Betting on Volatility: A Delta Hedging Approach

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Abstract

In the financial market, investors prefer to estimate the stock price based on historical volatility or implied volatility from the market. Consider a trader who believes the historical or implied volatility of an option is too high; the trader can capitalize on this information by selling the option short and buying the delta hedge. The question is which volatility to use in the delta hedge; the trader’s view on volatility $\sigma_T$ or the market’s implied volatility $\sigma_I$. This thesis investigates the performance, under different market conditions, of the two choices, the trader’s view $\sigma_T$ and the market’s view $\sigma_I$. Recently, traders in the option market try to bet on the volatility for the future prices, i.e. betting on a lower volatility. A number of delta hedging models will be discussed in this paper. We will see whether we use lower volatility in the trader’s view or the market’s implied volatility to forecast the asset price and hedge the risk. All the models in this paper will be built in a complete market. We will see some sensitive movements of the hedging options strategy after choosing different volatility. Moreover, after adding the transaction cost in every trading, we perhaps meet the bad results when applying delta hedge. How to manage the hedging schemes and what the number of maturity date we should take will be studied in this paper as well.

Keywords: delta hedge, trader’s view, market’s view, volatility, price simulation.
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Liang Zhong
Chapter 1

Introduction.

Option trading has had a long history since 332 B.C., when it was mentioned in Aristotle’s book named ‘Politics’. It records one of the seven sages of ancient Greece named Thales, who created a kind of call option for trading the rights to use olive presses for people who needed them at olive harvest times.

Modern options have their roots in the works of the French mathematician Louis Bachelier (1900), who is credited for being the first person to model the stochastic process now called the Brownian motion, which was a part of his PhD thesis ‘The Theory of Speculation’. Modern option trading was developed in 1973 when the Chicago Board Options Exchange (CBOE) and the Options Clearing Corporation (OCC) standardized the exchange traded call option, and when the put option was introduced by CBOE in 1977. Since that time, more and more exchanges have been set up and better computational models have been introduced.

In order to offset any unfavorable moves during trading, Greeks are significantly used in option hedging strategies. For instance, delta is used to hedge the directional risk, gamma describes how this risk changes when the price of the underlying asset changes, vega describes the volatility risk, and theta the time decay risk. We know that these Greeks are used based on the Black-Scholes (B&S) model. In particular, the volatility in the B&S model is interesting and important to study. In the past 10 years, people have become more and more interested in predicting the market by analyzing how traders bet on the volatility. Eric
Zitzewitz (2006) was able to construct an index,\(^1\) essentially based on such bets, that has significant incremental power in predicting volatility over the next day, week, or month and in predicting trending or mean reversal in the level of the Dow Jones Industrial Average (DJIA). Werner P. M. De Bondt (1993) found out that the subjective asset price distributions of non-experts are skewed in the opposite direction of their expectations. However, people’s beliefs come from experience, and always, these experiences are very useful to them [Murphy and Wright (1984)]. Consider a trader who has a belief that the volatility is not as high as the market’s volatility; this trader could sell an option short and buy a delta hedge at the same time, deposit the net profit into bank account, then repeat this procedure many times. Accordingly, forecasting the volatility based on people’s experience is very important for hedging an option.

On the other hand, to apply the hedging scheme, we need to define which volatility to use, the volatility in the trader’s view \(\sigma_T\) or in the market’s view \(\sigma_I\). The trader’s view \(\sigma_T\) comes from the trader’s belief in the financial market. In contrast, the market’s view \(\sigma_I\) is implied volatility estimated from the market data. In this paper, I will consider \(\sigma_T\) to be less than \(\sigma_I\), i.e. that the trader bets on the volatility to be lower than the market’s view, and show how the trader can evaluate the stock price and make profit by shorting the option and immediately purchasing a delta hedge. However, the speculator has to make up his mind about which volatility to use for rebalancing his portfolios. This thesis involves volatility choosing and model establishment for verifying this idea of an option hedging strategy. A plenty of results will be presented and illustrated in the following chapters and appendix.

Chapter 2 describes the theoretical knowledge I use and parameters assumptions. Chapter 3 defines the basic models for the hedge strategy with respect to different volatility choices. Chapter 4 introduces a stochastic volatility, which is in practice more approachable than the volatility being constant. In Chapter 5, I develop a modified model to control the trading times. Chapter 6 discusses how to search for the optimal trading times corresponding to different volatilities. Chapter 7 gives the results of hedging an option with price simulation in the market’s view. Chapter 8 presents some conclusions about the study of this paper. Finally, in the appendix, I present some results corresponding to slight parameter changes.

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\(^1\) Discovery Among the Punters: Using New Financial Betting Markets to Predict Intraday Volatility(July 2006), Eric Zitzewitz.
Chapter 2

Theory Background.

The Black-Scholes model is a mathematical description of the financial markets and derivative instruments. The Black-Scholes formula is widely used in the pricing of European options. This paper will discuss a plenty of usages of the delta hedging strategy, the delta being the first derivative of Black-Scholes price. In this chapter, basic background of price simulation, Black-Scholes model, delta hedge, and parameter definitions will be given.

2.1 Simulation of the asset price.

In the financial market, the asset price $S_t$ is modeled by a geometric Brownian motion (GBM). A GBM is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion\(^2\), also called Wiener process.

The stochastic price $S_t$ is said to be a GBM if the following stochastic differential equation (SDE) exists:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$S_0 = s_0$$

(1)

where $W_t$ is the Wiener process, $\mu$ is the percentage drift of return and $\sigma$ is the volatility of $S_t$.

---

\(^2\) Introduction to Probability Models by Sheldon M. Ross, 2007 Section 10
Some arguments for using a GBM for modeling asset prices follow:

- The expected returns of a GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of “roughness” in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

In order to find the solution of (1), I define 

\[ Z_t = \ln S_t, \text{ where } S_t \text{ is the solution of (1)}. \]

Then, we have:

\[
\begin{align*}
\frac{dZ}{S} &= \frac{1}{2} \left( \frac{-1}{S^2} \right) (dS)^2 \\
&= \frac{1}{2} (\mu dt + \sigma dW) + \frac{1}{2} \left( \frac{-1}{S^2} \right) \sigma^2 S^2 dt \\
&= (\mu dt + \sigma dW) - \frac{1}{2} \sigma^2 dt
\end{align*}
\]

Then we have the equation:

\[
\begin{align*}
dZ_t &= (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW_t \\
Z_0 &= \ln S_0
\end{align*}
\]

It is simple to integrate this to:

\[
Z_t = \ln S_0 + \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t
\]

Which means that \( S_t \) is given by:

\[
S_t = S_0 e^{\left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t}.
\]

I will present two volatilities used for simulating the asset price in this paper. One is \( \sigma_T \), the view of trader; the other is \( \sigma_I \), the implied volatility of the market. Both \( \sigma_T \) and \( \sigma_I \) are assumed to be constants. Chapters 3 to 6 will use the asset price simulation with \( \sigma_T \), and chapter 7 will simulate the price using \( \sigma_I \).

### 2.2 The Black-Scholes model

The Black-Scholes model (B&S) was first articulated by Fischer Black and Myron Scholes in their 1973 paper, “The Pricing of Options and Corporate Liabilities.” I use their model to price a European option contract, which gives the buyer the right to buy or sell the underlying asset at a pre-determined price at the maturity time. Nowadays, the B&S model is applied for pricing various derivatives, e.g. options on commodities and stock options. In order to introduce the basic model, we explicitly assume the following:

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3. Options, Futures, and other Derivatives (7th edition) by John Hull, 2009, Chapter 12


5. Options, Futures, and other Derivatives (7th edition) by John Hull, 2009, Chapter 13
• The stock price follows the GBM process developed in (1) with the constants $\mu$ and $\sigma$.
• The short selling of securities with full use of proceeds is allowed.
• There are no transaction costs or taxes. All securities are divisible.
• There are no dividends during the life of the derivative.
• There are no riskless arbitrage opportunities.
• Security trading is continuous.
• The risk free interest rate $r$ is constant, and is the same for all maturities.
• The market is risk neutral.

We will discuss in later chapters that some of these assumptions can be relaxed. For example, $\sigma$ can be a known function of $t$.

The Black-Scholes formula prices the European call option on a non-dividend-paying stock at time $t$ with:

$$c(S, \sigma, t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$  \hspace{1cm} (3)

where

$$d_1 = \frac{\ln(S/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$\Phi(x)$ is the cumulative normal distribution function for standardized normal distribution.

$K$ is the strike price, constant all the time.

$T$ is the maturity time, constant all the time.

$r$ is the risk free interest rate, constant all the time.

$c(S, \sigma, t)$ is the call price corresponding to $S, \sigma$, and $t$.

The way to derive the Black-Scholes formula can be found in John Hull’s book, Chapter 13. In next section, we will extend the use of the Black-Scholes formula for delta hedging.

### 2.3 Delta Hedging.

Delta is an important parameter in the pricing and hedging of options. It is the ratio of the change in the price of the stock option to the change in the price of the underlying stock. In mathematical terms, we can consider delta as the slope of the curve of the option price against the stock price:

$$\Delta = \frac{dc}{dS}$$ \hspace{1cm} (4)

where $c$ and $S$ are the same parameters as in formula (3).
Delta represents the number of units of the stock we hold for each option shorted in order to create a riskless portfolio. The construction of such a portfolio is referred to as delta hedging.\(6\)

The delta of a European call option is given with:

\[
\Delta = \frac{\partial c}{\partial S} = \Phi(d_1) \tag{5}
\]

Hence in this paper, the value of delta will be influenced by the parameters \(S, \sigma, \text{and } t\).

### 2.4 Parameters Assumption.

In this paper, I assume \(S_0 = 100, K = 100, r = 0.05, \mu = 0.08, \sigma = 0.5, \sigma_T = 0.3\) all the time. There have also been some results simulated with \(\sigma_T = 0.4\) in the appendix.

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\(^6\) Options, Futures, and other Derivatives (7th edition) by John Hull, 2009, Chapter11.6
Chapter 3

Basic Models

3.1 Trade with $\sigma_T$.

In the financial market, we usually purchase an option and wish to make profit at the maturity time. In case of the price bubble collapse, the trader would like to buy a delta hedge to reduce the directional trading risk. Commonly, the trader would use historical volatility or implied volatility estimated by the empirical data from the market to simulate the stock price, I named this kind of volatility be market’s view. Then the trader uses this market’s volatility to rebalance the underlying by the delta portfolios. This is delta neutral problem. On the part of investor’s view, the trader believes that the market is robustness and the stock price fluctuates more gently than the price estimated by the market’s view. Then the volatility of an option in trader’s view will be lower than the market’s view. The trader can capitalize this information by selling the option short and buy a delta hedge to make the net profit. In this section, I will use the trader’s view $\sigma_T$ to simulate the asset price and trade the delta hedge portfolio at each time point.

Suppose we have a call option with the price $C_0 = C(S_0, \sigma_I, 0)$ at time 0, $\sigma_I$ is the volatility calculated by the market. A trader, who believes that the volatility $\sigma_T$ in his/her view will be $\sigma_T < \sigma_I$. If the trader sells the option at time 0 with the price $C_0$, replicating the delta portfolio based on his/she view $\sigma_T$ with the price on $P_0 = C(S_0, \sigma_T, 0)$ at time 0. Hence in continuous time, we have the profit: $B_0 = C_0 - P_0$, at time 0, then $B_T = e^{rT}(C_0 - P_0)$, at time $T$. However we note that this strategy is not delta neutral. Because
\[ \frac{\partial v_T}{\partial s} = \Delta(0, \sigma_t) - \Delta(0, \sigma_T) \neq 0, \text{ since } \sigma_T < \sigma_t. \]  (6)

In practice, mostly we trade assets at discrete time points \(0 = t_0 < t_1 < \cdots t_n = T\). And in each period we buy \(\Delta(t_i, \sigma_T, S_i), i = 1 \cdots n\) shares of stocks, and sell \(\Delta(t_{i-1}, \sigma_T, S_{i-1})\) shares at the same time. Hence the net value can be derived:

\[
\begin{align*}
t & = 0: \quad B_0 = C_0 - P_0, \\
 t = t_1: \quad B_1 = e^{r \Delta t} B_0 - (\Delta(t_1, \sigma_T, S_1) - \Delta(t_0, \sigma_T, S_0)) \cdot S_1 \\
 t = t_2: \quad B_2 = e^{r \Delta t} B_1 - (\Delta(t_2, \sigma_T, S_2) - \Delta(t_1, \sigma_T, S_1)) \cdot S_2 \\
 & \vdots \\
 t = t_n: \quad B_T = e^{r \Delta t} B_{T-1} + \Delta(t_{T-1}, \sigma_T, S_{T-1}) \cdot S_T
\end{align*}
\]  (7)

where \(B_i, i = 0, 1 \cdots n - 1\) is the amount of net profit. If \(B_i\) is negative, means we have to borrow money from the bank. If \(B_i\) is positive, we deposit the profit into the bank account. \(S_i\) is the spot price of the asset at time \(t_i\). And at time \(t_n = T\), we sell the option with the volatility \(\sigma_T\) to get the final return.

Suppose both the volatility \(\sigma_T\) and \(\sigma_t\) be constant during the whole maturity time. Also the free interest rate \(r\) and the strike price \(K\) keep constant all the time. Then we can use the Black-Scholes call option formula (3) in Chapter 2.2:

\[
\begin{align*}
c(S_t, \sigma, t_i) & = S_t \Phi(d_1) - e^{-r(T-t)} \Phi(d_2) \\
\text{and} \quad d_1 & = \frac{1}{\sigma \sqrt{T-t}} \left\{ \ln \left( \frac{S_t}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) (T-t) \right\}, \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T-t}.
\end{align*}
\]

And use the first order derivatives of B&S formula to derive the delta portfolio \(\Delta(t_i, \sigma_T, S_i), i = 1 \cdots n\) for different time point. The future price \(S_t\) can be simulated by a trajectory of GBM introduced in Chapter 2.1. Here using \(\sigma_T = 0.3\) in (1) (some results from \(\sigma_T = 0.4\) are in appendix) and repeating the simulation 5000 times to get a varying stock price with respect to different time steps, i.e. \(n = 10; 20 \cdots 100\). Then apply these price simulations in function (7) for corresponding time. The computation results are given below.

Figure 1 shows the curves of final return \(B_T\) compare with the payoff \(\pi = \max (S_T - K, 0)\) (A.1 gives the results for \(\sigma_T = 0.4\) for time steps 10, 20, 50 and 100.
Figure 1: Blue line is the payoff of $\pi = \max(S_T - K, 0)$; Red line is: $B_T = e^{r\Delta t}B_{T-1} + \Delta(t - 1, \sigma_T, S_{T-1}) \cdot S_T$

Figure 1 shows the computation results with $\sigma_T = 0.3$, we can see that the final return at time $T$ always higher than the results from $\pi = \max(S_T - K, 0)$ for different time steps. That because we get the initial capital from selling the option in market’s volatility $\sigma_I$, which is higher than the trader’s view, and buy the option in $\sigma_T$ at the same time point, hence we make the net profit and repeat the procedure.

In case to know the differences between two curves in figure 1, I collect the mean and standard deviation (SD) in the following tables.

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>8.002676</td>
<td>3.147983</td>
</tr>
<tr>
<td>n=20</td>
<td>7.954631</td>
<td>2.280598</td>
</tr>
<tr>
<td>n=50</td>
<td>7.949745</td>
<td>1.471554</td>
</tr>
<tr>
<td>n=100</td>
<td>7.941089</td>
<td>1.047638</td>
</tr>
</tbody>
</table>

*Table 1: Mean and SD of the difference $d = e^{r\Delta t}B_{T-1} + \Delta(t - 1, \sigma_T, S_{T-1}) \cdot S_T - C_n$*

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>0.349310</td>
<td>0.137407</td>
</tr>
<tr>
<td>n=20</td>
<td>0.347213</td>
<td>0.099546</td>
</tr>
<tr>
<td>n=50</td>
<td>0.347000</td>
<td>0.064232</td>
</tr>
<tr>
<td>n=100</td>
<td>0.346622</td>
<td>0.045729</td>
</tr>
</tbody>
</table>

*Table 2: Mean and SD of the normalized difference $d = \frac{e^{r\Delta t}B_{T-1} + \Delta(t - 1, \sigma_T, S_{T-1}) \cdot S_T - C_n}{e^{r\Delta t}}$*
Table 1 collects the values of mean and standard deviation of the difference between two lines at time $T$. Table 1.2 collects the mean and standard deviation of the normalized difference, which is the difference at $T$ against the option price at $T$:

$$d = \frac{e^{\frac{\Delta t}{2}B_{t-1} + \Delta(t-1, \sigma_T, S_{T-1})}S_T - C_n}{e^{rt}C_0}$$

(8)

We can see from the table that the mean difference decreases slightly from 10 to 100, the value is approximately 7.94 under the assumptions in chapter 2. And the SD always keeps decreasing as the time steps get bigger as well, which means that the function (7) will move much more robust as time steps increasing. Also we can see the width of the histograms of the difference between two lines below with the value of normalized difference for $\sigma_T = 0.3$, in Figure 2.

From the histogram Figure 2, we can see clearly the width of the difference is getting smaller as time steps rising up, which means that the variance is changing smaller. The normalized mean difference value almost follows the shape of a normal distribution. As time steps increasing, the function (7) leads approximately to a steady return.

Generally, we know from the results above, if we trade an option with the price simulation of $\sigma_T$, and buy delta hedge using the same volatility, since $\sigma_T < \sigma$, we always have greater return than the payoff at the maturity time. Furthermore, as the time steps grow, the variance of the difference is moving down, and leading a stable return. However, what will happen if we estimate the stock price in $\sigma_T$, but exercise the delta hedge in $\sigma$ for each time point, whether we will still meet good final return? The comparison results will present in the next section.
3.2 Trade with $\sigma_T$.

Section 3.1 gives us the expected results. As an alternative consideration, I still simulate the future price in the trader’s view $\sigma_T$. Instead of hedging in $\sigma_T$, we rebalance the portfolio by purchasing a delta hedge in market’s view $\sigma_I$. Then we change $\sigma_T$ to $\sigma_I$ in the invest strategy model (7). Let’s compare with the return at the maturity time $T$ against the payoff $\pi = \max(S_T - K, 0)$. Will it be good as well? The following pictures show the results in different time steps.

**Figure 3**: Blue line is the payoff of: $\pi = \max(S_T - K, 0)$; Red line is: $B_T = \sigma_T^{\Delta t} B_{T-1} + \Delta(t - 1, \sigma_I, S_{T-1}) \cdot S_T$

Figure 3 shows the results of payoff at time $T$ for different time steps. It tells us that if we hedge in $\sigma_I = 0.5$, the final return against the payoff $\pi = \max(S_T - K, 0)$ always have a big difference when the asset price is in the interval [50,150]. After the stock price rising higher, this gap is changing asymptotic to 0. For small time steps, i.e. $n = 10; 20$, the final return is less than the payoff when the stock price is under 50. Compared with Figure 1, Figure 3 gives bigger difference between two curves around the strike price, but not always keeps stable all the time after the asset price rises up.

In order to know the value of difference, here presents two tables recording the results of mean and standard deviation for the difference between two lines bellow.
We see from above tables, the mean values are close to 7.93 under the assumptions for different time steps. While the SD of the difference decrease much more slowly compared with the SD in Table 1 and 2. The histogram of normalized difference below tells the distribution of the difference.

These histograms show wider width than Figure 2. The shape of the histogram is not perfectly match the normal distribution. It tells that we would not meet the final return as stable as using the trader’s view in the hedging strategy.

After changing to $\sigma_j$, although we could get the profit positive and similar with the results from table 1, we would bear higher risk of buying this delta hedge. On the other hand, the
SD also decrease as time steps grown, we could make steady profit if and only if we keep the trading period long enough. But in practice, we must pay tax, administration fee, transaction fee etc, for every trading, thus the extra fees are created. Then the next section will introduce how the results change when adding the fees.

3. 3 Adding transaction cost.

3.3.1 In the view $\sigma_T$.

In last two sections, I traded the asset without any costs, as a practical matter, the trader sometimes pays some extra fees to complete one trade. Not losing generality, let’s define all the other fees for each trade is transacted fee, which will be generated by buying and selling options at every trading point. In this paper, assume the parameter $\lambda$ be the transaction cost per unit of the asset, and assume the number of offsetting asset by buying and selling is always positive.

In this section, as 3.1, we believe our view $\sigma_T$ will be good during the trading. Then we modify the model (7) for adding the transaction cost $\lambda$:

$$
t = 0: \quad B_0 = C_0 - \Delta(t_0, \sigma_T, S_0) \cdot (S_0 + \lambda),
$$

$$
t = t_1: \quad B_1 = e^{r\Delta t}B_0 - \left( \Delta(t_1, \sigma_T, S_1) - \Delta(t_0, \sigma_T, S_0) \right) \cdot S_1 - \lambda \cdot |\Delta(t_1, \sigma_T, S_1) - \Delta(t_0, \sigma_T, S_0)|
$$

$$
t = t_2: \quad B_2 = e^{r\Delta t}B_1 - \left( \Delta(t_2, \sigma_T, S_2) - \Delta(t_1, \sigma_T, S_1) \right) \cdot S_2 - \lambda \cdot |\Delta(t_2, \sigma_T, S_2) - \Delta(t_1, \sigma_T, S_1)|
$$

$$
\vdots
$$

$$
t = t_n: \quad B_T = e^{r\Delta t}B_{t-1} + \Delta(t_{T-1}, \sigma_T, S_{T-1}) \cdot S_T - \lambda \cdot |\Delta(t_{T-1}, \sigma_T, S_{T-1})| \tag{9}
$$

All the parameters except $\lambda$ are the same as (7).

The future price $S_T$ can be simulated by a trajectory of GBM with $\sigma_T = 0.3$ for 5000 times.

For choosing the transaction cost, if I set $\lambda$ too small, it might be no any difference compare with no transaction fee as section 3.1, otherwise we would always get very bad negative difference if we let $\lambda$ too big. Through some testing, in this section I choose $\lambda = 1$ that will present us reasonable results, also I made some bigger transaction cost, i.e. $\lambda = 3$ in later section and A.2. After paying the transacted fee, the final return against the payoff at time $T$ will be smaller than the trading without any fee.
Figure 5: Blue line is the payoff: \( \pi = \max(S_T - K, 0) \); Red line is: \( B_T = e^{rT} B_{T-1} + \Delta(t_{T-1}, \sigma_T, S_{T-1}) \cdot (S_T - \lambda \cdot |\Delta(t_{T-1}, \sigma_T, S_{T-1})|) \), for \( \lambda = 1 \).

Compare with Figure 1, the gaps between two lines in Figure 5 are much smaller. The final return is less than the payoff when the asset price is under 50 at time \( T \). After that the differences change big when the asset price is up to the strike \( K = 100 \). Then the final return is moving close to the payoff since the stock price is higher than the strike. We note that, if the time steps rise to 100, here comes out coincidence between two curves when the asset price over the strike \( K = 100 \). We find that after paying the transacted cost for every trading, the net value cannot offset the transaction fees at some time point.

The following table collects the mean and SD compared with trading without the transaction cost.

<table>
<thead>
<tr>
<th>Time step</th>
<th>Transaction cost</th>
<th>No transaction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>n=10</td>
<td>5.846055</td>
<td>3.318483</td>
</tr>
<tr>
<td>n=20</td>
<td>5.349445</td>
<td>2.444485</td>
</tr>
<tr>
<td>n=50</td>
<td>4.551757</td>
<td>1.763203</td>
</tr>
<tr>
<td>n=100</td>
<td>3.547458</td>
<td>1.639327</td>
</tr>
</tbody>
</table>

*Table 5: Compare the difference of mean and SD for \( \lambda = 1 \).*

<table>
<thead>
<tr>
<th>Time step</th>
<th>Transaction cost</th>
<th>No transaction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>n=10</td>
<td>0.255176</td>
<td>0.144849</td>
</tr>
<tr>
<td>n=20</td>
<td>0.233499</td>
<td>0.106700</td>
</tr>
<tr>
<td>n=50</td>
<td>0.198680</td>
<td>0.076962</td>
</tr>
<tr>
<td>n=100</td>
<td>0.154844</td>
<td>0.071555</td>
</tr>
</tbody>
</table>

*Table 6: Compare the normalized difference of mean and SD for \( \lambda = 1 \).*
The above tables tell us that the mean with transaction cost is obviously smaller than the trade without any cost. The SD is greater than the one without extra fees. But it still keeps decline trend.

We will see more clearly about the distribution of the difference with transaction cost in the following histograms.

Figure 6 gives the distribution of the difference for different time steps. We see when the time steps are small, i.e. \( n = 10; 20 \), the histograms look similar as the one without any transaction cost, the shape almost normal distributed. Because few time steps cause little transaction costs. As \( n \) change to 50; 100, the histogram skewed left, and the width reduced. Because after adding the transaction fees, the final return might be less than the payoff sometimes, the peak point of the histogram moves to the left side along x-axis.

We note that, the profit would be influenced by other extra fees. Unless we invest the delta hedge in a short maturity time, or we try to buy and sell the hedge less, pay less transaction cost, then we may insure us against the negative return. Otherwise, if we have bad luck to meet the low asset price, we would lose our money.

In the next section, I will use the same scheme as section 3.2, but adding the transaction cost during the trading. The results will be shown in the following section.
3.3.2 In the view $\sigma_T$.

Consider the same condition in section 3.2. If we change the $\sigma_T$ to $\sigma_t$ in the function (9), using the price simulation by GBM in $\sigma_T = 0.3$. I repeat the same procedure 5000 times and get the results below. Here I still use the transaction cost $\lambda = 1$. (Some results about $\lambda = 3$ are in later section and A.3)

Figure 7 is the plot of final return against the payoff at time T. For different time steps, the final return is still less than the payoff when the asset price is lower than 50. And we get a little big difference between two curves in the price interval $[50, 110]$ as the same with Figure 5. But after that the difference is getting smaller, and two curves start to close since the price is growing. Also, the two curves have coincidence around the strike price $K = 100$ when $n = 100$. Next, I collected the mean and SD of the difference in the following table.

<table>
<thead>
<tr>
<th>Time step</th>
<th>Transaction cost</th>
<th>No transaction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>n=10</td>
<td>5.816607</td>
<td>3.309205</td>
</tr>
<tr>
<td>n=20</td>
<td>5.353889</td>
<td>2.535752</td>
</tr>
<tr>
<td>n=50</td>
<td>4.506760</td>
<td>1.799765</td>
</tr>
<tr>
<td>n=100</td>
<td>3.544519</td>
<td>1.654849</td>
</tr>
</tbody>
</table>

Table 7: Compare the difference of mean and SD in $\sigma_t$, for $\lambda = 1$. 

\[ \text{Figure 7: Blue line is the payoff: } \pi = \max(S_T - K, 0) ;\text{ Red line is: } B_T = e^{\lambda \Delta t} B_{T-1} + \Delta (t_{T-1}, \sigma_t, S_{T-1}) \cdot S_T - \lambda \cdot |\Delta (t_{T-1}, \sigma_t, S_{T-1})|, \text{ for } \lambda = 1. \]
### Table 8: Compare the normalized difference of mean and SD in $\sigma_r$, for $\lambda = 1$.  

The comparison tables above report the mean and SD of the difference between red line and blue line. The tables tell us the means of difference are obviously smaller than the ones without transaction costs. And compare with table 5 when we buy the hedge in $\sigma_r$, the mean values and SDs are very close. It also shows the same decline trend of SDs. The histograms show the distributions of the difference for different time steps $n$.  

<table>
<thead>
<tr>
<th>Time step</th>
<th>Transaction cost</th>
<th>No transaction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>n=10</td>
<td>0.253890</td>
<td>0.144444</td>
</tr>
<tr>
<td>n=20</td>
<td>0.233693</td>
<td>0.110684</td>
</tr>
<tr>
<td>n=50</td>
<td>0.196716</td>
<td>0.078558</td>
</tr>
<tr>
<td>n=100</td>
<td>0.154715</td>
<td>0.072233</td>
</tr>
</tbody>
</table>

We can see Figure 8 shows same story as hedging in $\sigma_r$. The shapes of distribution are close to normal distributed when the time steps are small. As $n$ rises up, the histograms skewed left and the peak point moves to left side along the x-axis. We will see some results about adding the transaction cost to $\lambda = 3$ in the following section.
3.3.3 Adding Bigger Transaction Cost

As discussed above, whenever trading with $\sigma_T$ or $\sigma_I$, the tables present us positive final mean differences of return. Now we increase the transaction cost to $\lambda = 3$. We may get negative difference. There are only tables collected the mean and SD for $\sigma_T$ and $\sigma_I$ respectively below. Other figures are in A.2 and A.3.

<table>
<thead>
<tr>
<th>Time step</th>
<th>Transaction cost</th>
<th>No transaction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>n=10</td>
<td>1.473100</td>
<td>3.732102</td>
</tr>
<tr>
<td>n=20</td>
<td>0.159406</td>
<td>3.112072</td>
</tr>
<tr>
<td>n=50</td>
<td>-2.403278</td>
<td>3.184459</td>
</tr>
<tr>
<td>n=100</td>
<td>-5.237400</td>
<td>3.869621</td>
</tr>
</tbody>
</table>

*Table 9: Compare the difference of mean and SD with $\sigma_T$, for $\lambda = 3$.*

<table>
<thead>
<tr>
<th>Time step</th>
<th>Transaction cost</th>
<th>No transaction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>n=10</td>
<td>1.594876</td>
<td>3.742493</td>
</tr>
<tr>
<td>n=20</td>
<td>0.145457</td>
<td>3.149260</td>
</tr>
<tr>
<td>n=50</td>
<td>-2.376468</td>
<td>3.108416</td>
</tr>
<tr>
<td>n=100</td>
<td>-5.142027</td>
<td>3.849335</td>
</tr>
</tbody>
</table>

*Table 10: Compare the difference of mean and SD with $\sigma_I$, for $\lambda = 3$.*

The means of difference from two tables have decrease trend as $n$ increasing. We note that, both table 9 and 10 give negative values of the average difference when $n = 50; 100$. And as $n$ increases, the negative value moves bigger, the SD is growing again as the time steps rising up. In A.2 and A.3, the corresponding results are showed in the figures.

3.4 Review.

This chapter introduces us a new price simulation by using $\sigma_T$, and discusses the results when applying delta hedge method under different conditions. The trading option models by using $\sigma_T$ or $\sigma_I$ show positive results as we expected in advance. It means that we could buy and sell with the price simulation in trader’s view to get better return at maturity time $T$. On the other hand, if we estimate the stock price with lower volatility, we would avoid high movements of the asset price and reduce the risk when buy the delta hedge.

As a trader, we wish make profit as much as possible. If we think the market’s view is too high, we then sell the option now to get the initial capital, and buy the delta hedge in the trader’s view. Without the transaction cost, we do get stable return in $\sigma_T$ than hedging in $\sigma_I$.

After adding extra fees, the histograms by using $\sigma_T$ and $\sigma_I$ both show the left skew. When we increased the transaction cost, the final return would be much less than the payoff. The SD decrease first and increase again as $n$ rises. Hence we cannot always keep good return unless we can control our trades. Moreover, during the trading period we could not ensure the volatility will keep constant all the time. In the next chapter, I will introduce a stochastic volatility into the trade strategy.
4.1 The volatility set up.

In real financial market, the volatility cannot keep constant all the time, it must be fluctuating. Suppose we still believe the market’s view is too high, but it will decrease randomly to the trader’s view. That is \( \sigma_J \) moving to \( \sigma_T \) followed the line between two volatilities, where we have \( \sigma_J > \sigma_T \). Then the stochastic volatility has a decline trend, noted it be \( \sigma_S \). We assume \( \sigma_S \) decreases with random noise \( \omega \), and \( \omega \) is normal distributed with the function \( N(0, \varepsilon) \), \( \varepsilon \) is the variance and small enough. In this part, I will choose \( \varepsilon = 0.01 \) in the computation. Now the stochastic volatility is a function depending on time \( t \), the derivation of stochastic volatility is function (10) below.

\[
\begin{align*}
t_0 &= 0, \sigma_S(t_0) = 0.5 \\
t_1, \quad \sigma_S(t_1) &= \sigma_S(t_0) + \frac{\sigma_T - \sigma_I(t_0)}{T}(t_1 - t_0) + \omega \\
&\vdots \\
t_n &= T, \sigma_S(t_1) = \sigma_S(t_{n-1}) + \frac{\sigma_T - \sigma_I(t_0)}{T}(t_n - t_{n-1}) + \omega 
\end{align*}
\]

Where \( \omega \sim N(0, \varepsilon) \), \( \varepsilon \) is small enough.

Figure 9 gives the decreasing trend of \( \sigma_S \).
Figure 9 shows the jumping path of $\sigma_S$. In this paper, $\sigma_S$ has decline trend with normal noise. So we can see the curve of $\sigma_S$ decreases follow the linear line from $\sigma_I$ to $\sigma_T$ with respect to the corresponding time point.
4.2 Computation Results in $\sigma_S$.

Now we apply $\sigma_S$ instead of $\sigma_T$ for the corresponding time point in the function (9), using $\varepsilon = 0.01$.

\[
\begin{align*}
\text{t} = 0: & \quad B_0 = C_0 - \Delta(t_0, \sigma_S(t_0), S_0) \ast (S_0 + \lambda), \\
\text{t} = t_1: & \quad B_1 = e^{r\Delta t}B_0 - (\Delta(t_1, \sigma_S(t_1), S_1) - \Delta(t_0, \sigma_S(t_0), S_0)) \cdot S_1 - \lambda \cdot |\Delta(t_1, \sigma_S(t_1), S_1) - \Delta t_0, \sigma_S(t_0, S_0)|, \\
\text{t} = t_2: & \quad B_2 = e^{r\Delta t}B_1 - (\Delta(t_2, \sigma_S(t_2), S_2) - \Delta(t_1, \sigma_S(t_1), S_1)) \cdot S_2 - \lambda \cdot |\Delta(t_2, \sigma_S(t_2), S_2) - \Delta t_1, \sigma_S(t_1, S_1)|, \\
\vdots
\end{align*}
\]

\[
\text{t} = t_n: \quad B_T = e^{r\Delta t}B_{T-1} + \Delta(t_{T-1}, \sigma_S(t_{T-1}), S_{T-1}) \cdot S_T - \lambda \cdot |\Delta(t_{T-1}, \sigma_S(t_{T-1}), S_{T-1})|.
\]

(11)

As we did in chapter 3, first let’s see the curves of the final returns against the payoff at time $T$.

Figure 10: payoff at maturity $T$ with the stochastic volatility, $\lambda = 1$

Record the mean value and SD in the following tables.

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>5.941675</td>
<td>3.394599</td>
</tr>
<tr>
<td>n=20</td>
<td>5.492917</td>
<td>2.596241</td>
</tr>
<tr>
<td>n=50</td>
<td>4.896377</td>
<td>1.872899</td>
</tr>
<tr>
<td>n=100</td>
<td>4.069742</td>
<td>1.755384</td>
</tr>
</tbody>
</table>

Table 11: Mean and Variance of difference for stochastic volatility, for $\lambda = 1$
Despite the curves’ difference in figure 10 is similar with the results in chapter 3.3, we can see the mean differences are better by applying $\sigma_5$ than using $\sigma_1$ or $\sigma_2$ in table 11 and 12. And the SD is decreasing as time steps increases. Furthermore, the histograms below show good distribution of the difference.

**Table 12: Normalized Mean and Variance of difference for stochastic volatility, for $\lambda = 1$**

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>0.259349</td>
<td>0.148171</td>
</tr>
<tr>
<td>n=20</td>
<td>0.239761</td>
<td>0.113324</td>
</tr>
<tr>
<td>n=50</td>
<td>0.213723</td>
<td>0.081751</td>
</tr>
<tr>
<td>n=100</td>
<td>0.177641</td>
<td>0.076621</td>
</tr>
</tbody>
</table>

Compare with the histogram figure 6 in section 3.3.1, when we use $\sigma_5$ to buy and sell the option, the effect of left skew of the final difference is reduced; regarding the histogram figure 8 in section 3.3.2, we note the width in figure 11 is wider, the distribution weaken the left skew with respect to different time steps.

By contrast with figure 11, now increase the transaction cost $\lambda = 3$ in function 11, and report the histograms in figure 12.
Figure 12: Histogram of the normalized difference, $\lambda = 3$

We find that the width of histogram is decreasing when the time steps is varying from 10 to 50, and increasing again as the trading step equal 100. The peak of the histogram is moving to left side as $n$ increasing. For time steps 10 to 50, we almost receive stable final difference since the time steps are under 50, but the histogram has left skew when $n = 100$.

4.3 Review.

This chapter defines the varying volatility $\sigma_S$ from $\sigma_f = 0.5$ to $\sigma_T = 0.3$. If we trade with a suitable transaction cost during the maturity time, we could receive reasonable final return, and keep the variance of the mean difference small and decreasing. But after increasing the transaction cost, we almost keep good return in small time steps, the negative difference will come out when the time step is big, even more, the SD of difference will get wider again. On the other hand, if we can cancel some bad trades, and only buy and sell in the day we think it is good for trading, we will save our transaction cost into account. In the next chapter, I will introduce a new method to manage our trade.
Chapter 5

Conditional Trading.

When we buy and sell the options at the discrete time points, we pay the transaction cost for every trading. The portfolios of the option depend on the volatility and asset price at each time $t$. Sometimes we might meet very bad results due to the stock price is too low, we have to trade a small number of portfolios, paying for the transaction costs to increase the hedging error. In this case, we may consider we do not need trade at this ‘bad’ time, and just keep the underlying until the time when the asset price rise to a reasonable value, then we continue the trading.

5.1 Trading in $\sigma_T$.

In this section, I consider the trading by using the volatility $\sigma_T$. Apply the same function as (9),

$t = 0$: $B_0 = C_0 - \Delta(t_0, \sigma_T, S_0) \cdot (S_0 + \lambda)$,
$t = t_1$: $B_1 = e^{r \Delta t} B_0 - (\Delta(t_1, \sigma_T, S_1) - \Delta(t_0, \sigma_T, S_0)) \cdot S_1 - \lambda \cdot |\Delta(t_1, \sigma_T, S_1) - \Delta(t_0, \sigma_T, S_0)|$
$t = t_2$: $B_2 = e^{r \Delta t} B_1 - (\Delta(t_2, \sigma_T, S_2) - \Delta(t_1, \sigma_T, S_1)) \cdot S_2 - \lambda \cdot |\Delta(t_2, \sigma_T, S_2) - \Delta(t_1, \sigma_T, S_1)|$

In each step the delta portfolio $\Delta(t_i, \sigma_T, S_i)$ is depending on the time $t$ and future price $S$. I set a conditional value for verifying the portfolio whether decide to trade, noted the conditional value be $\rho$. If $\Delta(t_i, \sigma_T, S_i) > \rho$, we exercise the option as function (9), otherwise
we hold our money to the next time point, and check the amount of delta portfolio whether exceed $\rho$ or not, repeat the same strategy. Hence I add the conditional value into function (9):

$$\text{If } t = t_i:$$

$$\Delta(t_i, \sigma_T, S_i) > \rho$$

$$B_i = e^{r\Delta t} B_{i-1} - \left( \Delta(t_i, \sigma_T, S_i) - \Delta(t_{i-1}, \sigma_T, S_{i-1}) \right) \cdot S_i - \lambda \cdot |\Delta(t_i, \sigma_T, S_i) - \Delta(t_{i-1}, \sigma_T, S_{i-1})|$$

$$\text{Else } B_i = e^{r\Delta t} B_{i-1}.$$  

For the numerical computation I still simulate the future price with 5000 times in $\sigma_T$, using $\lambda = 1$, $n$ from 10 to 100, and any other parameters are the same. For choosing of $\rho$, we know the initial portfolio $\Delta(t_0, \sigma_T, S_0) = 0.6234$, and $\Delta(T, \sigma_T, S_0) = 0.5126$, for $t_0 = 0, T = 1, S_0 = 100$. To make sure our trading is reasonable judged by $\rho$, I choose $\rho = 0.55$ in this paper.

The following figures and tables are the results calculated by applying condition (12).

![Figure 13: payoff of conditional trading at maturity $T$](image)

From figure 13 we can see the difference of return and payoff change obviously bigger than the model we use in chapter 3.3.1. Which means after adding some trade conditions, we mitigate the cost of transaction efficiently, and then we would receive bigger profit at the final time point. However, we also need care about when the asset price is very low, i.e. under 70, we would meet very bad return which lead to negative difference between two curves.
The histograms show obviously skewed right tendency, and the tables above tell the results of the mean and SD. Although they have much greater means than the values without using the conditional value $\rho$, the SDs are much bigger compared with the amount from table 5 and 6. Moreover, we note that the SD is increasing as the time steps rising up.

Superficially, we would have big profit in this model for doing fewer trading and paying less transaction cost, but not exactly. In this strategy, we choose the time to purchase and sale an option when we think it is mature. However, we find the SD of the final difference is very big. Figure 13 describes when the asset price is low enough, such as fewer than 70 or even lower, we will meet very bad final return. In this case, whatever the conditional value we choose, the extra fees cannot be offset unless the stock prices rise to a reasonable level.

Let’s see the results hedge in the stochastic volatility $\sigma_S$ in the next part.
5.2 Conditional Trading with Stochastic Volatility.

Chapter 4 gives good results when I use stochastic volatility, next I apply $\sigma_S$ for the corresponding time steps with the conditional function (12).

Figure 15: payoff of conditional trading at maturity $T$

Compare with figure 13, the difference of final return against the payoff at time $T$ is smaller in figure 15. As the stock price increasing, the final return is asymptotically getting close to the payoff line ($\sigma_S$ is decreasing from 0.5 to 0.3). When the asset price is lower than 70, the final return is smaller than the payoff. I also collect the mean and SD of the difference in the tables and plot the normalized difference histograms below.

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>11.532376</td>
<td>11.492882</td>
</tr>
<tr>
<td>n=20</td>
<td>14.254087</td>
<td>14.157576</td>
</tr>
<tr>
<td>n=50</td>
<td>16.376304</td>
<td>16.881091</td>
</tr>
<tr>
<td>n=100</td>
<td>17.717474</td>
<td>19.416456</td>
</tr>
</tbody>
</table>

Table 15: Mean and Variance of difference

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>0.503379</td>
<td>0.501655</td>
</tr>
<tr>
<td>n=20</td>
<td>0.581391</td>
<td>0.576428</td>
</tr>
<tr>
<td>n=50</td>
<td>0.714812</td>
<td>0.736846</td>
</tr>
<tr>
<td>n=100</td>
<td>0.773353</td>
<td>0.847512</td>
</tr>
</tbody>
</table>

Table 16: Normalized Mean and Variance of difference
Table 15 and 16 give the mean values and SDs. The means and SDs are both smaller than trading in $\sigma_T$, and the SD is increasing from $n = 10$ to 100 as well as we trade in $\sigma_T$. The histograms of figure 16 also show right skew, even though the width of shape is increasing, these histograms describe more stable distribution than using $\sigma_T$.

5.3 Review

As discussed after adding the transaction cost, the trader would receive very bad return as time steps increase in last chapter. Therefore I set a conditional value to control the trading in order to increase the efficiency of the hedging strategy. Unfortunately, sometimes we still meet very bad return due to the final asset price is low enough. Then the investing risk in this method is increasing hugely. I wander that, if we could have good control of the trading, such as maximum the final difference and minimum the variance, we can compute the optimal trading time steps. I will develop an optimization problem for this delta hedge strategy in the next section.
Chapter 6

Optimal Trading Time Steps.

During the trading period, we will meet big variance and bad negative value of the difference between the final return and the payoff at maturity time after adding transaction cost. However we note that the mean difference and SD are decreasing first for some time steps, the SDs will rise again when the trading steps $n$ is big enough. Therefore, I think there is an optimal time step to make the mean difference maximum and minimum the variance of the difference. This chapter will discuss to find this optimal trading days in the following way.

6.1 Method Establishment.

Consider function (9), for different time steps $n$ we have a series of corresponding final return $B_T$ at the maturity time $T$ and the payoff $C_T$ at time $T$. We hope to record the difference which we denote it be $V_T = B_T - C_T$. Assume here the value $V_T$ only depends on variable $n$, any other parameter is fixed, then we wish to find the value:

$$max_n E[V_T(n)] - \gamma \cdot Var[V_T(n)]$$  \hspace{1cm} (13)

Where $\gamma$ here is a parameter need to fix.

During the computation, I let $n$ vary from 10 to 100 with the gap 10, and choose the $\gamma$ be (0.1, 0.2, 0.3, 0.4, 0.5, 1.0, 1.5, 2.0, 2.5) firstly, simulate the future price with 1000 times, the transaction cost $\lambda = 1$, other parameters are the same. Using function (9) to calculate $B_T$. 

- 29 -
and $C_T$, and put the results in (13). Now I got a table of the optimal $n$ corresponding different $\gamma$:

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>15</td>
<td>15</td>
<td>30</td>
<td>50</td>
<td>40</td>
<td>45</td>
<td>60</td>
<td>70</td>
<td>70</td>
</tr>
</tbody>
</table>

Table 17: optimal $n$ with different $\gamma$ for $\lambda = 1$

In addition, I plot the expectation and variance of the difference against different time steps in Figure 17:

From the table and the figure we can see when the $\gamma$ is small, the time step is short, while it gives a mostly reliable optimal $n$s during the interval 0.3 to 1.5, and after $\gamma$ increases higher than 1.5, the $n$ rise to a great value. It also describes in the figure that the curve of expectation value is much lower than the curve of variance at beginning when $n$ vary from 10 to 20, after that the variance line falls off under the expected line. It tells us if we choose $\gamma$ small, function (13) is always maximal for $n$ is as smaller as possible. Otherwise if we choose $\gamma$ too big, the difference is maximal for $n$ as bigger as possible. Thus we have to choose $\gamma$ in a reasonable interval, i.e. the value from 0.3 to 1.5.

6.2 Results from Computation.

First I still use the volatility $\sigma_T$ in the function (9), and put the results in the function (13), see the results of choosing $\gamma = 0.4, 0.5, 1, 1.5$ below.
From figure 18 we found the peak point of the curve is moving to the right side along x-axis as $\gamma$ increasing. For $\gamma = 1$ and 1.5, the two curves are very flat when the time step is big. That’s because the region of time steps is too short for the simulation. Now for the aim of more precisely, I still simulate the future prices for 1000 times, but increase the time steps from 10 to 500 for every 10 units, using $\gamma = 0.5$, other parameters are the same. The computation result is in the following figure.

Figure 18: $E[V_T] - \gamma \cdot Var(V_T)$

Figure 19: Optimal time steps $n$
We see that increasing the time steps, the curve shows a peak point around \( n = 50 \). Through the computation, the optimal step is \( n = 40 \).

Next we turn to compute the optimal time steps with stochastic volatility. I apply the function (11) and condition (13). The parameter \( \gamma = 0.5 \), I got the results below.

\[
E[V_T] = \gamma \cdot Var(V_T) \text{ with stochastic volatility, } \gamma = 0.5.
\]

When we apply the stochastic volatility, the optimal time step is \( n = 60 \). We see that no matter which the volatility we use, and how long the total time steps we choose to observe, we will get optimal time step around \( n = 50 \).
Chapter 7

Simulation In The Market’s View $\sigma_I$.

From chapter 1 to 6, I use the volatility $\sigma_T$ for the price simulation $S$ during the trading period $T$. Because of we believe our expected volatility $\sigma_T$ will be lower than the volatility from the real market one $\sigma_I$. Although the discussions in the former chapters look superficially attractive, we must consider if our assumption with $\sigma_T$ is wrong in the asset price simulation. Therefore I come to use $\sigma_I$ for the price simulation to see what the different will bring to us. In the method establishment, all the calculations are the same except the asset price simulation in $\sigma_I$.

7.1 Trading in the view $\sigma_T$.

Now use the new simulation of asset price in $\sigma_I$, The price simulation function is the same as (1):

$$dS_t = \mu S_t dt + \sigma_I S_t dW_t$$
$$S_0 = s_0$$

Follow the basic model in chapter 3.1, the results will give in the following figures and tables.
Figure 21: Blue line is the payoff of $\pi = \max(S_T - K, 0)$; Red line is: $V_T = e^{r\Delta t}B_{T-1} + \Delta(t - 1, T, S_{T-1}, S_T)$; Both $ST$ simulated with $\sigma = 0.5$.

Compare with figure 1 in chapter 3.1, figure 21 almost gives two coincide lines. But when the asset price is lower than 60, the final return will be less than the payoff for the different time steps.

Figure 22: Histogram of the normalized difference $= \frac{e^{r\Delta t}B_{T-1} + \Delta(t - 1, S_{T-1}, S_T) - C_n}{e^{rT}C_0}$.
In histogram figure 22, we see the obvious left skew. That because we will get negative difference between two curves in figure 21 for $\sigma_T < \sigma_f$, which causes the left tail. The mean value and SD of the differences are recorded in the table 18 and normalized in the table 19. We see that all the mean values for different time steps are close to 0, but the SD is a little high, despite the SD is decreasing as time steps increase.

### 7.2 Trading in the view $\sigma_f$.

This part we use the new asset price simulation in $\sigma_f$, and rebalance portfolios with the market view $\sigma_f$ in the trading strategy. Therefore we find it is the perfectly delta neutral problem. Hence the return curve coincides with payoff line, the mean difference must be very close to zero, and the histograms should perfectly agree with normal. Thus, in this section I only give the plot when $n = 100$. The Figure 23 shows the delta neutral results.

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n=10$</td>
<td>-0.012569</td>
<td>6.599958</td>
</tr>
<tr>
<td>$n=20$</td>
<td>-0.109087</td>
<td>5.507433</td>
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<td>$n=50$</td>
<td>0.046585</td>
<td>4.471593</td>
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<tr>
<td>$n=100$</td>
<td>0.016835</td>
<td>4.115472</td>
</tr>
</tbody>
</table>

**Table 18: Mean and Variance of difference**

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n=10$</td>
<td>-0.000549</td>
<td>0.288083</td>
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<tr>
<td>$n=20$</td>
<td>-0.004762</td>
<td>0.240395</td>
</tr>
<tr>
<td>$n=50$</td>
<td>0.002033</td>
<td>0.195181</td>
</tr>
<tr>
<td>$n=100$</td>
<td>0.000735</td>
<td>0.179637</td>
</tr>
</tbody>
</table>

**Table 19: Normalized Mean and Variance of difference**

The mean of difference is 0.001368, The standard deviation is 1.686609.
7.3 Adding transaction cost.

In this part I use function (9) in chapter 3.3.1, but simulate the asset price $S$ with $\sigma_I$.

After adding the transaction cost, the curve of final return is always under the payoff line at time $T$. As the price $S$ rising higher, two curves are getting together. However, the mean difference will be always negative after adding the transaction cost.
The histograms show left skew to us, as the time step goes up, this skewness moves larger. We know from the table above that as time steps increasing, the mean of difference is decreasing, which means that if we trade the option in a longer days, we would pay more transaction cost, and the difference of return would get bigger. Also we note that the SD is decreasing from $n = 10$ to 50, and it is increasing again when $n = 100$.
7.4 Trading in stochastic volatility.

Turn to trade the delta hedge in stochastic volatility $\sigma_S$, where $\sigma_S$ is derived in chapter 4.1, using the trading strategy function (11) in chapter 4.2. The following plots show the results from 10 to 100.

![Plots showing asset price at maturity $T$ and profit for different $n$ and $\lambda$.]

**Figure 26: payoff at maturity $T$ with the stochastic volatility**

Figure 26 is the results of final return against the payoff at $T$. The red line is almost under the blue line. Which means even though we hedge in $\sigma_S$, we always meet negative differences. We will see the histogram and the mean value and SD collected in the tables below.
The histogram still shows skewed left for each time step, but it’s not as skew as figure 25 in the last section. Then the mean differences are bigger compared with table 20. We received the negative amount of average difference as well, and this mean value is decreasing as \( n \) increasing for adding the transaction fee in more trading days. Also, the SDs decline first, and rise up again as \( n \) getting larger.

\[ \text{Table 22: Mean and Variance of difference} \]

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>-2.130144</td>
<td>5.803413</td>
</tr>
<tr>
<td>n=20</td>
<td>-2.563762</td>
<td>4.336111</td>
</tr>
<tr>
<td>n=50</td>
<td>-3.702098</td>
<td>4.050671</td>
</tr>
<tr>
<td>n=100</td>
<td>-4.630756</td>
<td>4.426337</td>
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</tbody>
</table>

\[ \text{Table 23: Normalized Mean and Variance of difference} \]

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
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<tbody>
<tr>
<td>n=10</td>
<td>-0.092979</td>
<td>0.253314</td>
</tr>
<tr>
<td>n=20</td>
<td>-0.111906</td>
<td>0.189268</td>
</tr>
<tr>
<td>n=50</td>
<td>-0.161594</td>
<td>0.176808</td>
</tr>
<tr>
<td>n=100</td>
<td>-0.202129</td>
<td>0.193206</td>
</tr>
</tbody>
</table>
7.5 Conditional Trading.

Apply the function (12) derived in chapter 5.1, I still use the conditional value $\rho = 0.55$. Trading with condition function (12) in $\sigma_T$.

![Graphs showing payoff of conditional trading at maturity $T$](image)

*Figure 28: payoff of conditional trading at maturity $T*

We note that the gap between two lines is smaller than using the simulation price in $\sigma_T$ in chapter 5.1. When the time step is increasing, the final return (red line) is getting close to the payoff (blue line).
When we decide whether to trade with a conditional value as chapter 5, we see that the histograms come out skewed right, and we can make positive difference. It is the same result as we got in chapter 5. However, the mean value is greater than the results in chapter 5, and also meets much larger SD value in the simulation of $\sigma_1$, which means that we would bear higher risk than we expected. It is not good for trading in the market.

If we use stochastic volatility $\sigma_5$ by the trading function (12), the results are reported below:
Figure 30: Payoff of conditional trading at maturity $T$

Figure 31: Histogram of the normalized difference
Table 24: Mean and Variance of difference

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>8.995692</td>
<td>18.844931</td>
</tr>
<tr>
<td>n=20</td>
<td>12.812923</td>
<td>22.890606</td>
</tr>
<tr>
<td>n=50</td>
<td>14.639621</td>
<td>23.625658</td>
</tr>
<tr>
<td>n=100</td>
<td>18.951279</td>
<td>30.461165</td>
</tr>
</tbody>
</table>

Table 25: Normalized Mean and Variance of difference

<table>
<thead>
<tr>
<th>Time step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>0.392655</td>
<td>0.822566</td>
</tr>
<tr>
<td>n=20</td>
<td>0.559274</td>
<td>0.999156</td>
</tr>
<tr>
<td>n=50</td>
<td>0.639008</td>
<td>1.031241</td>
</tr>
<tr>
<td>n=100</td>
<td>0.827208</td>
<td>1.329605</td>
</tr>
</tbody>
</table>

Figure 30 shows the final return against the payoff at time \( T \). The differences between two lines are much smaller than figure 28. The histogram figure 31 also gives right skew of the difference value, and the width of the figure shape is changing bigger and bigger as \( n \) increasing. The mean differences are smaller than trading in \( \sigma_T \). Even though the SD for corresponding time step is smaller than the value in table 22 and 23, these SDs are still very big compared with chapter 5.2 when we simulated the asset price in \( \sigma_T \).

7.6 Optimization of Trading Time.

In former sections we simulate the future price \( S \) in the market view \( \sigma_T \), all the model’s derivations are the same as former discussion, we have similar results as chapter 3 to 6. Moreover adding the transaction cost causing negative difference and the SD increase again as time steps increase. Therefore we can search for the optimal time step as we did in chapter 6. The method is the same as chapter 6.1, First let’s hedge with \( \sigma_T = 0.3 \).

Figure 32: \( E[V_T] - \gamma \cdot Var(V_T) \)

In figure 32, the left plot is computed for \( \gamma = 0.5 \), got \( n = 30 \); the right plot is \( \gamma = 1 \), got \( n = 60 \).

Next, I calculate the optimal time step with stochastic volatility \( \sigma_S \).
In figure 33, the left one is computed for $\gamma = 0.5$, received $n = 30$; the right plot is $\gamma = 1$, received $n = 60$.

With different volatilities choices, we find the optimal step in the interval $[30, 60]$.

7.7 Review.

As an alternative choice, I simulated the asset price $S$ in the market’s view $\sigma_I$ in this chapter. Compare with the results by using $\sigma_T$, this trading strategy will be a delta neutral problem without any transaction cost. We will meet zero difference between final difference and payoff, or even negative difference. And we will always receive negative difference after adding the transaction cost. However we can set a conditional value to verify the delta portfolio, and decide whether to trade. We would have met high risk since the SD value is big in this method. As a contrast choice, we would like to use $\sigma_S$ into the hedge strategy, leading better return to us. No matter how we choose the volatility to manage our investment, control the maturity time in the interval [30, 60] is a good choice.
Chapter 8

Conclusions.

In this paper, all the models are based on Black-Scholes formula, derived follow the delta hedging scheme in a complete financial market. Chapter 3 discusses the basic aspects of the trading model with price simulation in trader’s view \( \sigma_T \). If I trade without any cost, the price simulation in \( \sigma_T \) seems very good to use. Whereas adding the transaction fees, the final return sometimes would arrive at very bad values, and the variance goes to a big value. Then I consider to rebalance the portfolios in the market’s view \( \sigma_T \). I note that the final difference were close to replicate the trading in \( \sigma_T \). But after adding the transaction cost, I have gotten worse final return than hedging in \( \sigma_T \) since the distribution of the difference has greater width. Both distributions of difference between two curves in using \( \sigma_T \) and \( \sigma_I \) show left skew in the histograms. Chapter 4 presents a kind of stochastic volatility \( \sigma_S \) which has decreasing trend from \( \sigma_I \) to \( \sigma_T \). I use this volatility to rebalance the model with the price simulation in \( \sigma_T \). Theoretically speaking, the volatility moving follow the law of decreasing trend, the final return would be reasonable and stable than we using the volatility constant. I found the distribution of histograms using in \( \sigma_S \) show smaller left skewness when adding the transaction cost in the hedging strategy. However, no matter what the volatility I choose, I found increase the transaction cost would create bad return for every trading. Thus I come to consider about manage the trading strategy in chapter 5, setting a conditional value to judge whether to hedge in this time point. By using this idea, the trader can avoid some trades which he thinks it is bad, and save the transaction fees in account. Moreover, for the aim to control the risk of trading and maximize the expected return, I establish an optimization model in chapter 6. I found the time step around 50 units is the best choice for this delta hedging strategy under the limitations in this thesis. On the other hand, if I use the
market’s view \( \sigma_f \) to simulate the asset price, the problem changes to delta neutral whatever the volatility be used. Because the asset price estimated by using bigger volatility \( \sigma_f \), and the delta change is very small, then the final difference is close to zero. In chapter 7, if I buy and sell an option without using \( \sigma_f \), the histogram of final difference comes out left skew. And the distribution of difference in using \( \sigma_f \) is normal distributed.

To sum up, I found smaller volatility \( \sigma_T \) for using GBM to simulate the stock price would give better final return for the option trading contrast with using \( \sigma_f \) in this thesis. As a trader in a real financial market, he/she would like to bet on the lower volatility to buy delta hedge and make net profit. In order to be more precisely, considering the transaction cost in the hedging process, I found using the price simulation in \( \sigma_T \) and replicating the delta portfolio with the same volatility will be a good choice. In addition, on the matter of increasing the transaction fees, I can choose some conditional value to judge the delta portfolio and decide whether to buy and sell the asset at this time point. Furthermore, I can optimal the trading time steps by maximizing the final difference and minimizing the variance of this difference. Under the assumptions of this paper, I found \( n = 50 \) is a best choice for the trading with all the volatilities using in this paper.

There are certain amounts of truth in the given results above, but I still have a problem about regarding to the usage of simulation in trader’s view in the real market: many factors influence the price jumping problem, we cannot guarantee the volatility be constant or decreasing trend from the market view to our view as we thought all the time. As a matter of fact, the financial market is not complete: the asset price perhaps doesn’t move like GBM in a continuous time; this price can be influenced by government’s policies [Wolfers, J. and Zitzewitz E. (2005)], or nature cases out of control, etc. How to build a model more precisely and consider more factors in the rebalancing strategy will be further questions to study. Otherwise, the method developed in this paper shows a case when we forecasting the volatility be lower than the market’s view. In practice, the mean value of stochastic volatility would be close to the market’s implied volatility, and then we could use trader’s view to simulate the stock price, and predict the final return in more insurance way.

Generally speaking, we could try to use lower volatility than the market’s implied one to simulate the asset future price in real trading, but during the hedging process, we’d better keep one eye on the real asset price movement. If the simulation prices move opposite with the real price, we must to stop our hedging and think about whether we need to persist in this trading.
References


Djehiche, B. (2009), *Stochastic Calculus, An Introduction with Applications*, Stockholm, KTH, Division of Mathematical Statistics


Appendix

A.1 Trade in the model (7) with $\sigma_T = 0.4$

Final return against the payoff at time $T$ by using function (7) with $\sigma_T = 0.4$.

Figure A.1: Blue line is the payoff of $\pi = \max(S_T - K, 0)$; Red line is: $V_T = e^{rt}B_{T-1} + \Delta(t - 1, \sigma_T, S_{T-1}) \cdot S_T$. Both $S_T$ simulated with $\sigma_T = 0.4$. 
Mean and SD values and normalized values for $\sigma_T = 0.4$, using function (7).

<table>
<thead>
<tr>
<th>Time Step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>4.034974</td>
<td>4.124897</td>
</tr>
<tr>
<td>n=20</td>
<td>4.152110</td>
<td>3.085996</td>
</tr>
<tr>
<td>n=50</td>
<td>3.842640</td>
<td>1.938010</td>
</tr>
<tr>
<td>n=100</td>
<td>3.983373</td>
<td>1.338024</td>
</tr>
</tbody>
</table>

*Table A.1: Mean and SD of the difference $d = e^{\sigma_T B_{T-1} + \Delta(t-1, \sigma_T, S_{T-1} \cdot S_T - C_n)}$*

<table>
<thead>
<tr>
<th>Time Step</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=10</td>
<td>0.176123</td>
<td>0.180048</td>
</tr>
<tr>
<td>n=20</td>
<td>0.181236</td>
<td>0.134701</td>
</tr>
<tr>
<td>n=50</td>
<td>0.167728</td>
<td>0.084593</td>
</tr>
<tr>
<td>n=100</td>
<td>0.173871</td>
<td>0.058404</td>
</tr>
</tbody>
</table>

*Table A.2: Mean and SD of the normalized difference $d = \frac{e^{\sigma_T B_{T-1} + \Delta(t-1, \sigma_T, S_{T-1} \cdot S_T - C_n)}}{e^{\sigma_E \epsilon_0}}$*

Histogram for distribution of the difference, using function (8) with $\sigma_T = 0.4$.

*Figure A.2: Histogram of the normalized difference $d = \frac{e^{\sigma_T B_{T-1} + \Delta(t-1, \sigma_T, S_{T-1} \cdot S_T - C_n)}}{e^{\sigma_E \epsilon_0}}$ for $\sigma_T = 0.4$.\*
A. 2 Using transaction cost $\lambda = 3, \sigma_T = 0.3$

Final return against the payoff at time $T$ by using Function (9), choose $\lambda = 3$.

![Graphs showing final return against the payoff at time $T$](image)

*Figure A.3: Blue line is the payoff: $\pi = \max(S_T - K, 0)$; Red line is: $B_T = e^{rT}B_{T-1} + \Delta(t_{T-1}, \sigma_T, S_{T-1}) \cdot S_T - \lambda \cdot |\Delta(t_{T-1}, \sigma_T, S_{T-1})|$, for $\lambda = 3$."

Mean and SD values compared with no transaction cost for $\lambda = 3$

<table>
<thead>
<tr>
<th>Time step</th>
<th>Transaction cost</th>
<th>No transaction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>$n=10$</td>
<td>0.064300</td>
<td>0.162903</td>
</tr>
<tr>
<td>$n=20$</td>
<td>0.006958</td>
<td>0.135839</td>
</tr>
<tr>
<td>$n=50$</td>
<td>-0.104901</td>
<td>0.138999</td>
</tr>
<tr>
<td>$n=100$</td>
<td>-0.228608</td>
<td>0.168906</td>
</tr>
</tbody>
</table>

*Table A.3: Compare the normalized difference of mean and SD between [1.2] and [3.1], for $\lambda = 3$."

- 51 -
Histogram of normalized difference for $\lambda = 3$.

$$d = \frac{e^{\lambda B_{T-1} + \delta(t_{T-1}, \sigma_T, S_{T-1})} - e^{\delta(t_{T-1}, \sigma_T, S_{T-1})}}{e^{\lambda T_{\alpha}}}. $$

### A.3 Using transaction cost $\lambda = 3, \sigma_I = 0.3$

Results for simulation of $\sigma_I$, trading with transaction cost $\lambda = 3$.

<table>
<thead>
<tr>
<th>Time step</th>
<th>Transaction cost</th>
<th>No transaction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean SD</td>
<td>Mean SD</td>
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<tr>
<td>n=10</td>
<td>0.069615 0.163357</td>
<td>0.345780 0.170718</td>
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<td>n=20</td>
<td>0.006349 0.137463</td>
<td>0.347891 0.137434</td>
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<td>n=50</td>
<td>-0.103731 0.135680</td>
<td>0.351718 0.118094</td>
</tr>
<tr>
<td>n=100</td>
<td>-0.224445 0.168020</td>
<td>0.350340 0.107679</td>
</tr>
</tbody>
</table>

*Table A.4: Compare the normalized difference of mean and SD with Table 2.2, for $\lambda = 3$.***
Figure A.5: Blue line is the payoff: $\pi = \max(S_T - K, 0)$; Red line is:
$B_T = e^{rT}B_{T-1} + \Delta(t_{T-1}, \sigma_1, S_{T-1}) \cdot S_T - \lambda \cdot |\Delta(t_{T-1}, \sigma_1, S_{T-1})|$, for $\lambda = 3$.

Figure A.6: Histogram of the normalized difference for $\lambda = 3$

$$d = \frac{e^{rT}B_{T-1} + \Delta(t_{T-1}, \sigma_1, S_{T-1}) \cdot S_T - \lambda \cdot |\Delta(t_{T-1}, \sigma_1, S_{T-1})|}{e^{rT}C_0}$$