee model is a very appropriate model to price bonds and bond futures. With a tree of 300 steps, the futures and bond prices were very well fitted to the market prices. The small difference between the theoretical and the markets futures prices could stem from the difference in term structure of interest rate that the market utilizes. In the zero curve constructed in section (3.4.1), we applied the Eonia swap rates, which are interbank interest rates. We made the decision that the Eonia swap rates would be a good replacement for the short term government rates, since they have very low credit risk, just like the government bonds. However, usually there is a slight difference in these two rates, which might have caused the gap between the market and theoretical futures price. Further, the constant volatility that is assumed in this thesis, is another factor that causes discrepancies between
the market and the theoretical futures price.

5.1 Suggestions for future studies

Some ideas for future investigation

- studying the bond and futures prices by another term structure, i.e. the Hull-White model.

- instead of assuming a constant volatility, calculate the implied volatility through the OGBL which is an option with the Euro-Bund contract as the underlying. In this case, we would need to employ two different volatilities since the implied volatility from OGBL is only valid for the period until the futures maturity. The 'forward volatility' needed for the period between the futures maturity and the bonds maturity can be obtained from the 10-year cap or equivalent measure

- fitting the implied volatility from OGBLs mentioned above with different strikes.

- studying a time-varying volatility that is fitted at more time points than the two mentioned above.

- derive the zero curve through another interpolation method.

- investigate the influence of the volatilities on the futures price.

- employing an alternative source for short term rates for the first two years, i.e. repo rates. The repo rates are rates at which one prime bank offers funds in euro to another prime bank.
Bibliography


