Optimal proxy-hedging of options on illiquid baskets

Peter Larsson Leonhard Flohr

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Abstract

When hedging a written option on an equity basket one could use the stock and option markets of each of the components, and the risk-free bond market, to create a replicating portfolio. However, when the market components are illiquid - or non-existent - one need to proxy hedge. In that case the trader has to consider both the transaction costs and the variance of the difference between the basket and the hedging portfolio. A common strategy among practitioners is to hedge basket options with index futures of which the basket is a subset. The motive for this strategy is simplicity and low transaction costs. Other alternatives are to use liquid stocks with similar characteristics, such as a liquid subset of the basket or stocks in the same industry, and vanilla options written on these.

This thesis aims to evaluate the performance of different proxy-hedging strategies given the level of the transaction costs at the time of the rebalance of the hedging portfolio. The transaction costs are thought of as half of the bid-ask spread, i.e. the difference between the frictionless price and the buy or the sell price. The strategies will be evaluated under assumptions of the Black-Scholes framework.

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Contents

1 Introduction

A basket option is a financial derivative that has several underlying assets, such as equities, indices or currencies. There are many different types of basket options and they are often traded over the counter. The pay-off function of basket options often have exotic features such as path-dependence or barriers. In this thesis, a European basket option with three correlated underlying assets will serve as an example in an attempt to make the contents easier to grasp.

Pricing and hedging a basket option is not a trivial task. Correlations between underlying assets must be considered and the underlying distribution must be approximated since the sum of lognormal random variables is no longer lognormal distributed. This means that there does not exist an analytical expression for the price of a basket option. There are some papers treating this subject, for example the Nengjiu Ju approximation model presented in [6] and the moment matching methods presented in [4], that both approximates the price of a basket option.

The Black-Scholes option pricing formula assumes that it is possible to trade continuously under no transaction costs in order to perfectly replicate the payoff of the option. However, in reality, neither of this is possible. In the presence of transaction costs, there is a trade-off between hedging often to improve replication of the payoff, and hedging less to decrease the level of transaction costs.

This thesis aims to evaluate the performance of different strategies to hedge basket options. In contrast to many of the papers published in this area, transaction costs will not be neglected. The transaction costs are thought of as half of the bid-ask spread, which is treated as a free parameter. The bid-ask spread will be varied from 0 % to 5 % in order to illustrate how illiquidity affects the performance of the hedge. Both static, i.e. discrete hedging at predetermined fixed points in time, and dynamic hedging methodologies will be treated and compared. When hedging at discrete points in time, Leland derived a modified pricing formula by transforming the volatility, see [7] and [1]. In the one-dimensional case, his results will be evaluated and related to the results presented in this thesis.

There are several models used by practitioners to model the market, like the famous Black-Scholes model [3] in a multidimensional version, the Heston model that incorporates stochastic volatility and the Bates model which basically ads jumps to the Heston model. For simplicity, stochastic volatility will not be considered in this thesis and therefore the hedging strategies will be evaluated under the Black-Scholes framework. This implies that neither volatilities nor correlations will depend on time.

This thesis is organized as follows: In Section 2 the theoretical background will be treated. Numerical examples are presented in Section 3. Finally, Section 4 summarizes and concludes the thesis.

2 Theory

2.1 Pricing and Hedging in One Dimension

2.1.1 The Black-Scholes Model

Consider a financial market consisting of only two assets, a risk free asset and a stock. The price process B and S is associated with the risk free asset and the stock respectively. According to Black-Scholes [3], the dynamics is given by:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$
$$dB_t = rB_t dt$$

where σ, α and r are deterministic constants representing the volatility of the underlying asset, the local mean rate of return of S and the risk free interest rate respectively. W is a Brownian process.

The most important derivative that can be priced using the Black-Scholes model is the European call option. A European call option with exercise price K and time of maturity T written on an underlying asset S is defined as [2]:

The holder of the option has the right to buy one share of the underlying stock at price K at time T from the underwriter.

The value of the European option is completely defined by the underlying asset S, which is why it is called a derivative instrument or a contingent claim. A contingent claim with time of maturity T is called a simple T-claim if it is of the form:

$$X = \Phi(S(T))$$

The function Φ is called the contract function and for a European call option it is given by:

$$\Phi(x) = \max[x - K, 0]$$

2.1.2 Hedging Options with a Replicating Portfolio

Suppose that we have sold a European call option, X, with strike price K and maturity T at time $t = 0$. Furthermore suppose that we have charged the arbitrage free price $C(S, T, K, \sigma)$. The portfolio is rebalanced at the following discrete points in time: $t_0 = 0, t_1, ..., t_{n-1}$, where $t_j = jT/n$.

The value process for the hedge portfolio, given that we hedge using the underlying asset and a risk free asset, is then given by:

$$H(t_j) = h_0(t_j)B(t_j) + h_1(t_j)S(t_j)$$

Since we have also sold a call option, C, the value of our total portfolio is given by:

$$V(t_j) = h_0(t_j)B(t_j) + h_1(t_j)S(t_j) - C(t_j, S(t_j))$$

We are interested in the hedging errors, i.e. the distribution of $V(T) = H(T) - C(T)$ where $V(T)$ is the value of the total portfolio at time T, $H(T)$ is the value of the hedge portfolio at time T, and $C(T)$ is the value of the call option at time T.

The hedging weights $h_0(t_j)$ and $h_1(t_j)$ is given by [2]:

$$\begin{cases} h_0(t) = \frac{f(t,S(t)) - S(t)\frac{\partial f}{\partial S}(t,S(t))}{B(t)} \\ h_1(t) = \frac{\partial f}{\partial S}(t, S(t)) \end{cases}$$

where $h_0(t_j)$ denotes the number of bonds and $h_1(t_j)$ the number of stocks. $f(t, S(t))$ is the price of the T-claim, in this case the Black-Scholes formula for a plain vanilla call option. The derivative of the option price with respect to the price of the underlying asset, $\frac{\partial f}{\partial S}(t, S(t))$, is called the delta and determines how many stocks you should buy to replicate the derivative.

2.1.3 Dynamic Hedging

An alternative to hedging at discrete points in time is to hedge dynamically. The point is to only rebalance the portfolio when it is necessary. This requires some condition to be defined that determines when to hedge. Throughout this thesis, this condition is defined as follows:

$$abs\left(\frac{(h_1 - \Delta_t) \cdot S_t}{c_t}\right) > b \tag{2.1}$$

Here h_1 is the number of stocks held in the hedging portfolio just before a possible rebalance of the portfolio, Δ_t is the delta at time t and c_t the price of the option to be hedged at time t, S_t is the price of the underlying asset and b the boundary. The bound is expressed relative to the option price at time t so that it is not sensitive to movements in the stock price.

2.1.4 Leland's Transformation of Volatility

The assumption of no transaction costs in the Black-Scholes framework is not very realistic. The hedging of a derivative requires frequent trading which will increase costs, otherwise the hedge will not be particularly accurate. There is some research in the field of pricing and hedging derivatives in presence of transaction costs, starting with Leland [7]. He showed that a call option could be priced according to Black-Scholes formula even in a world where transaction costs exist, but the volatility must be transformed to incorporate this. The transformation Leland suggested was:

$$\sigma_A = \sigma\sqrt{1+A} \tag{2.2}$$

$$A = \sqrt{\frac{2}{\pi}} \cdot \frac{k}{\sigma\sqrt{\delta_t}} \tag{2.3}$$

Here σ is the volatility of the underlying asset, A is the so called Leland's number, δ_t is the interval between two rebalances of the portfolio and k is the relative bid-ask spread. A large Leland number corresponds to large transaction costs, i.e. high frequency of rebalances of the hedging portfolio. The model is restricted to derivatives with convex payoff functions, such as plain vanilla calls and puts. Whalley and Willmot [9] presented another transformation that is valid for arbitrary payoff functions:

$$\sigma(\Gamma) = \sigma\sqrt{1 + A \cdot sign(\Gamma)}$$

$$\Gamma = \frac{\partial^2 f}{\partial S^2}(t, S(t))$$

where $f(t, S(t))$ is the price of the derivative and S is the spot price of the underlying asset at time t. However, their model is restricted to Leland numbers that are less than one, which implies that it is only valid when the transaction costs are relatively small. Only Leland's original transformation will be considered in this thesis.

2.2 Pricing and Hedging Basket Options

A basket option is an option written on several underlying assets. A European basket call option's payoff function is given by:

$$BC(T) = \max\left(\sum_{i=1}^{n} w_i S_i(T) - K, 0\right)$$

where w_i is the weight of asset i, $S_i(T)$ is the price of asset i at maturity T and K is the strike.

2.2.1 Dynamics of the Multidimensional Brownian Motion

As in the one-dimensional case of the Black-Scholes model, the volatility is assumed to be constant. Analogously in the multidimensional case, the correlation matrix and the volatilities of the underlying assets are assumed to be constant over time. The risk neutral dynamics is in the multidimensional Black-Scholes given by:

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) \tag{2.4}$$

where $S_1, ..., S_i, ..., S_n$ are stocks and $W_1, ..., W_i, ..., W_n$ are independent standard Brownian motions. The short interest rate is denoted by r and q_i is the dividend paid by asset i. The covariance between asset i and j is denoted by σ_{ij}. For convenience, the underlying assets are assumed to not pay any dividend i.e. $q_i = 0$. Following solution to the geometrical Brownian motion is used to simulate prices:

$$S_i(T) = S_i(0) \exp\left[\left(r - q_i - \frac{1}{2}\sigma_{ii}^2\right) T + \sum_{j=1}^{n} \sigma_{ij} W_j\right] \tag{2.5}$$

Under the martingale measure $\mathbb{P}$ the risk neutral valuation of the European basket call option at time t is given by:

$$BC(t) = e^{-r(T-t)} E\left[\left(\sum_{i=1}^{n} w_i S_i(T) - K\right)^{+}\right]$$

2.2.2 Calculations of the Greeks

The delta in the multivariate case, i.e. the partial derivative of the option price with respect to the underlying spot price of asset i, is calculated by numerical methods. The reason for this is that no analytical solution exists since the weighted sum of lognormal variables is not lognormal distributed. The partial derivatives are approximated with the finite difference ratio:

$$\Delta_{S_i} = \frac{\partial BC}{\partial S_i} \approx \frac{BC(S_1, ..., S_i(1+h), ..., S_n) - BC(S_1, ..., S_i(1-h), ..., S_n)}{2hS_i}$$

where we set $h = 10^{-6}$. The gamma is calculated in the same manner with the following finite difference ratio:

$$\Gamma_{S_i} = \frac{\partial^2 BC}{\partial S_i^2} \approx$$

$$\frac{BC(S_1, ..., S_i(1+h), ..., S_n) - 2BC(S_1, ..., S_i, ..., S_n) + BC(S_1, ..., S_i(1-h), ..., S_n)}{(hS_i)^2}$$

where BC is the price of the derivative and S_i is the spot price of the underlying asset i, see [5].

2.2.3 Nengjiu Ju Approximation Model

The Nengjiu Ju model approximates the price of a basket option via Taylor expansions. It is an accurate model that is analytical. The complete derivation is quite cumbersome and therefore this text will focus mainly on the results. In the presentation of the model below, the author uses a parameter z that is assumed to be equal to 1. The reason for its existence is that a Taylor expansion around zero volatilities is not applicable because the volatility is different for each underlying stock. To overcome this problem, a fictitious market is considered where all the volatilities are scaled by a parameter z. See [6] for further details.

The price of a basket call is given by:

$$BC = e^{-rT}(\sum_{i=1}^{N} w_i S_i(1,T) - K)^+ = e^{-rT}\left[e^{X(1)} - K\right]^+ =$$
$$\left[U_1 e^{-rT} N(y_1) - K e^{-rT} N(y_2)\right] + \left[e^{-rT} K \left(z_1 p(y) + z_2 \tfrac{dp(y)}{dy} + z_3 \tfrac{d^2 p(y)}{dy^2}\right)\right]$$

where

$$\begin{aligned}
&y = \log(K)\\
&y_1 = \tfrac{m(1)-y}{\sqrt{v(1)}} + \sqrt{v(1)}\\
&y_2 = y_1 - \sqrt{v(1)}\\
&z_1 = d_2(1) - d_3(1) + d_4(1)\\
&z_2 = d_3(1) - d_4(1)\\
&z_3 = d_4(1)
\end{aligned}$$

and

$$d_1(z) = \frac{1}{2}(6a_1^2(z) + a_2(z) - 4b_1(z) + 2b_2(z)) -$$
$$-\frac{1}{6}(120a_1^3(z) - a_3(z) + 6(24c_1(z) - 6c_2(z) + 2c_3(z) - c_4(z)))$$

$$d_2(z) = \frac{1}{2}(10a_1^2(z) + a_2(z) - 6b_1(z) + 2b_2(z)) -$$
$$(\frac{128a_1^3(z)}{3} - \frac{a_3(z)}{6} + 2a_1(z)b_1(z) - a_1(z)b_2(z) + 50c_1(z) - 11c_2(z) + 3c_3(z) - c_4(z))$$

$$d_3(z) = (2a_1^2(z) - b_1(z)) - \frac{1}{3}(88a_1^3(z) + 3a_1(z)(5b_1(z) - 2b_2(z)) + 3(35c_1(z) - 6c_2(z) + c_3(z)))$$

$$d_4(z) = (\frac{-20a_1^3(z)}{3}) + a_1(z)(-4b_1(z) + b_2(z)) - 10c_1(z) + c_2(z))$$

and

$$\begin{aligned}
c_1(z) &= \frac{z^6}{48A^5(0)} E[A'^4(0)A''(0)] = -a_1(z)b_1(z) \\
c_2(z) &= \frac{z^6}{144A^4(0)} (9E[A'^2(0)A''^2(0)] + 4E[A'^3(0)A^{(3)}(0)]) \\
c_3(z) &= \frac{z^6}{48A^3(0)} (4E[A'(0)A''(0)A^{(3)}(0)] + E[A''^3(0)]) \\
c_4(z) &= \frac{z^6}{72A^2(0)} E[(A^{(3)}(0))^2] = \frac{z^6 U_2^{(3)}(0)}{12U_2(0)} = a_1(z)a_2(z) - \frac{2}{3}a_1^3(z) - \frac{1}{6}a_3(z)
\end{aligned}$$

where

$$\begin{aligned}
E\left[A'^2(0)A''^2(0)\right] &= 8\sum_{ijkl=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{S}_l\bar{\rho}_{il}\bar{\rho}_{jk}\bar{\rho}_{kl} + 2U_2'(0)U''_2(0) \\
E\left[A'^3(0)A^{(3)}(0)\right] &= 6\sum_{ijkl=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{S}_l\bar{\rho}_{il}\bar{\rho}_{jl}\bar{\rho}_{kl} \\
E\left[A'(0)A''(0)A^{(3)}(0)\right] &= 6\sum_{ijk=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{\rho}_{ik}\bar{\rho}_{jk}^2 \\
E[A''^3(0)] &= 8\sum_{ijk=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{\rho}_{ij}\bar{\rho}_{ik}\bar{\rho}_{jk}
\end{aligned}$$

and

$$\begin{aligned}
b_1(z) &= \frac{z^4}{4A^3(0)} E[A'^2(0)A''(0)]a \\
b_2(z) &= \frac{z^4}{8A^2(0)} E[A''^2(0)] = \frac{z^4 U''_2(0)}{4A^2(0)} = a_1^2(z) - \frac{1}{2}a_2(z)
\end{aligned}$$

where

$$E[A'^2(0)A''(0)] = 2\sum_{ijk=1}^{N} \bar{S}_i\bar{S}_j\bar{S}_k\bar{\rho}_{ik}\bar{\rho}_{jk}\frac{1}{2}a_2(z)$$

and

$$a_1(z) = z^2 m'(0) = -\frac{z^2 U_2'(0)}{2U_2(0)}$$
$$a_2(z) = z^4 m''(0) = 2a_1^2 - \frac{z^4 U''_2(0)}{2U_2(0)}$$
$$a_3(z) = z^6 m^{(3)}(0) = 6a_1 a_2 - 4a_1^3 - \frac{z^6 U^{(3)}{}_2(0)}{2U_2(0)}$$

where

$$U_2(0) = \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j$$
$$U_2'(0) = \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j (\bar{\rho}_{ij})$$
$$U''_2(0) = \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j (\bar{\rho}_{ij})^2$$
$$U_2^{(3)}(0) = \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j (\bar{\rho}_{ij})^3$$

and

$$m(z^2) = 2\log U_1 - 0.5\log U_2(z^2)$$
$$v(z^2) = \log U_2(z^2) - 2\log U_1$$

where

$$U_1 = \sum_{i=1}^{N} \bar{S}_i = A(0) U_2(z^2) = \sum_{ij=1}^{N} \bar{S}_i \bar{S}_j e^{z^2 \bar{\rho}_{ij}}$$

$m(z^2)$ and $v(z^2)$ is the mean and the variance respectively of a random variable $Y(z)$, $p(y)$ is the normal density with mean $m(1)$ and variance $v(1)$ and

$$A(z) = \sum_{i=1}^{N} w_i S_i(z, T)$$

and therefore the payoff of a basket option can be written as:

$$BC(T) = (A(1) - K)^+$$

Finally

$$X(z) = \log(A(z))$$

See appendix 5.1 for a validation of the model.

2.2.4 Hedging Non-linear Risk

Delta hedging of options does not capture the non-linear risk of the Black-Scholes model. This can be done by making the hedging portfolio gamma neutral. For this purpose it is not enough to hedge the basket option with the underlying assets, non-linear instruments such as options needs to be included in the hedging portfolio.

The greeks are calculated numerically as described in Section 2.2.2. When hedging with options on the underlying assets we also need the single stock greeks derived from the Black-Scholes model. For a European call and put option with strike price K and time to maturity T we have the following relations:

$$\Delta_{Call} = \frac{\partial C}{\partial S} \approx N(d_1) \qquad \Delta_{Put} = \frac{\partial P}{\partial S} \approx -N(-d_1)$$

$$\Gamma_{Call} = \Gamma_{Put} = \frac{\partial^2 C}{\partial S^2} \approx \frac{\varphi(d1)}{s\sigma\sqrt{T-t}}$$

The notation is the same as in the Black-Scholes formula.

In this case we use one call option C_i for every underlying asset i to gamma neutralize the basket option. The strike price is updated at every point in time and set to a relative distance from the spot price at time t to create a realistic scenario and to avoid large gamma values close to maturity.

$$K_{C,i}(t) = S_i(t)(1 - 0.01)$$

The value of the total portfolio is then given by:

$$V(t) = h_0(t)B(t) + h_1(t)S(t) + h_2(t)C(t) - BC(t)$$

Here h_0 is the amount invested in the bank account B, h_1 is the vector of quantities invested in the stocks $S(t) = (S_1(t), S_2(t), ..., S_n(t))$ included in the basket, h_2 is a vector of quantities invested in the call options on these stocks and BC is the price of the basket option at time t. The first step to make the portfolio gamma neutral is to calculate the partial second derivatives with regard to every component i of the basket.

$$\begin{aligned}\frac{\partial^2 V(t)}{\partial S_i(t)^2} &= \frac{\partial^2}{\partial S_i(t)^2}\left\{h_0(t)\,B(t) + h_1(t)\,S(t) + h_2(t)C(t) - BC(t)\right\} = \\ &= h_{2,i}(t)\Gamma_{C,i}(t) - \Gamma_{BC,i}(t) = 0\end{aligned}$$

The solution is given by

$$h_{2,i}(t) = \frac{\Gamma_{BC,i}(t)}{\Gamma_{C,i}(t)}$$

The portfolio given by the quantities h_1 and h_2 has a delta calculated as follows:

$$\begin{aligned}&\tfrac{\partial V(t)}{\partial S_i(t)} = \tfrac{\partial}{\partial S_i(t)} \left\{ h_0\,(t)\,B(t) + h_1\,(t)\,S(t) + h_2(t)C(t) - BC(t) \right\} = \\ &= h_{1,i}(t) + h_{2,i}(t)\delta_{C,i}(t) - \delta_{BC,i}(t) = 0\end{aligned}$$

This gives h_1, the number of stocks needed to hedge the linear risk:

$$h_{1,i} = \delta_{BC,i}(t) - h_{2,i}(t)\delta_{C,i}(t)$$

In the last step, h_0 is calculated so that the portfolio becomes self financing.

2.3 Proxy Hedging

When the payoff of an option cannot be perfectly replicated, the trader can only rely on approximate hedging methods. There are several possible ways of proxy-hedging and only a few examples will be presented in this thesis. When some underlying assets are illiquid, one strategy is to construct a hedging portfolio that consists of some subset of underlying assets that are traded more frequently. This might also be a good strategy if the option has a large number of underlying assets and therefore a hedge that is simpler is favourable. Principal component analysis can be used to determine a set of underlying assets that explain the variance in the total basket to a certain degree, and therefore are suitable to include in the hedge. If the option is written on assets that are highly correlated with a certain index, trading the index is a natural component in the hedging portfolio. The methods introduced here are independent of the construction of the hedging portfolio. That means that for example futures on indices, underlying assets that are a subset of the basket or any assets suitable to hedge with can be chosen arbitrarily.

2.3.1 Principal Component Analysis

One approach to proxy hedge a basket option is to hedge with a subset of the underlying assets, as well as options written on these. By principal component analysis it is possible to calculate how much of the total variation of the basket that is explained by a certain subset of underlying assets. The techniques are rather simple and easy to implement. Using PCA it is possible to pick underlying assets that are traded frequently and that have a high explanatory value of the basket option as a whole.

Fist we decompose the covariance matrix into its eigenvalues and eigenvectors:

$$\Sigma = \Gamma \Lambda \Gamma^T$$

where $\Lambda = diag(\lambda_1, ..., \lambda_N)$ and

$$\Gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} \end{pmatrix}$$

The eigenvalues and the corresponding eigenvectors are sorted so that $\lambda_1 > ... > \lambda_N$. The covariance matrix considered here is modified which is motivated by the fact that the weights and individual asset volatilities affect the price of the basket option to a large extent. The following covariance matrix is used:

$$\Sigma = \begin{pmatrix} \omega_1^2\sigma_1^2 & \omega_1\omega_2\sigma_1\sigma_2\rho_{12} & \cdots & \omega_1\omega_N\sigma_1\sigma_N\rho_{1N} \\ \omega_1\omega_2\sigma_1\sigma_2\rho_{12} & \omega_2^2\sigma_2^2 & \cdots & \omega_2\omega_N\sigma_2\sigma_N\rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1\omega_N\sigma_1\sigma_N\rho_{1N} & \omega_2\omega_N\sigma_2\sigma_N\rho_{2N} & \cdots & \omega_N^2\sigma_N^2 \end{pmatrix}$$

The principal component transformation is linear and defined as:

$P = \Gamma^T(S - \mu)$, where $S = (S_1, ..., S_N)^T$ is the price vector and μ is the mean vector of S.

The cumulative explanatory value of the first N_1 principal components are measured as:

$$\pi_{N_1} = \frac{\sum\limits_{j=1}^{N_1} \lambda_j}{\sum\limits_{j=1}^{N} \lambda_j}$$

It is a relative measure of how much of the total variance the first N_1 principal components explain. The covariance between the price vector S and the principal components P are:

$$\begin{aligned} & Cov(S, P) = E[SP^T] - E[S]E[P^T] \\ & = E[SS^T\Gamma] - \mu\mu^T\Gamma \\ & = \Sigma\Gamma \\ & = \Gamma\Lambda\Gamma^T\Gamma \\ & = \Gamma\Lambda \end{aligned}$$

This implies that the correlation r_{ij} between the principal component i and the price S_i is calculated as follows:

$$r_{ij} = \rho_{S_i,P_j} = \frac{\gamma_{ij}\lambda_j}{({\sigma^2}_{S_i}\lambda_j)^{\frac{1}{2}}} = \gamma_{ij}\left(\frac{\lambda_j}{{\sigma^2}_i}\right)^{\frac{1}{2}}$$

The subset of underlying assets to use in the hedging portfolio is chosen by calculating the cumulative r^2 for each asset with the principal components chosen above. This is done by calculating:

$$r_i^2 = r_{i,1}^2 + ... + r_{i,N_1}^2$$

where $i = 1...N$

The underlying assets with the highest r_i^2 is chosen. For further details about how to use PCA to select a subset of underlying assets to hedge basket options, see the article by Xia Su [8].

2.3.2 Quadratic Hedging

This method aims to minimize the following objective function:

$$\min_w E\left[\left(\sum_j^m w_j F_T^j - f\left(S_T^1, S_T^2, ..., S_T^n\right)\right)^2\right]$$

where F_T^j denotes the spot price of the hedging instrument j at time T, w_j is the associated weight and $f\left(S_T^1, S_T^2, ..., S_T^n\right)$ is the payoff function of the derivative. The weights that minimize the mean of the squared hedging error are explicitly given by:

$$w = \Sigma_F^{-1} \Sigma_{Y,F} \tag{2.6}$$

where

$$\Sigma_F = \begin{pmatrix} Cov(F_T^1, F_T^1) & \cdots & Cov(F_T^1, F_T^j) & \dots & Cov(F_T^1, F_T^n) \\ \vdots & \ddots & \vdots & & \vdots \\ Cov(F_T^i, F_T^1) & & Cov(F_T^i, F_T^j) & & Cov(F_T^i, F_T^n) \\ \vdots & & \vdots & \ddots & \vdots \\ Cov(F_T^n, F_T^1) & \cdots & Cov(F_T^n, F_T^j) & \cdots & Cov(F_T^n, F_T^n) \end{pmatrix}$$

$$\Sigma_{Y,F} = \begin{pmatrix} Cov(F_T^1, f(S_T^1 ..., S_T^n) \\ Cov(F_T^2, f(S_T^1 ..., S_T^n) \\ \vdots \\ Cov(F_T^n, f(S_T^1 ..., S_T^n) \end{pmatrix}$$

The weights w given by the expression stated above, is the regression coefficients of the basket onto the components of the hedging portfolio.

2.3.3 Optimization

An extension of the quadratic hedge approach is to use numerical optimization to determine the optimal quantity of assets needed to proxy hedge the basket option. The advantage of the numerical approach compared to the analytical solution in the quadratic hedging ratio is that it can be implemented on almost any objective functions e.g. mean, variance or a combination of both. The method also takes into account the correlation between the underlying assets of the basket and will not suffer from problems with extreme or missing solutions. In this thesis the sum of the mean and the variance of the hedging errors is minimized with regard to the weights $w = (w_1, ..., w_i, ..., w_n)$.

$$\min_w E\left[\left(\sum_j^m w_j F_T^j - f\left(S_T^1, S_T^2, ..., S_T^n\right)\right)^2\right] + E\left[\sum_j^m w_j F_T^j - f\left(S_T^1, S_T^2, ..., S_T^n\right)\right] \tag{2.7}$$

The same notation as in the Section 3.3.1 is used. In practice this is a non-linear problem where there is no analytic solution to the objective function - both the expected value and the variance of the basket option needs to be either approximated or simulated. To speed up calculation the objective function is approximated with Monte Carlo simulation with 1'000 simulations where all calculations are vectorized. The optimization is performed without constraints and calculates the hessian numerically to obtain the optimal solution. The drawbacks when optimizing on a non-linear objective function is the time consuming calculations, the risk of several local minimum and that the method produces approximate solutions.

2.3.4 Proxy Delta Hedging

A natural approach to construct a proxy hedge is to start with the ordinary principals of delta hedging. Imagine that a subset of underlying assets is tradable and that the hedging portfolio is to be constructed by these. Using PCA techniques can be useful to find such a subset of underlying assets that has a high explanatory value of the basket as a whole. A consequence of hedging with a subset of underlying assets is that the dimension of the hedging portfolio is reduced. It is therefore not possible to be partially delta-neutral in all underlying assets when delta hedging.

One possible solution is to solve a system of equations where the weights of the assets that are traded in the hedging portfolio are scaled to make the value of the whole position approximately globally delta neutral. First, the partial derivatives of the option value with respect to the traded underlying assets' spot prices are calculated. As previously, this is done by pricing with the Nengjiu Ju model and then calculating the finite difference ratios that approximates the first order derivatives. Then, imagine that n assets are traded and m assets are not a part of the hedging portfolio (denoted by NotTraded below). The total dimension of the basket is n+m. The following system of equations is solved:

$$
\begin{gathered}
h_1 S_{1,Traded} + ... + h_i S_{i,Traded} + ... + h_n S_{n,Traded} = \frac{\partial BC}{\partial S_{1,NotTraded}} \cdot S_{1,NotTraded} \\
\vdots \\
h_1 S_{1,Traded} + ... + h_i S_{i,Traded} + ... + h_n S_{n,Traded} = \frac{\partial BC}{\partial S_{i,NotTraded}} \cdot S_{i,NotTraded} \\
\vdots \\
h_1 S_{1,Traded} + ... + h_i S_{i,Traded} + ... + h_n S_{n,Traded} = \frac{\partial BC}{\partial S_{m,NotTraded}} \cdot S_{m,NotTraded}
\end{gathered}
$$

The solution to this system is the weights $h_1 ... h_i ... h_n$. If $n < m$ this system will be overdetermined and the system is solved by OLS. However, the number of assets that are traded must be less than or equal to the number of non-traded assets for the system to not have an infinite number of solutions, assuming that the determinant of the matrix consisting of the stock prices of the assets that are traded is different from zero. These new weights are being added to the original weights, i.e. the partial deltas, to achieve a hedge whose delta is approximately equal to the global delta of the basket.

3 Numerical Examples

3.1 Delta Hedging in One Dimension

Following examples illustrates the replication of a European call option using ordinary delta hedging in the Black-Scholes framework. The geometric Brownian motion is simulated with parameters $\mu = 0.05$ and $\sigma = 0.3$. The hedging portfolio is rebalanced at fixed discrete points in time as well as dynamic points in time. When increasing the number of hedges, the hedging error should converge to zero, at least when transaction costs are neglected. The hedging error is computed in each trajectory and finally the distribution function is approximated. This can be done arbitrarily good by increasing the number of simulations. The call option is assumed to have a maturity of one year, $T = 1$, and a strike price of $K = 100$. The price of the underlying stock today is $S_0 = 100$. Given the parameters stated above, the price of the call option is 14.23 at $t = 0$ according to the Black-Scholes formula.

3.1.1 Density of the Hedging Errors without Transaction Costs

According to Black-Scholes, the option with the paramaters given above can be perfectly replicated by trading the underlying stock. However, this assumes that it is possible to trade continuously without any transaction costs. These assumptions are unrealistic. First the assumption of continous trading is relaxed. In other words, it is possible to trade assets for free but the position can only be rehedged at a limited number of discrete points in time. How does this affect the distribution of the hedging errors?

Below are the approximate density functions of the hedging errors when the time step is $n = 100$, $n = 500$ and $n = 1000$. This means that the position is rehedged 100, 500 and 1000 times respectively at equidistant discrete points in time. The hedging errors are presented as relative to the original price of the option. To illustrate the effect of varying the number of hedges, T-distribution functions were fitted to the original data with the following parameters:

Parameter	n=100	n=500	n=1000
μ	0.00067	0.00026	3.24323e-006
σ	0.06018	0.02542	0.01848
ν (degrees of freedom)	6.28762	4.59847	5.22229

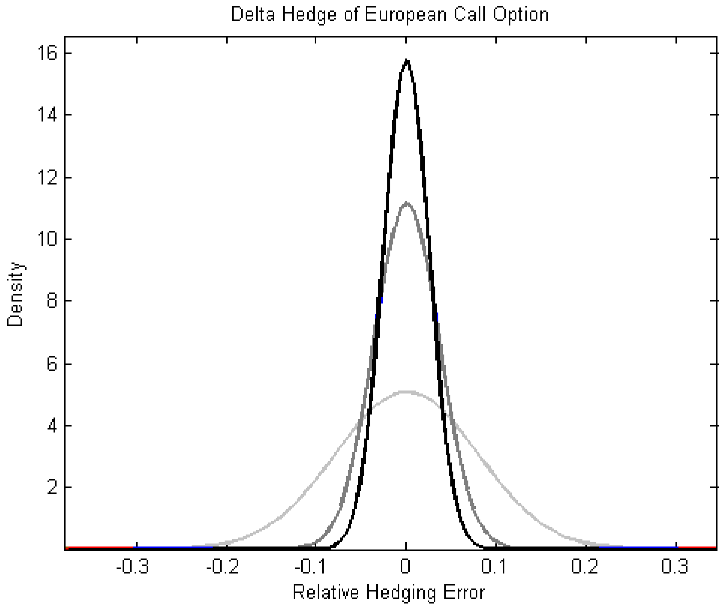

Figure 1: Hedging errors with 5000 simulations and 100 (light grey), 500 (grey) and 1000 (black) time steps.

As expected, the variance of the distribution of the hedging errors decreases as the number of rebalances of the hedging portfolio increases.

3.1.2 Static and Dynamic Delta Hedging in One Dimension

Now consider a world with transaction costs. The transaction costs are thought of as half of the stock's bid-ask spread and the same option as described above is delta hedged. The bid-ask spread, denoted by ϵ in this thesis, is treated as a free parameter that is varied from 0 % up to 5 %. The geometric Brownian motion is simulated 5000 times with parameters $\mu = 0.05$ and $\sigma = 0.3$. The option is hedged in two different ways. First by varying the number of hedges according to:

$Hedges =$

$(5, 15, 25, 35, 45, 55, 65, 75, 85, 95, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000)$

A second way of controlling the transaction costs is to set a re-hedging condition - a dynamic hedge. In this case a relative bound of the delta of the option is used to determine whether to re-hedge the portfolio or not. See equation 2.1 for further details about the rehedging condition.

The dynamic hedge is tested using different boundaries with the same parameters in the Black-Scholes model as before. The number of time steps, i.e. the number of possible hedges, is set to 365 to reduce the computational time. The bid-ask spread is varied from 0 % to 5 % in whole percentage units and the delta bound is varied from 1.41 to 0.001 with step length 0.1.

Figure 2 below shows both the static and the dynamic hedge in the same graph where the lower x-axis shows the number of hedges in the static case and the upper x-axis shows the delta bound in the dynamic case. The relative mean of the hedging errors decrease rather rapidly when the bid-ask spread increases, especially when rebalancing the hedging portfolio many times.

The darkest lines (black) illustrates the case when the transaction costs are zero, i.e. when the bid-ask spread is zero. It is worth noting that the second derivative of the relative mean of the hedging errors with respect to the number of hedges is positive in the static case. The function with respect to the number of hedges is in other words concave. However, it is negative in the delta bound case. For all bid-ask spreads, the delta bound hedge performs better in terms of relative hedging errors. This is because it reduces the number of actual rebalances of the hedging portfolio and therefore the transaction costs are lower.

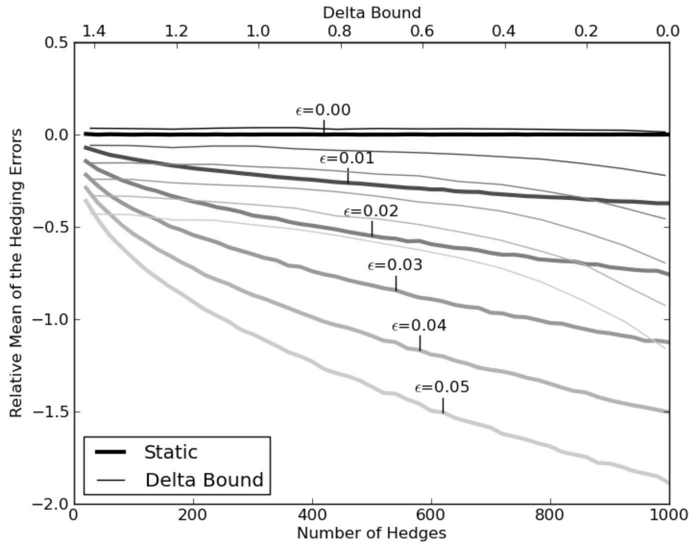

Figure 2: Mean of the hedging errors using static hedge (thick lines and lower x-axis) and dynamic hedge (thin lines and upper x-axis) for different bid-ask spreads.

To form some opinion about the distribution of the relative hedging errors, some measure of dispersion has to be evaluated. Figure 3 shows the relative standard deviation with respect to the number of hedges and the delta bound in the static and the dynamic case respectively. In the static case, the standard deviation of the hedging portfolio increases fast as both the number hedges and the bid-ask spread increases. When the bid-ask spread is 0, the relative standard deviation converges to zero as the number of hedges increases. This result is in line with what was shown in Section 3.1.1. However, when transaction costs are considered, there seems to be an optimal point in terms of number of hedges that minimizes the relative standard deviation of the hedge. It is also notable that this optimal point is shifted to the left as the bid-ask spread increases. When the number of hedges is larger than the point that minimizes the relative standard deviation, there seems to be an almost linear relationship between the relative standard deviation and the number of hedges.

In the delta bound case, the plot looks rather different. As long as the bid-ask spread is low, approximately below 2 %, it is optimal to have a small bound, which implies that the portfolio is rebalanced as often as possible. When the delta bound is zero, the standard deviation is the same as when hedging 365 times in the static case. If the bid-ask spread is higher than around 2 %, the function becomes concave and there seems to be a delta bound that minimizes the relative standard deviation.

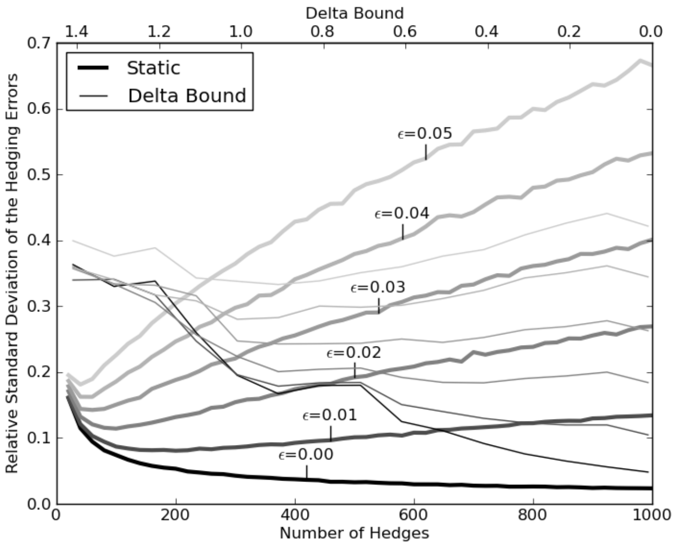

Figure 3: Standard Deviation of the hedging errors using static hedge (thick lines and lower x-axis) and dynamic hedge (thin lines and upper x-axis) for different bid-ask spreads.

3.1.3 Comparison of the Static Delta Hedge and Leland's Model

The Leland's transformation of volatility, see equation 2.2 and 2.3, is a way of pricing options with transaction costs in the standard Black-Scholes formula. The transformation of volatility takes two parameters except for the original volatility, the relative bid-ask spread and the time-interval between the rebalances of the hedging portfolio. If Leland's model is correct, the results should look similar to the results of the delta hedge in the static case. The difference in price of the option when pricing with the transformed volatility instead of the original volatility is considered to be the transaction costs according to Leland, and is calculated as follows:

$$\frac{C(\sigma) - C(\sigma_L)}{C(\sigma)}$$

The plot below shows the relative mean of the hedging errors in the static delta hedge case and the relative transaction costs according to Leland. Leland's transformation of volatility does not capture all of the costs when delta hedging the option, especially when the bid-ask spread is high. Therefore it seems as the model underestimates the costs. Leland published his work in 1985 but his model has later been critisized in several articles, see [10].

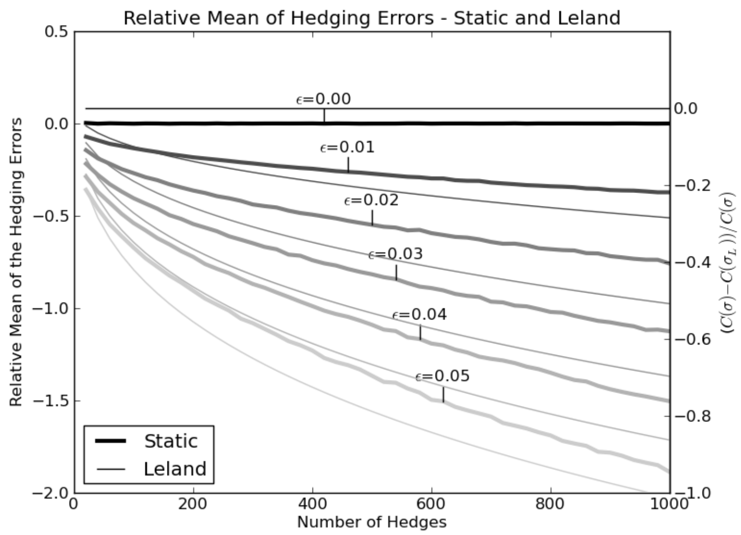

Figure 4: Relative mean of the hedging errors using static hedge (thick lines and left y-axis) and theoretical Leland value (thin lines and right y-axis) for different bid-ask spreads.

3.2 Hedging Basket Options

3.2.1 Static and Dynamic Delta Hedging

The following section deals with the problem of delta hedging a basket option, i.e. the multivariate case. The Nengjiu Ju approximation model is used to price the option. The underlying assets are simulated using the multivariate Black-Scholes model stated in equation 2.4 and 2.5. The hedging strategies are eveluated using a basket option written on three underlying assets. Their prices today are:

$$s = (s_1, s_2, s_3) = (110, 75, 125)$$

All assets are assumed to have a volatility of 0.3. The strike of the basket is $K = 100$ and the weights of the underlying assets are:

$$w = (w_1, w_2, w_3) = (0.4, 0.4, 0.2)$$

The time to maturity is 1 year and the interest rate $r = 5\%$. The correlation matrix is assumed to be:

$$\begin{pmatrix} 1.00 & 0.50 & 0.15 \\ 0.50 & 1.00 & 0.60 \\ 0.15 & 0.60 & 1.00 \end{pmatrix} \tag{3.1}$$

With these parameters, the price of the basket call option at time $t = 0$ is 11.17. As in the one-dimensional problem, the basket option is first hedged at fixed points in time. Due to very heavy computations, the number of simulations is restricted to 500. In the static case, the relative mean and standard deviation of the hedging errors is calculated for different number of hedges:

$Hedges =$

$(5, 15, 25, 35, 45, 55, 65, 75, 85, 95, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000)$

In the multivariate dynamic hedge, the same delta bound as in the one dimensional case is used, see equation 2.1. The number of possible hedges in this case is fixed to 365, i.e. once a day assuming that it is possible to trade all days during a year. The delta bound is varied from 1.41 to 0.01 with step length 0.1. As previously, the results will be presented for bid-ask spreads of 0 %, 1 %, 2 %, 3 %, 4 % and 5 % and the spread is assumed to be equal for all underlying assets in the basket and the vanilla options on these assets.

First the distribution of the hedging errors is studied when hedging 100 times with bid-ask spread $\epsilon = 0.03$. The histograms below are created from 1'000 simulations. The hedging errors and transaction costs are expressed relative to the initial price of the basket option. In Figure 5 the data is split into two parts. The right histogram shows the distribution of the hedging errors without transaction costs and the left histogram the distribution of the transaction costs.

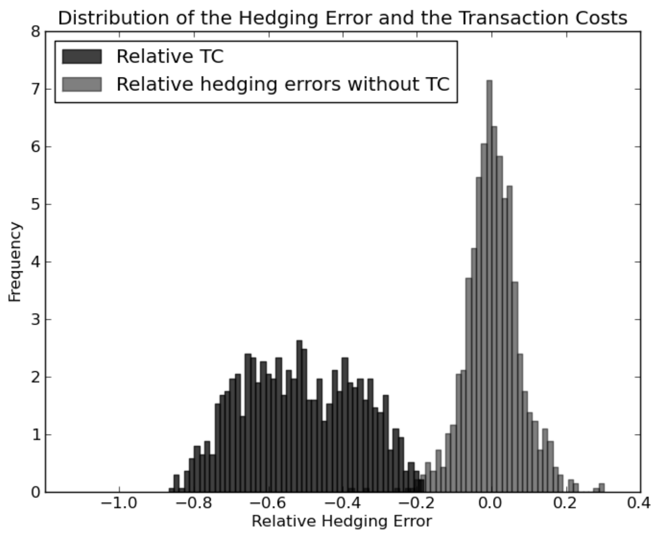

Figure 5: The distribution of the hedging errors without transaction costs (grey) and the transaction costs (dark grey) with bid-ask spread $\epsilon = 0.03$.

In Figure 6 the data is merged to the distribution of the hedging errors *with* transaction costs. When hedging under transaction costs the distribution is shifted to the left and the standard deviation increases.

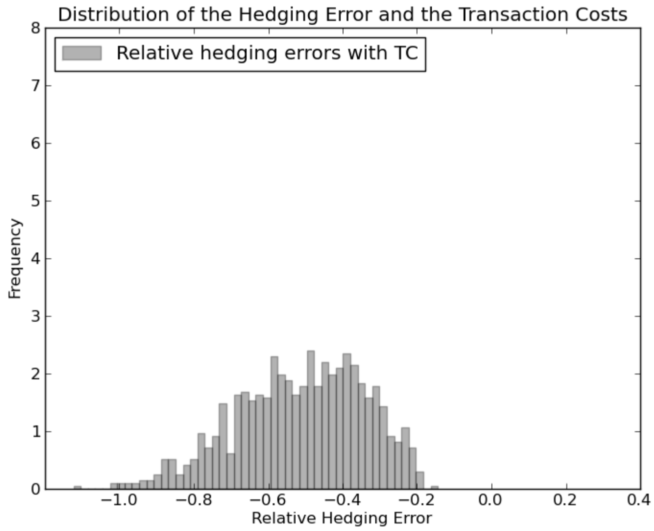

Figure 6: The distribution of the hedging errors with transaction costs (light grey) with bid-ask spread $\epsilon = 0.03$.

Figure 7 shows the relative mean of the hedging errors in both the static and the dynamic case. As expected, the relative mean of the hedging errors decay with higher bid-ask spread in both cases. The most negative point is observed when hedging as many times as possible for the highest possible cost, i.e. when the spread is 5 %. The dynamic hedge is not as sensitive to changes in transaction costs as the static hedge and in general results in lower hedging errors, except for the case when the delta bound is very high, i.e. when the hedging portfolio is rebalanced very few times. In the dynamic case, the graphs make a parallell shift downwards when the bid-ask spread increases.

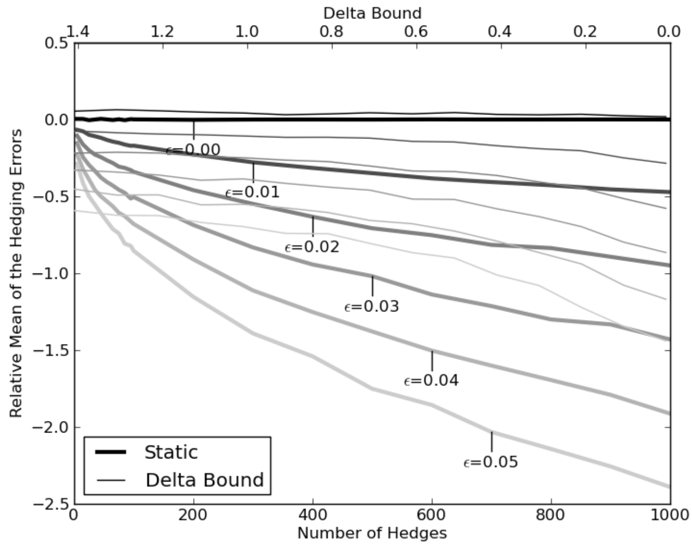

Figure 7: Relative mean of the hedging errors using static hedge (thick lines and lower x-axis) and dynamic hedge (thin lines and upper x-axis) for different bid-ask spreads.

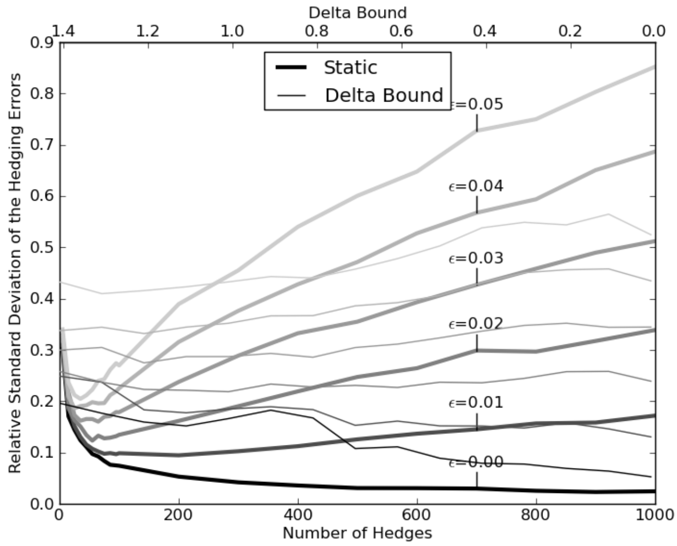

Figure 8: Standard deviation of the hedging errors using static hedge (thick lines and lower x-axis) and dynamic hedge (thin lines and upper x-axis) for different bid-ask spreads.

The standard deviation of the hedging errors increase with higher bid-ask spread in both the static and the dynamic case. The shape of the curves are very similar to the ones in the one dimensional case. In the static case a minimum occurs for low number of hedges when transaction costs are included. In the delta bound case however, the level of the bound is not that important for the performance of the hedge.

3.2.2 Static Delta and Delta-Gamma Hedging

When hedging non-linear risk, one strategy is to use vanilla options on the underlying assets in the basket as described in Section 2.2.4. The basket option is hedged with call options with strikeprice $K_t = S_t \cdot (1 - 0.01)$ to avoid extreme gamma values close to maturity. The figure below shows the relative mean of the hedging errors for the delta hedge in thick lines and for the delta-gamma hedge in thin lines for the same bid-ask spreads as before.

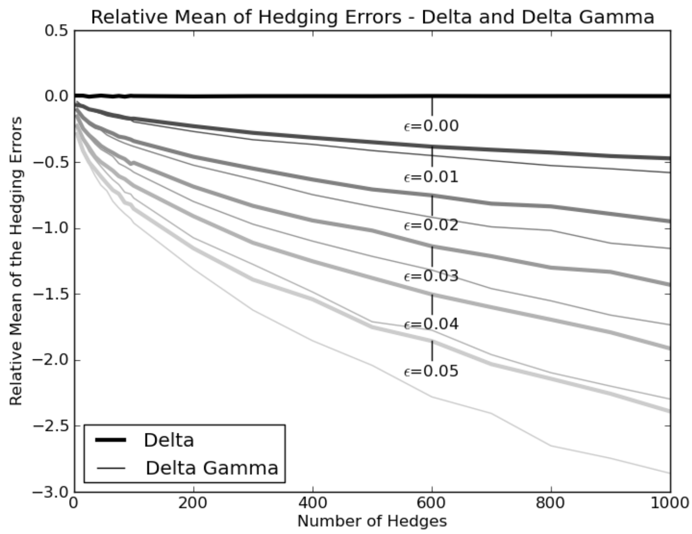

Figure 9: Relative mean of the hedging errors using delta hedge (thick lines) and delta gamma hedge (thin lines) for different bid-ask spreads.

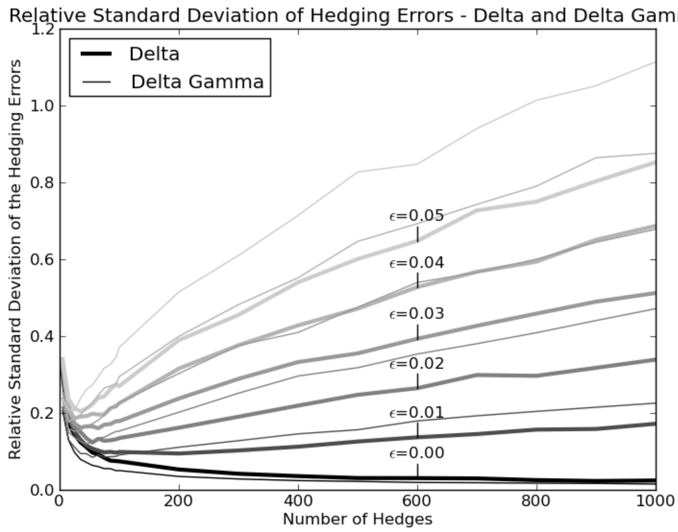

Figure 10: Standard deviation of the hedging errors using delta hedge (thick lines) and delta-gamma hedge (thin lines) for different bid-ask spreads.

In this particular example, the difference between the delta- and the delta-gamma-hedge is not big. The relative mean of the hedging errors are almost identical. However, a gamma-neutral hedging portfolio should be protected against small changes in the delta, and therefore the need of rebalancing the portfolio should be less. That implies that the gamma-hedge could decrease the total transaction costs. We know that the gamma approaches infinity when the option is near maturity. In this example, we have delta and gamma neutralized the portfolio in each fixed point in time, even close to maturity. This might have an effect on the results.

To test if the delta-gamma hedge performs better than the delta hedge when hedging fewer times we compared the two strategies on a shorter interval, varying the number of hedges between 5 and 100 with step length 10. As seen in the figure below, the delta-gamma hedge performs slightly worse than the delta hedge.

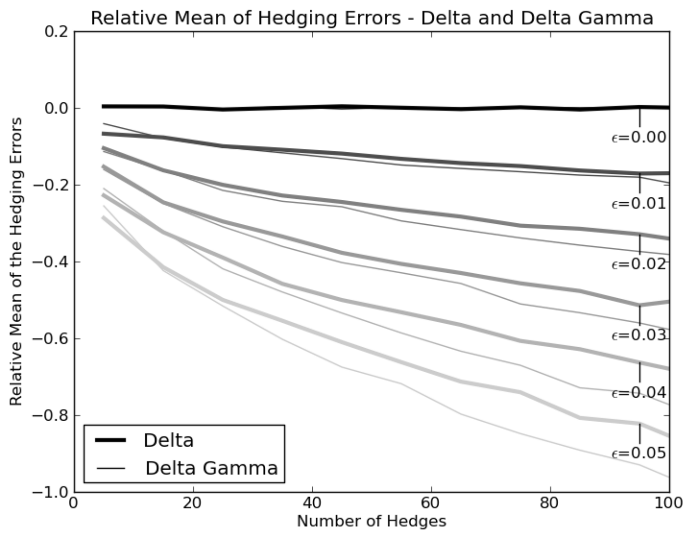

Figure 11: Relative mean of the hedging errors using delta hedge (thick lines) and delta-gamma hedge (thin lines) for different bid-ask spreads.

It can be observed in Figure 12 that the delta-gamma hedge provides a hedge with lower standard deviation when the number of hedges are small, especially when the transaction costs are low.

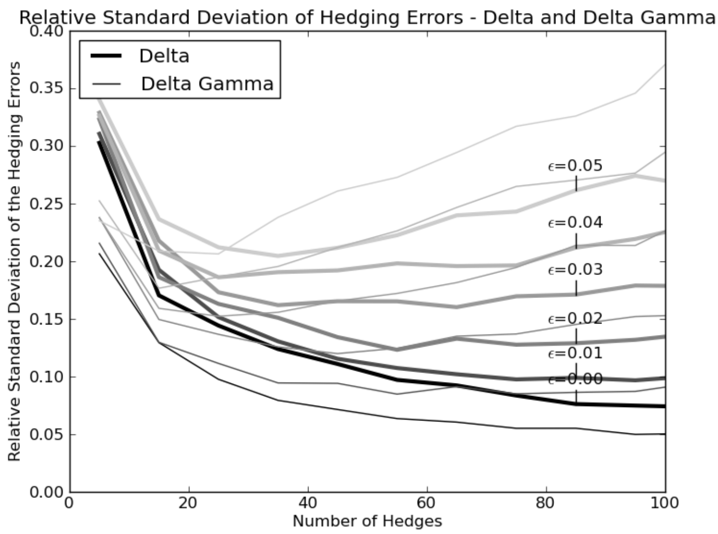

Figure 12: Standard deviation of the hedging errors using delta hedge (thick lines) and delta gamma hedge (thin lines) for different bid-ask spreads.

3.3 Performance of the Proxy Hedging Portfolios

The task is to hedge the basket of correlated underlying assets with a hedging portfolio that consists of a subset of the underlying assets. The size of the subset is determined by how much of the variation in the basket that some of the underlying assets can explain. In this example, it turned out that asset 1 and 2 explained as much as 95 % of the variation in the basket. This was calculated with principal component analysis. The same option as in Section 3.2.1 is used in the following examples.

3.3.1 Quadratic Hedging

The hedging portfolio is rebalanced at fixed points in time and the number of total rebalances during the lifetime of the option is varied from 5 to 95 in steps of 10 and from 100 to 1000 in steps of 100. Due to the high computational time the number of total simulations is set to 500. The calculations takes around 24 hours to compute on an intel dual core processor.

In each time step, Σ_Y and $\Sigma_{Y,F}$ are calculated using Monte-Carlo simulation. In some more detail, 10'000 scenarios of the underlying assets' spot prices in the next time step are simulated and the value of the option is calculated in each scenario. The spot prices of the components of the hedging portfolio must also be simulated, but since the hedging portfolio consists of a subset of underlying assets in this particular example, no further simulation is necessary. The covariance matrix of the components of the hedging portfolio and the covariances between these components and the derivative to be hedged are now calculated. The weights that minimize the quadratic objective function stated in Section 3.3.1 is now given by equation 2.6. Figure 13 shows the relative mean of the hedging errors with respect to the number of hedges.

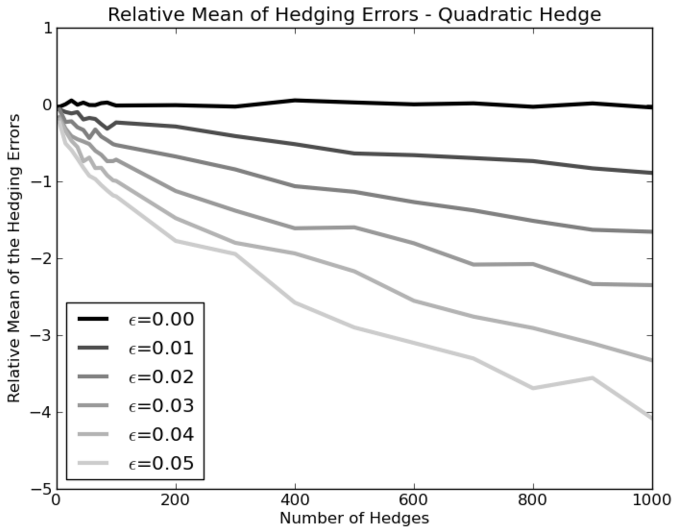

Figure 13: Relative mean of the hedging errors using Quadratic Hedging.

As before, the relative mean of the hedging errors is calculated by dividing the mean of the hedging errors with the price of the option at time zero, which is 11.17. The plot looks very similar to the results of the static delta hedge with the exception that the graphs are shifted downwards. Since this is a proxy-hedge, it is expected to perform somewhat worse than the static delta-hedge. When proxy-hedging in reality, the instruments used to hedge with are hopefully rather liquid and therefore the potential to create hedging portfolios that performs well in terms of relative mean of the hedging errors are good using quadratic hedging.

The relative standard deviations of the hedging errors are rather high. Overall, the relative standard deviations are approximately 4 times larger in comparison to the static delta hedge. When the bid-ask spread is high, the relative standard deviation increases fast as the number of hedges becomes larger.

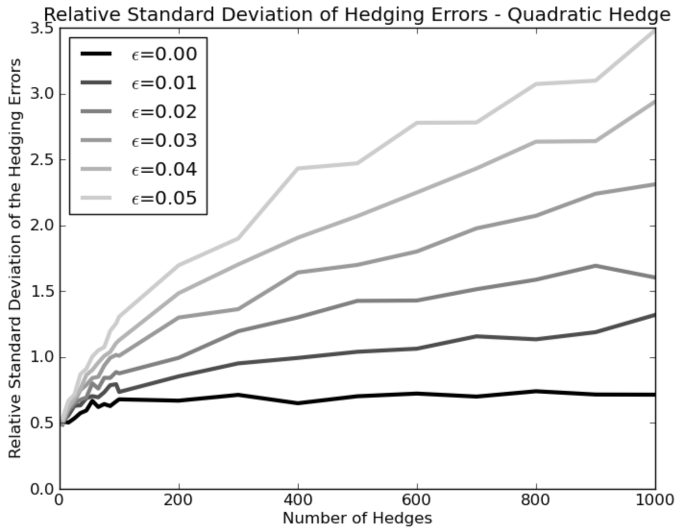

Figure 14: Relative standard deviation of the hedging errors using Quadratic Hedging.

These results show that proxy-hedging in this way is associated with pretty high uncertainness, especially when proxy-hedging using assets that are illiquid.

3.3.2 Optimization

Once again, the same basket and parameters defined in Section 3.2.1 is used to evaluate the optimization technique. The basket option is re-hedged in every time step using the first two assets in the underlying basket. In practice the optimization is performed for every re-hedge by first simulating the difference between the hedging portfolio and the basket option at the time of maturity. The objective function, in this case the sum of the mean and the variance of the hedging error is then calculated from a 1'000 simulations. The same seed is used in every calculation so that the objective function behaves as a analytical function. The minimization then needs 3-5 iterations to calculate the optimal weights.

500 trajectories are simulated for every bid-ask spread and hedging interval. As before the performance of the hedge is measured by calculating the relative mean and standard deviation of the difference between the option and the hedging portfolio upon maturity of the option. Note that unlike earlier examples, the option is only re-hedged up to 500 times because of high computational time.

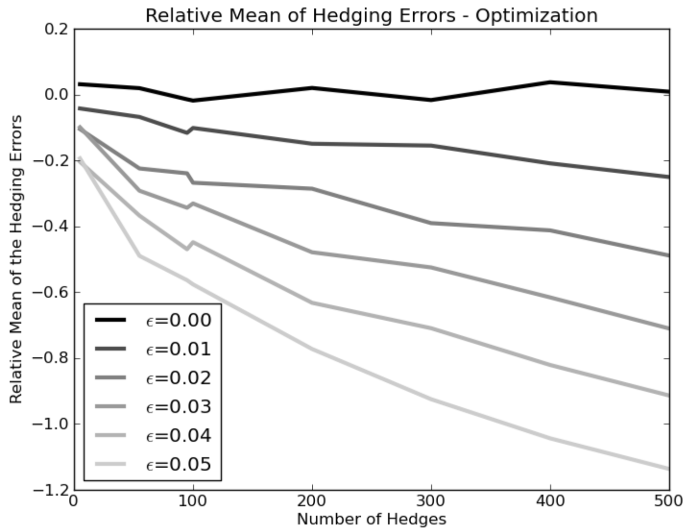

Figure 15: Relative mean of the hedging errors using optimization.

The relative mean of the hedging errors shows the same characteristics as the plots of the static delta and delta-gamma hedge. An interesting observation is that the optimization technique outperforms the delta hedge for high bid-ask spreads and large number of hedges. This despite the fact that optimized hedge only uses two out of three assets. This is because the optimized hedge use lower quantities of stocks to hedge the basket option which leads to lower transaction costs.

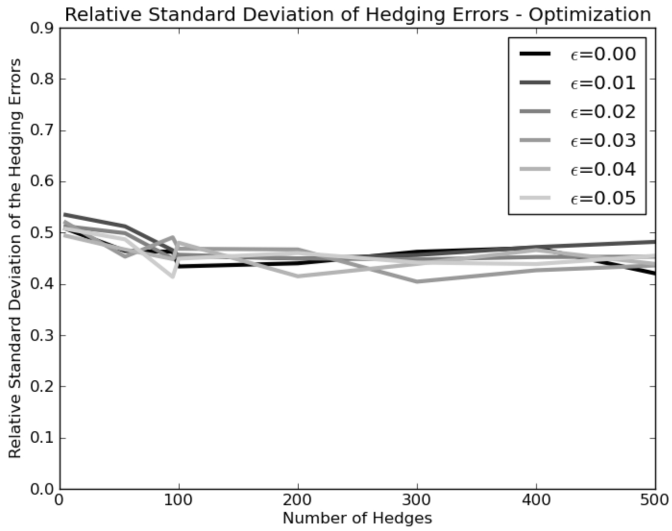

Figure 16: Relative standard deviation of the hedging errors using optimization.

So far all figures of the relative standard deviation of the hedging errors shows increasing values for higher bid-ask spreads. Despite noisy data, in this case a clear decline of the standard deviation is achieved when increasing the number of hedges - a direct effect of the objective function. It seems as if the bid-ask spread does not affect the standard deviation as in other hedging strategies.

3.3.3 Proxy Delta Hedge

The plot below shows the relative mean of the hedging errors for different bid-ask spreads and with respect to the number of hedges. This hedge performs only slightly worse than the static delta hedge when trading in all underlying assets are possible. In comparison to the quadratic hedge, the difference is rather large in favour of the proxy delta hedge.

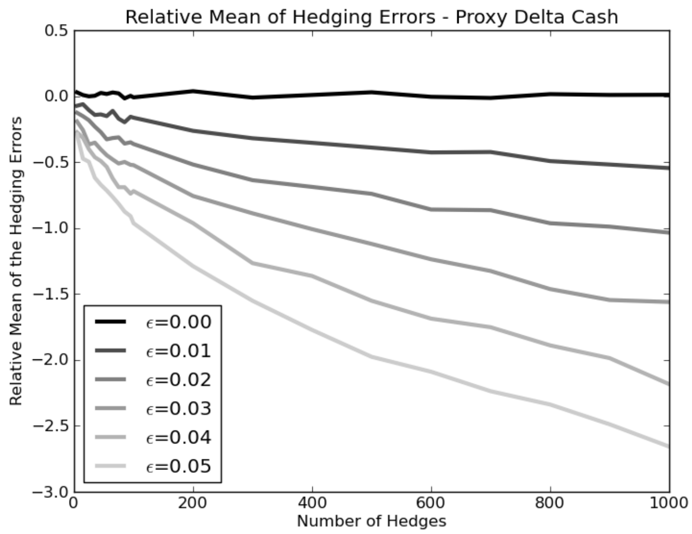

Figure 17: Relative mean of the hedging errors using proxy delta hedge.

Inspecting Figure 18, it is notable that the relative standard deviations are a bit higher in comparison to the static delta hedge, as expected when performing a proxy hedge. In general, around twice the relative standard deviation is observed overall. When the bid-ask spread is zero, the standard deviation do not converge as clearly to zero as the number of hedges increases and there are no obvious points around 40-50 hedges that minimizes the standard deviation of the hedging errors.

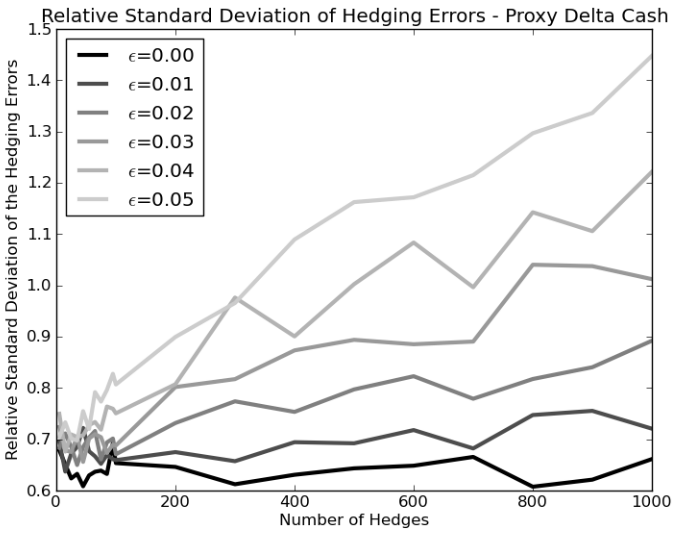

Figure 18: Relative standard deviation of the hedging errors using proxy delta hedge.

4 Summary and Conclusions

This thesis compares and evaluates various strategies for proxy-hedging basket options in a Black-Scholes framework where transaction costs are taken into consideration. The report can be categorized into three different parts. First, in Section 3.1, static and dynamic delta hedging is implemented in the one-dimensional case. In the second part, Section 3.2, the multidimensional case is treated where a basket option is delta and delta-gamma hedged using all underlying assets. These two sections builds up to proxy-hedging - hedging a basket option using assets that are a subset of or disjoint with the baskets underlying assets.

In Section 3.1.2 the static and dynamic hedge is evaluated on a simple call option. The relative mean of the hedging errors decreases rather rapidly when the bid-ask spread increases, especially when the hedging portfolio is rebalanced many times. For all bid-ask spreads, the delta bound hedge performs better in terms of relative hedging errors. This is because the bound reduces the number of actual rebalances of the hedging portfolio and therefore the transaction costs are lower.

In the static case, the standard deviation of the hedging portfolio increases fast as both the number hedges and the bid-ask spread increases. When the bid-ask spread is 0, the relative standard deviation converges to zero as the number of hedges increases, which is expected. However, when transaction costs are considered, there is an optimal point in terms of number of hedges that minimizes the relative standard deviation of the hedge. When the number of hedges is larger than the point that minimizes the relative standard deviation, an almost linear relationship between the relative standard deviation and the number of hedges can be observed.

In the delta bound case, the results are rather different. When the bid-ask spread is low, approximately below 2 %, it is optimal to have a small bound, which implies that the portfolio is rebalanced as often as possible. If the bid-ask spread is higher than around 2 %, the function becomes concave and there seems to be a delta bound that minimizes the relative standard deviation.

The Leland's transformation of volatility, see equation 2.2 and 2.3, is a way of pricing options with transaction costs in the standard Black-Scholes formula. If Leland's model is correct, the results should look similar to the results of the delta hedge in the static case. The results show that Leland's transformation of volatility do not capture all of the costs when delta hedging the option, especially when the bid-ask spread is high. Therefore it seems as if the model underestimates the costs.

In Section 3.2.1 the static and dynamic hedge is evaluated in the multi-dimensional case. The Nengjiu Ju approximation model is used to price the basket option and the underlying assets is simulated using the multivariate Black-Scholes model. As expected, the relative mean of the hedging errors decays with higher bid-ask spread in both cases. The dynamic hedge is not as sensitive to changes in transaction costs as the static hedge and in general results in lower hedging errors, except for the case when the delta bound is very high. The standard deviation of the hedging errors increases with higher bid-ask spread in both the static and the dynamic case. In the static case a minimum occurs for low number of hedges when transaction costs are included. In the delta bound case however, the level of the bound is not that important for the performance of the hedge.

Hedging non-linear risk is discussed in Section 2.2.4. In terms of relative mean of the hedging errors the delta-gamma hedge performs slightly worse than the delta hedge. However, the standard deviation is lower when the number of hedges is small. This is especially the case when the transaction costs are low.

The proxy-hedging strategies are evaluated with a hedging portfolio that consists of asset 1 and 2 in the basket. The quadratic hedge, described in Section 3.3.1, results in very high relative means and standard deviations of the hedging errors. Therefore, this method cannot be recommended, at least not in this context.

Another approach introduced is to minimize equation 2.7. The optimization technique outperforms the delta hedge for high bid-ask spreads and large number of hedges. This is because this method buys lower quantities of stocks to hedge the basket option with, which leads to lower transaction costs. On the contrary to all other hedging strategies evaluated in this thesis, the relative standard deviation of the hedging errors declines, in the presence of transaction costs, when the number of hedges increases. The bid-ask spread did not affect the standard deviation as in other hedging strategies.

The proxy delta hedge, discussed in Section 3.3.3, performs only slightly worse than the static delta hedge when trading in all underlying assets. In general, around twice the relative standard deviation is observed overall using this method. In comparison to the quadratic hedge, the difference is rather large in favour of the proxy delta hedge.

The results show that transaction costs complicates the hedging process. When the bid-ask spread rises, the hedging portfolio should be re-hedged very few times to minimize the variance. When proxy hedging the challenge is to consider the correlation between the hedging assets and the underlying assets of the basket. The easiest way to do this is to state and solve an optimization problem. Quadratic hedging provides an analytical solution that minimizes the variance of the hedging error. However, this method has shown to be unstable because of extreme solutions in some cases. Instead a numerical optimization technique is tested where the sum of the mean and the variance is minimized. The method outperforms the analytical approach.

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5 Appendix

5.1 Validation of the Nengjiu Ju Approximation Model

The Nengjiu Ju approximation model for pricing basket options is validated by using two similar Monte Carlo methods. Both methods are based on the multidimensional geometrical Brownian motion defined in equation 2.4. In the first approach, N normal distributed random variables are generated, and then the cholesky decomposed covariance matrix is used to produce correlation between the random variables. These correlated values are inserted directly into the solution stated in equation 2.5. The advantage of this method is that it can be fully vectorized and hence significantly reduce time consumption. The second approach is based on a discretization of the problem according to an Euler iterative approximation scheme. In the two dimensional case the process is defined as:

$$dS_{t_{i+1}} = S_{t_i} + rS_{t_i}dt + \sigma S_{t_i}\left[\rho dW^1_{t_i} + \sqrt{1-\rho^2}dW^2_{t_i}\right]$$

where $dW^j_{t_i}$ represents the normal distributed Brownian increments. The iterative methodology makes this simulator slow when generating trajectories with small time steps. However the method is more general and can easily be adapted to consider stochastic volatilities or processes with jumps.

The expected value of the basket option is then simply approximated using the discounted mean of the simulated prices inserted into the payoff function. The figure below illustrates the convergence of the two Monte Carlo methods and also the Nengjiu Ju approximation of the basket price.

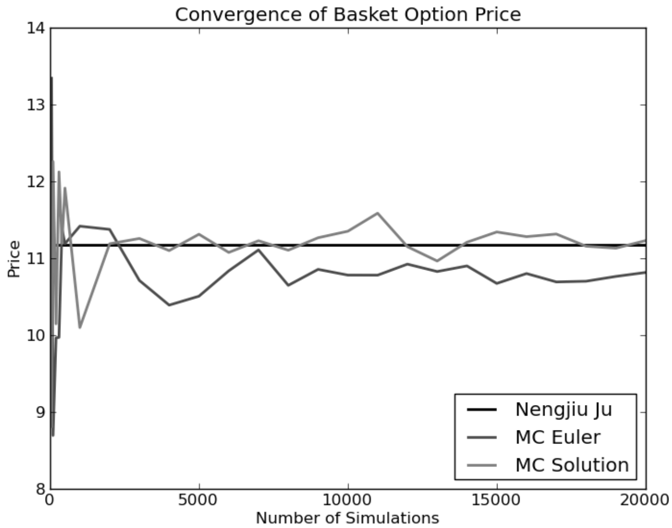

Figure 19: Convergence of Monte Carlo prices.

The Monte Carlo method where the solution to the geometrical Browninan motion is used converges to the Nengjiu price. The Euler approximation seems to underestimate the price of the basket option.

Model	Number of simulations	Price
Nengjiu Ju Approximation	-	11.1713
Monte Carlo - Simulation from solution	100 000	11.1612
Monte Carlo - Euler approximation	100 000	10.8990

From the results we conclude that the Nengjiu Ju Model serves as a good approximation to price basket options.