# A pricing and performance study on auto-callable structured products

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Stockholm, May 2012

## Abstract

We propose an algorithm to price and analyze the performance of auto-callable structured financial products. The algorithm contains Monte-Carlo simulations in order to reproduce, as probable as possible, a future product. This model is then compared to other, previously presented models. The different in-data parameters together with a time dependency study is then performed to evaluate what one might expect when investing in these products. Numerical results conclude that, the risks taken by the investor closely reflect the potential return for each product. When constructing these products for the near future, one must closely evaluate the demand from the investors i.e. evaluate the level of risk that the investors are willing to take.



**Keywords :** Structured products, Auto-callable products, Monte-Carlo methods, pricing algorithms, time series, performance analysis.

# Acknowledgements

I want to thank my supervisors Professor Boualem Djehiche at KTH Royal Institute of Technology, Professor Bernard Lapeyre at Ecole des Ponts ParisTech and Alexander Frick at SIP Nordic Fondkommission AB for their great guidance and the many valuable discussions during this master thesis. Many thanks also to Tor Nordqvist at Handelsbanken for valuable comments.

Also I want to thank all staff at SIP Nordic Fondkommission AB, for making me feel very welcome, great great team.

Stockholm, May 2012

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# Table of contents

## Contents

# List of Tables

# List of Figures

# 1 Introduction

SIP Nordic Fondkommission is a company selling structured financial products for institutional and private investors with operations in the Nordic region, the UK and the US.

With the Royal Bank of Scotland as their sole provider of these structured products, there have been an increasing interest in being able to benchmark the different products with internal pricing programs as well as conducting a performance study which includes how their products perform in general and also a research on how the different parameters associated with these products affects the price and performance.

Three methods have been analyzed and tested in order to model the algorithm and conduct the performance study mentioned above, these models are:

- The Flexible PDE method
- The one-step survival and GHK Importance Sampling method
- The Monte-Carlo Path Generation With Stochastic Parameters method (Generalized method)

## Outline

Starting with a product description, presenting structured financial products and also the auto-callable structured products, we introduce the products in general and how they can offer a return for the investor.

To understand the methods that have been used and assumptions that have been made, the different models for pricing these products will be theoretically presented.

Together with this theoretical presentation, a theoretical study is conducted on how one can extend the model, so that it is possible to further adjust the scenario simulations, this is done in order to reproduce a model, as similar as possible, to what one might expect from the real product.

After this theoretical study, we will present the different major products offered by SIP Nordic Fondkommission AB, Stockholm. Their special features and properties will here be explained, as they will be further examined in the last main part in this thesis, the numerical results.

In the numerical results part, we will study the effectiveness of the different models suggested. We will also study how well the prices converge to the products offered by SIP Nordic Fondkommission AB and how an investor can expect the possible returns, given certain risks.

One of the predetermined questions was how the investor can increase the return by selling the product before maturity, given certain scenarios. To approach this question, a time dependency study will be conducted where the different parameters have been varied.

Finally a conclusion and possible further studies will be presented for the reader who whishes to continue in this field.

# 2 Structured products

Structured products, also called principle protected notes (PPN), have for some time been an increasingly more popular form of investment for large investors and also private investors due to their potential upside and limited downside. Nowadays with an relatively volatile world market, many investors, institutional and private, accept a less significant yield in exchange for a more or less predetermined measurable risk.

## 2.1 "Normal" structured products

A "normal" structured product is an investment form where the variations are limited only by the fantasy of the issuer and the current demands from the investors.

The standard product, the principle protected note, is an investment form, which is a combination of mainly two products.
The first item is an zero-coupon bond, $Z_t$, also called a $T$-bond, which is a contract that guarantees the holder the value 1 to be paid on the date $T$. The price at time $t$ of such bond is often denoted $p(t, T)$. The second item in this product will be the an option or an set of options of some underlying assets. (See Björk [2] for more details)

Usually, the difference between the bond price and the face value of the bond is the amount invested in the risky assets, that way, the complete product will be capital guaranteed seen in the long perspective as the investor will, in the worst-case-scenario, receive the notational amount i.e. the face value of the bond.

This above mentioned ratio of the amount invested in the risky part and the difference between the bond price and the face value is known as *the participation rate*. It is an important part of the product description as this is an adjustable parameter where the issuer can offer a more competitive product by increasing, and also increase margins by decreasing, this ratio.

A higher participation rate demands an increased investment in the risky asset, which will not be covered by the capital guarantee, but may result in a higher yield, if markets develop as anticipated by the buying investor. See Figure 1 for a detailed description of the "normal" or "classic" structured product.

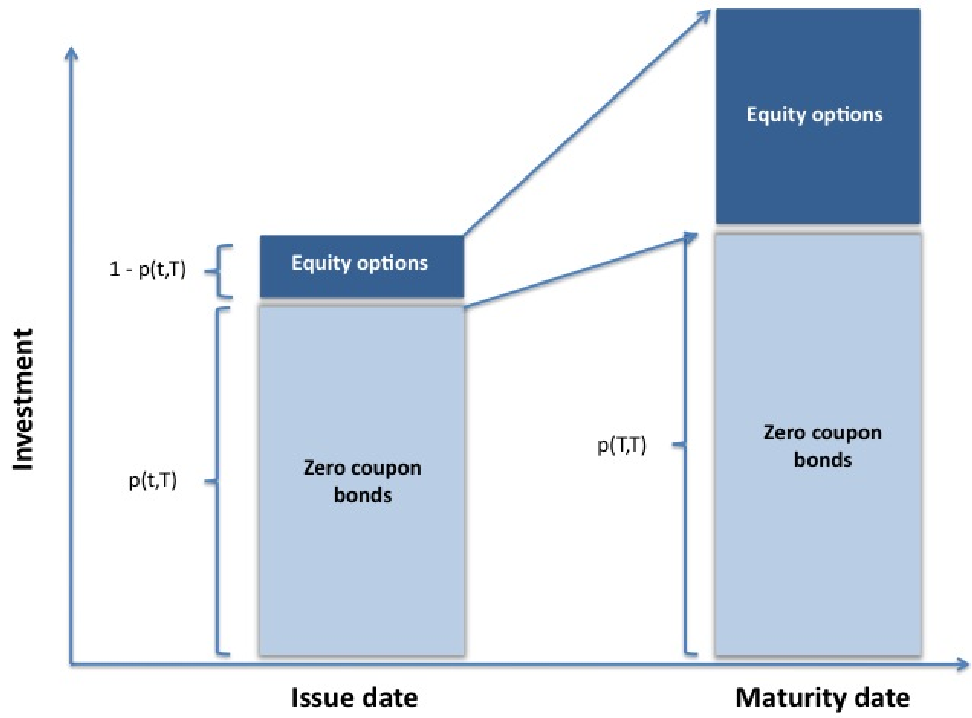


Figure 1: "Normal" principle protected structured product with bonds and options

We can see how the zero-coupon bond makes the product capital protected. The difference between the capital invested and the zero-coupon bonds, *the equity options part*, is then the possible profit, this part will then add value to the product that will be reimbursed at maturity. Figure 1 illustrates the scenario with 100 % participation rate assuming investment is 1, also equal to $p(T, T)$.

## 2.2 Auto-callable structured products

Since its first issue in the U.S. by BNP Paribas in August, 2003 (see [6]), the Auto-callable structured products has gained popularity and they are today an often seen investment vehicle in any investors portfolio.

An auto-callable structured product may pay out a predetermined fixed accumulative coupon plus notational value based on the evolution of the underlying assets, which may be one, or several, known as Uni- or Multivariate Auto-callable options. There exists several forms of these auto-callable products and we will come back to some of these, more popular variations later on in this thesis.

The potential payouts on any of the predetermined observation dates are determined using a reference index to which one compare the underlying assets and, assuming they are above this level, also known as trigger index, a coupon plus nominal value is reimbursed to the investor and all future cash-flows are cancelled. As the coupon normally accumulate until the end of the product, this "standard" auto-callable structure is often called a Snowball Effect and Worst-of put. Bouzoubaa and Osserian precisely present a more detailed discussion on this subject (see [4]).

The product may be seen as a mixture of a zero-coupon bond, but with a stochastic maturity, together with these auto-callable options generating the accumulative coupons and finally a sold sell-option. This composition makes them somewhat capital protected in the sense that, in order for the investor to loose the entire investment, the worst performing underlying has to fall 100 % of its initial value. Of course, as this is theoretically possible, these products are not categorized as capital protected unless the risk barrier that we can see in Figure 2, is set to 0 %.

An obvious upside is the fact that the product, as the name is suggesting, might be auto-called in advance, meaning that the investor will receive his or hers nominal value plus an accumulative coupon prior to the maturity.
These products offer the investor an attractive mitigated level of risk while having a relatively high return. The downside is the sold sell option, in case of an drastic loss for one of the assets, where the asset at the day of maturity, is traded below the predetermined risk barrier, the payout will be as if the investor initially invested the notational amount in the worst performing asset.

In this thesis, we will be covering the more commonly used products, which are the auto-called structured products with discrete call dates together with some modern mutations that have been innovated by banks to meet investors demand.

As mentioned before, the more popular traded products today are the multi-asset worst-of auto-callables. As the name indicates, these products reimburse the investor depending on the evolution of the different underlying assets and more specifically the worst-off underlying asset. As described in Figure 2, the investor can be auto-called if the worst performing asset is above a predetermined level, $C$, at one of the predetermined discrete call dates.

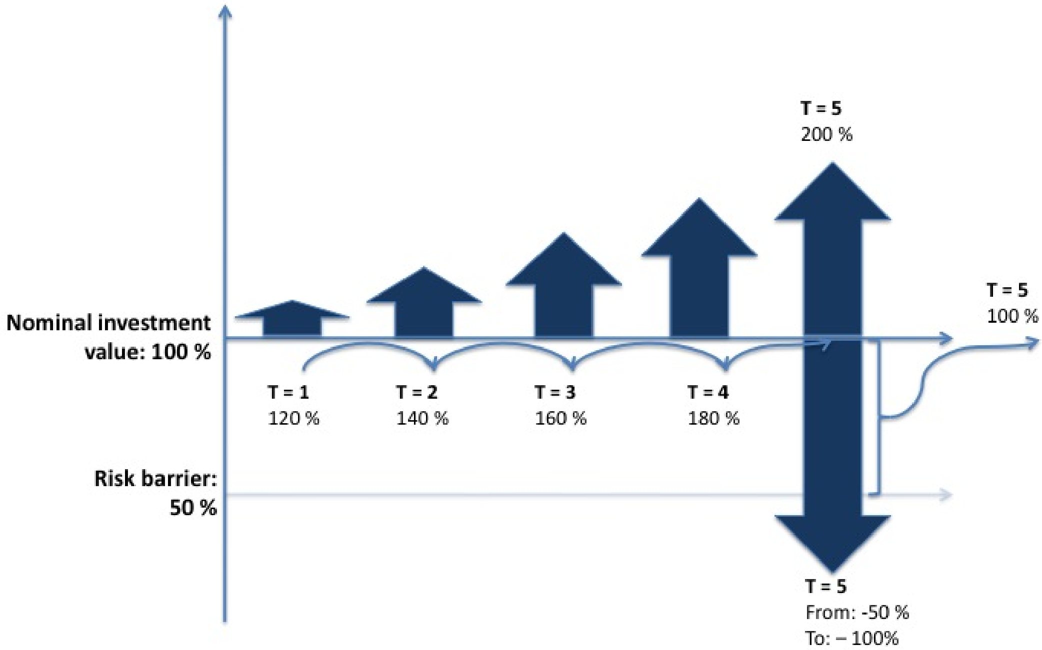


Figure 2: An auto-callable structured product.

Shown in Figure 2 above, the product continues as long as any of the underlying assets are being traded below the reference value, these values are most often their respective initial value at the issue date.

Either the investor will receive any of the up arrows above or, the investor will, at year $T = 5$, receive between 0 and the whole investment sum. This value can be either the nominal investment if the assets are traded between the risk barrier and initial values or, equivalent to the loss of the worst-performing asset.

# Pricing models and their underlying theory basics

These types of products are similar to a basket of exotic options, namely rainbow options or digital options together with some special features.

These products do not have a simple closed form solution to their payout function due to their complexity, possible early maturity and path dependency. Therefore, a different pricing approach must be used compared to the more frequently traded options.

Following are the most applicable methods that, at today's date and to the authors knowledge, have been developed. This together with a more generalized method that has been constructed solely for this product pricing mission as requested by the employer.

## 3 The Flexible PDE method

It was recently presented by Deng, Mallett and McCann that one can use a flexible Partial Differential Equation (PDE) to sole discrete dates auto-called products using the finite difference method. (See [6])

This method proved to be useful but only with one single underlying asset. This general PDE could then be solved by rewriting the Black-Scholes formula into an ordinary heat differential equation.

This is a very effective method as it can be solved using already well-known methods to determine an expected price for such an product without having to simulate any scenarios.

Though, as this type of products with one single underlying asset only represent a very small part of the auto-callable market, and not offered at all by SIP Nordic, it will be ignored in this thesis as it will fill no or little purpose for the desired end results.

## 4 The one-step survival and GHK Importance Sampling method

Alm, Harrach, Harrach and Keller (see [1]) together with the work done by Glasserman and Staum (see [9]) showed a comparable approach where Monte-Carlo simulations were used to calculate prices for auto-callable products.

The main ambition here was to obtain prices with as few Monte-Carlo paths as possible, making the model very effective.

What has been presented is a modern way of pricing auto-callable structures which is very effective due to its way of calculating probabilities out of paths generated as Eq. (5) below.

Much in line with the assumptions that will be further presented below as the generalized method will be described, the model simulates scenarios based on two underlying assets with payout function as described in Eq. (29).

To find an expected current value of the possible payout, the expected discounted payoff at time $t$ is calculated using

$$PV_t = \mathbb{E}[Q(S_1^1, S_1^2, ..., S_2^1, ...)]. \tag{1}$$

Where $Q$ denotes the discounted payoff of a multivariate auto-callable option.

As often when using Monte-Carlo methods to simulate scenarios to reproduce a certain product, we approximate the present value as an average of the different scenarios as Eq. (2) below

$$PV_t = \frac{1}{N}\sum_{n=1}^{N} Q(S_1^1, S_1^2, ..., S_2^1, ...)]. \tag{2}$$

What distinguish differs in this one-step survival and GHK Importance Sampling method and the more generalized model described below is the method of how to relate the value of each possible payout and also the Monte-Carlo path generation.

As we might expect, it can be inefficient to generate paths for each asset $S^1$ and $S^2$ if the structure is auto-called before maturity. Therefore, if this can be avoided, only for scenarios when the product has not been auto-called, the paths will be generated until maturity of the product.

As seen in Eq. (5), these paths are created with a stochastic part, possibly generated using samples from the standard normal distribution so that the variance equals $\sigma\sqrt{dt}Z_j$ and, if $u_j$ is drawn from a uniform distribution over [0,1], then $z_j$ is calculated using the inverse cumulative standard normal distribution of $u_j$.

Now, applying the one-step survival technique (see [9]) improves the previously mentioned Monte Carlo simulation by sampling only paths which stays below the auto-call barrier for all observation dates.

It should be remembered that a more thorough walk-through of the underlying theory while creating a model from the bottom up will be presented below as the generalized model is described.

In order to sample non-triggering paths, we calculate the probability that the assets $S^1$ and $S^2$, generated both by Eq. (5) below, and with correlation matrix $\rho$ as Eq. (33), stays below this trigger level $S_{ref}$, in the next time step, more clearly formulated as

$$P\left\{max\left\{\frac{S^1_{j+1}}{S^2_{ref}}, \frac{S^1_{j+1}}{S^2_{ref}}\right\} < B \mid (S^1_j, S^2_j) = (s^1_j, s^2_j)\right\} = \Phi_\rho(Ref^1_j, Ref^2_j). \quad (3)$$

Where $\Phi_\rho(C^1_j, C^2_j)$ is the bivariate cumulative normal distribution with correlation $\rho$, we will come back to this distribution further on.

To more easily explain the non-trigger one-step survival technique while simulating the paths presented by Glasserman and Staum, we quickly look at the case where we have a single underlying asset and where the product is only dependant of one path.

Similarly as to what we later will do with the multivariate care, we sample, instead from the uniform distribution over [0,1], we limit the possible sampling vector by multiplying the uniform distribution with the above mentioned probability as Eq. (3), for not hitting the trigger.

In accordance with the one-step survival technique, we have that this probability is intuitively calculated as

$$p_j = \Phi\left\{\frac{ln(\frac{S_{ref}}{s_j}) - (\mu - \frac{\sigma^2}{2})(t_{j+1} - t_j)}{\sigma\sqrt{t_{j+1} - t_j}}\right\}. \quad (4)$$

Using that

$$e^{(\mu - \frac{\sigma^2}{2})(t_{j+1} - t_j)) + \sigma Z_j \sqrt{t_{j+1} - t_j}}. \quad (5)$$

One can see how this probability in Eq. (4) is derived from Eq. (5).

We will come back to the build-up of this path generation as we present the more generalized model for pricing path dependent products.

To intuitively understand the path construction, Figure 3 below shows how these paths are stochastically generated and might give an better understanding of why this method is so useful

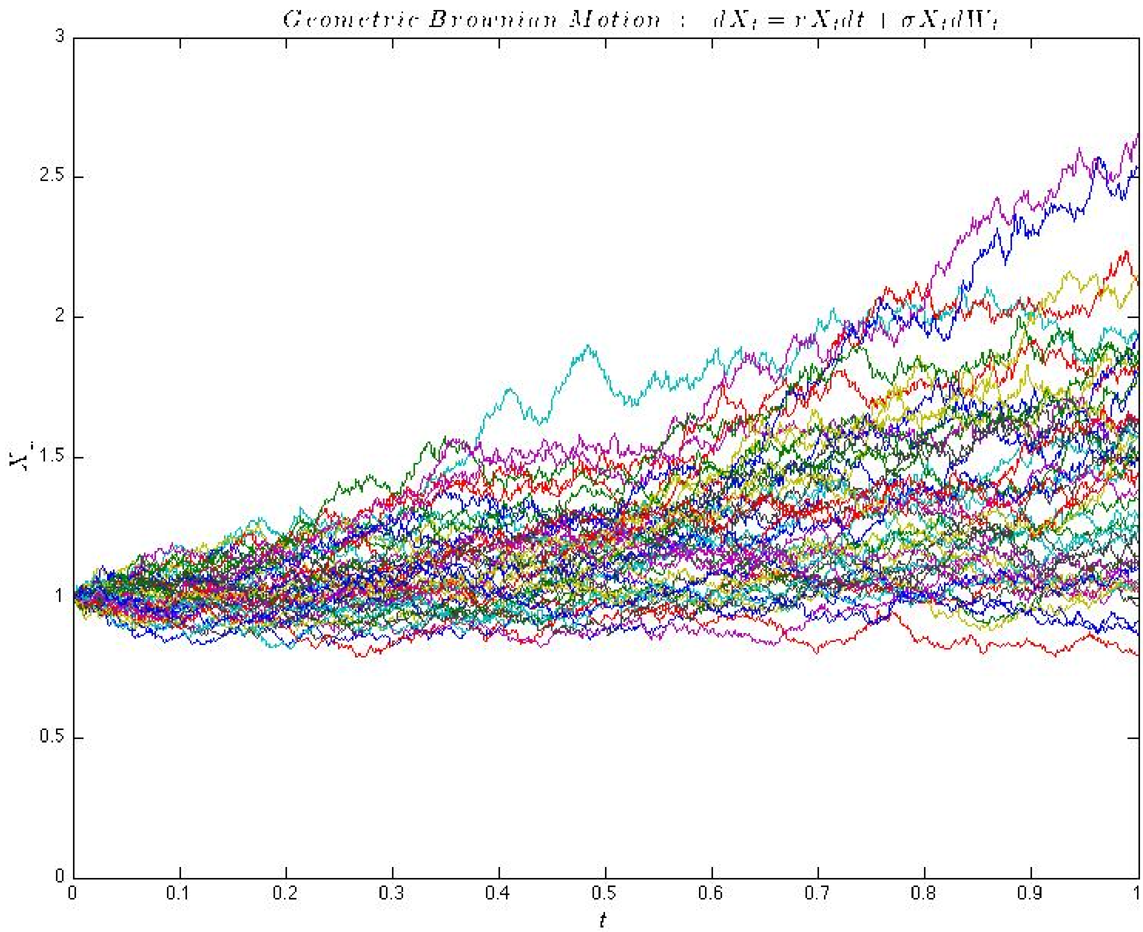


Figure 3: GBM paths generated by MatLab

Now, instead of sampling from the uniform distribution, we limit the model so that we sample only from the vector [0,$p_j$] i.e. only when the path, later paths when using several underlyings, stays below the trigger level at the different observation dates $t_j$.

Just as before with the multivariate case, we want to sample $z_j$ only when path is below trigger level so therefore, as

$$z_j = \Phi^{-1}(p_j u_j). \tag{6}$$

With Eq. (6) and Eq. (4) above, we see that the sampling from $Z_j$ is done from a truncated univariate standard normal distribution

$$Z_j < \frac{ln(\frac{S_{ref}}{s_j}) - (\mu - \frac{\sigma^2}{2})(t_{j+1} - t_j)}{\sigma\sqrt{t_{j+1} - t_j}}. \tag{7}$$

In order not to have a bias result, we have to balance for each time step as we correct for the missing barrier hits, this can more clearly be understood in Eq. (13). Now we will study the same situation but for the multivariate case.

Similarly to the univariate case in Eq. (7) above, we sample for each asset in the non-triggering area using, for two underlying assets

$$Z_j^k < Ref_j^k = \frac{ln(\frac{S_{ref^k}}{s_j^k}) - (\mu_k - \frac{\sigma_k^2}{2})(t_{j+1} - t_j)}{\sigma_k\sqrt{t_{j+1} - t_j}}, \quad k = 1, 2. \tag{8}$$

Now we can calculate the desired probability stated in Eq. (3) above using this technique.

One problem with the equation above is that this method forces us to evaluate the bivariate cumulative normal distribution for every observation time and every sample, which will be very time consuming. It is now that the GHK Importance Sampling part of the model name shows its importance.

We can now sample one dimension after the other, for an exact derivation of how this sampling is possible we refer to the reference. (See [1])

Presented in a shorter way, we use the standard transformation to uncorrelated normal distributions

$$z^1 = y^1 \ , \ z^2 = \rho y^1 + \sqrt{1 - \rho^2} y^2. \tag{9}$$

An additional problem now is the fact that, as we have an additional condition on the second sampling due to the fact that the first sampling has to stay below the trigger level, the truncation condition for $y^2$ now becomes active so the model now lacks the stability with respect to differentiation.

In order to obtain a Lipschitz continuous parameterization for the second sample with respect to the first one, we rotate the parameters so that we have

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}. \tag{10}$$

Due to the nature of the transformation, the sampling of the second underlying will depend smoothly on the sample of the first one. Hence, this rotation of the system is implemented. This will rotate the sampled values to ensure that the bound on the second sampled underlying does depend Lipschitz continuously on the first one. Since this rotation does not affect the function of the Brownian motion, it is a possible solution to the sampling-dependency.
We have now presented the problem so that we can generate paths and calculate probabilities, as done in the following part.

Using the standard transformation and rotation we have that, together with Eq. (3), the survival condition where $S^1$ and $S^2$ are strictly below the trigger level becomes

$$x^2 < max\left\{\frac{C_1^1 - x^1 cos\alpha}{sin\alpha}, \frac{C_1^2 - \rho x^1 cos\alpha + \sqrt{1-\rho^2}x^1 sin\alpha}{\rho sin\alpha + \sqrt{1-\rho^2}cos\alpha}\right\} = C_1. \tag{11}$$

By taking half the angle between the two bounding lines that is the original problem and the reason for the transformation and rotation (for further reading this is clearly illustrated in the reference see [1]) we find $\alpha$ to be

$$\alpha = \frac{1}{2}\left\{\frac{\pi}{2} - arctan\left\{-\frac{\rho}{\sqrt{1-\rho^2}}\right\}\right\}. \tag{12}$$

With correlation $\rho$. Now that we have presented the different sampling steps, the price is calculated as Eq. (13).

As mentioned before, to not have a bias result and in line with the one-step survival strategy, we have that, for each Monte-Carlo simulation

$$Q(s_1, ..., s_m) = \tag{13}$$
$$L_m e^{-r(t_m - t_0)} q\left\{\frac{s_m^1}{S_{ref}^1}, \frac{s_m^2}{S_{ref}^2}, \dots \frac{s_m^n}{S_{ref}^n}\right\} + \sum_{j=0}^{m-1} L_j(1-p_j)e^{-r(t_{j+1}-t_0)}Q_{j+1}.$$

Where $L_j =_{i=o}^{j-1} p_j$ is the important multiplication so that we do not have an bias result as the probability must sum up to 1 for all possible events.
The functions $q$ and $Q$ defines how the investor will be reimbursed either in case the product is auto-called or, for the second part of the function, if the product goes on until maturity and any of the underlying assets are below the trigger level and possibly also below the risk-level.

Due to the nature of the algorithm, the pricing function is limited to two underlyings as stated before but, for the sake of clarity, the equation is written as if we have an arbitrary number of underlyings as in the generalized method presented below.

Finally when we have priced the product given the different payoffs that was distributed for one certain simulation, we can calculate the average price of the product, this can be done using Eq. (2) above.

Numerical results of the effectiveness and comparisons with the next method will be presented in the numerical results chapter.

# 5 The Monte-Carlo Path Generation With Stochastic Parameters method (Generalized method)

As mentioned above, these auto-callable products differs from other structured products due to the non-existence of solvable closed-form solutions. (See [5]) With an increased computer capacity, the Monte-Carlo simulation approach has become more and more popular while pricing these products as it, fairly intuitively, reflect the most probable end result that the creator can imagine and therefore, he/she can price thereafter.

In order to compute prices for these auto-callable products with discontinuous payouts, certain underlying theories will be presented. (See [7])

Most importantly, we will base the pricing on the pricing function that, in our case, will be a mathematical expression of the future possible payouts that the investor can expect. This function is presented and explained in Eq. (29) below. It should be mentioned that this pricing function can be varied as to reflect the construction of the product, some of the most important variations will be presented in chapter 7 below.

Previous introductions of for example how we generate the paths representing the underlying assets will now be further analyzed as they serve as important keystones in the modeling of a pricing algorithm.

Starting with the most important component in this analysis, together with the pricing function, we will initially study the generation of the underlying assets, as they will form the base, together with the randomness built in the Monte-Carlo methods. This is done so that we can well understand the build-up of the model. We will also study how and why we can assume different hypothesis and use different tools that will form the important keystones. (See [2])

In accordance with Björk (see [2]), we consider an asset something that can be represented by a stochastic process $X$, where its local dynamics can be approximated using a time-series equation as described in Eq. (14) below

$$X(t + \Delta t) - X(t) = \mu(t, X(t))\Delta t + \sigma(t, X(t))Z(t). \tag{14}$$

Where $\mu(t, X(t))$ is the drift term and $\sigma(t, X(t))$ the diffusion term, this is also the stochastic part in the equation.

With $\Delta t \to 0$, and integrate the Eq. (14) , we obtain

$$X(t) = a + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s). \tag{15}$$

The first part of Eq. (15) above can be seen as an ordinary Riemann integral but then, for the second part, we introduce the well-known Itô calculus. This is done so that we can develop a corresponding differential calculus for the non-solvable integral. As more commonly written in financial literature, we will describe the equation as follows

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \;\; X(0) = 0. \tag{16}$$

This will form a fundamental base as most of the assets are described with this logic behavior. In order to develop a usable calculus, we will define that $Z = f(t, X(t))$. $Z(t)$ then has a stochastic differential that is given by the Itô's single dimensional formula fully stated in Eq. (17) below (See [17])

$$\begin{gathered} df(t, X(t)) = \\ \left\{ \frac{\partial f}{\partial t}(t, X(t)) + \mu(t)\frac{\partial f}{\partial x}(t, X(t)) + \frac{1}{2}\sigma(t)^2 \frac{\partial^2 f}{\partial t^2}(t, X(t)) \right\} dt + \sigma(t)\frac{\partial f}{\partial x}(t, X(t))dW(t). \end{gathered} \tag{17}$$

The implementation of the different path-dependent assets will play an important role in the modeling. In line with standard financial modeling, the Geometric Brownian Motion (GBM) will be used for this very purpose. The GBM is a rare case as it is a solution to the stochastic differential Eq. (16) above. It is because of this that it is often used while modeling assets paths.

The exact definition of the Geometric Brownian Motion is stated as a more simple written form in Eq. (18) below

$$dX_t = \mu X_t dt + \sigma X_t dW_t. \tag{18}$$

As introduced before, the $W$ is here simply the Wiener process generating the randomness in the equation. Shown by Björk, it seems logic to assume that $X$ is a solution to $Z = ln(X)$.

The use of the Itô formula above then gives us

$$dZ = \frac{1}{X}dX + \frac{1}{2}\left\{-\frac{1}{X^2}\right\}(dX)^2. \tag{19}$$

After rewriting the different parts and using that $Z = ln(X)$ , we have that

$$X_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{20}$$

This is the standard model we will use in order to generate the different assets that will make up the base for each scenario while structuring the products.

We have rejected the possibility of calculating a relevant price using a closed-form solution approach, this would not be possible as mentioned before. Instead, we will have to identify the payout function and find the discounted price or value today, by determining how each possible payout add value, or decrease when any asset ends up below the risk barrier.

To do so, we will also look at the possible add-ons that one might expect, in order to make the model somewhat more predictive.

## 5.1 Stochastic drift and/or volatility

One natural attribute that one could implement in the model would be to make the drifts and volatilities for the underlying assets stochastic.

Staring with a model to predict the future interest rate that can be used to calculate a plausible discount value for each year, we introduce the Vasiček model.

### 5.1.1 The Vasiček model

This short rate model is derived using the affine term structure and the term structure for the Vasiček model. If the reader wishes a more profound derivation of the affine term structure and how to reach the end bond pricing equation, a precise derivation has been presented in the indicated relevant reference. (See [2])

Shortly presented, we have that the term structure and dynamics of the rate is given by

$$dr = (b - ar)dt + \sigma dW. \tag{21}$$

With, as we have seen before, a drift and a stochastic part.

If we let the price of a $T$-bond has a form as Eq. (22)

$$p(t, T) = F(t, r(t); T). \tag{22}$$

where $F$ is a smooth function of three real variables and has the form as Eq. (23)

$$F(t, r(t); T) = e^{A(t,T)-B(t,T)r}. \tag{23}$$

Comparing with the system that solves the affine term structure, we see that for the Vasiĉek model

$$\begin{cases} B_t(t,T) - aB(t,T) = -1 \\ B(T,T) = 0 \end{cases}$$

and

$$\begin{cases} A_t(t,T) = bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0 \end{cases}$$

Solving for $A$ and $B$, we have that

$$p(t,T) = e^{A(t,T)-B(t,T)r}. \tag{24}$$

where

$$\begin{cases} A(t,T) = \dfrac{(B(t,T) - T + t)(ab - \frac{1}{2}\sigma^2}{a^2} - \dfrac{\sigma^2 B^2(t,T)}{4a} \\ B(t,T) = \dfrac{1}{a}(1 - e^{-a(T-t)} \end{cases} \tag{25}$$

The final model will then generate predictions of the forthcoming rate with $b$ being the long-term mean level and $a$ is the reversion. The results of the use of this model will be presented in the numerical results chapter.

### 5.1.2 The Heston Model

Another approach to once more broaden the possible variations of the model has been to implement a stochastic volatility. (See [10])

In contrast to the ubiquitous Black-Scholes-Merton model, the following model proposed by Stephen L. Heston takes into account a stochastic volatility.

The stochastic paths are given by Eq. (26) below

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1. \tag{26}$$

With the volatility given by

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2. \tag{27}$$

And finally the correlation between these stochastic processes are given by

$$dW_t^1 dW_t^2 = \rho dt. \tag{28}$$

$S_t$ denotes the price, $V_t$ the volatility process, $W_t^1, W_t^2$ are two correlated Brownian motion processes. $\kappa$ denotes the rate of reversion, $\mu$ the expected return on stock, $\theta$ the long run variance and $\sigma$, in this model, the volatility of volatility.

Also this method will be further evaluated, as we will study the possible implementation when pricing products later on in this thesis.

## 5.2 Pricing function using Monte-Carlo path generation

Now leaving the description of the theory behind the underlying building blocks for the creation of this algorithm, we will instead redirect the focus on how to actually implement a method so that we can price these products with an effective and generalized approach.

Similar to what was described by Fries and Joshi (See [7]) and also Alm, Harrach, Harrach and Keller (See [1]), a pricing model that describes the payout of an auto-callable product would be based on the possible payouts of the specific product.

Given a tenor structure where $T_1 < . . . < T_{n+1}$, we assume we are looking at a multivariate auto-callable option with $n$ underlying assets and with the, from an investors perspective of the discounted value, following risk neutral payout structure

$$Q((S_1^1), ...(S_1^n)) = \begin{cases} e^{-r(t_j - t_0)} Q_j \; if \; M_i < B \leq M_j \; \forall i < j \\ e^{-r(t_m - t_0)} q \left( \dfrac{S_m^1}{S_{ref}^1}, ... \dfrac{S_m^n}{S_{ref}^n} \right) \; if \; M_j < B \; \forall \; j = 1, ..., m \end{cases} \tag{29}$$

Where $M_j$ indicates the minimum value of the relative loss between the different underlying compared to their initial reference value.
$Q_j$ is here the predetermined payoff if every asset is above the reference value, if, at maturity date, some of the assets are still below this trigger-level, then the investor is reimbursed depending on the final value of the different assets, we call this function $q$ and comes back to this function in the model description.

With this defining how the initial investment can increase or not, depending on the trajectories of the different assets, we can use Monte-Carlo simulations to describe how one would expect the future cash flow, given certain assumptions based on historical data.

This Monte-Carlo method serves to determine different probabilities of when we could expect the product to be auto-called.

We can see in Figure 4 and Figure 5 how the paths converges towards the drift for each path, it is by using these path generations that we can estimate reasonable prices for different set ups of products.

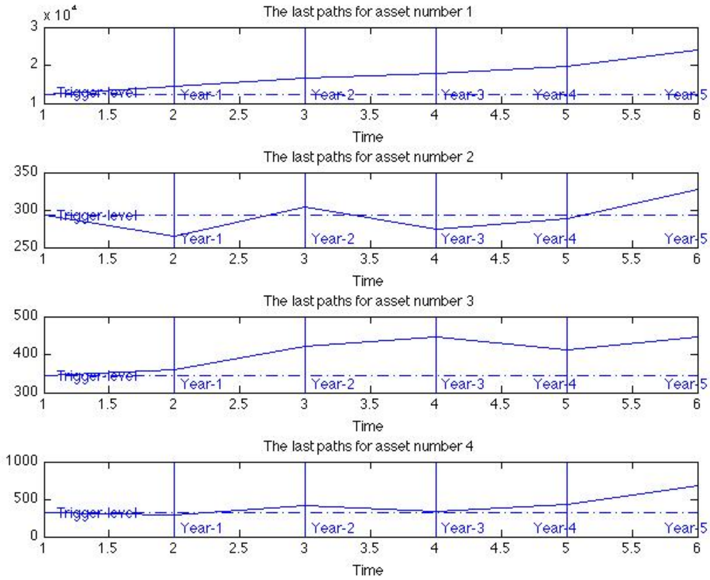


Figure 4: The paths for 1 simulation with 4 underlying assets

And then we have the same presentation but with 100 000 simulations

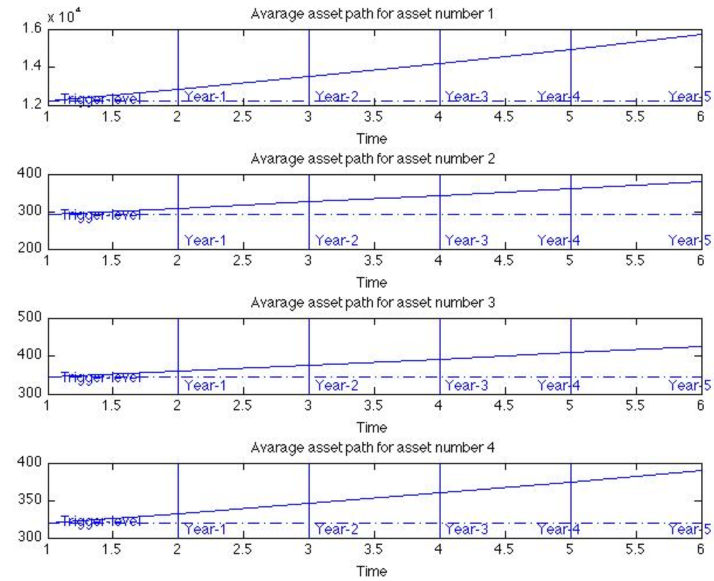


Figure 5: The average paths for 100 000 simulation with 4 underlyings'

To better understand the methodology we can see that, by using this algorithm, we evaluate what happened, or how the investor was reimbursed, for each of the different simulations.

As indicated by the bar graph in Figure 6 below, we see how the 100 000 scenarios are spread out as they were auto-called after 1,2,3,4,5 years indicated on the X-axis. The sixth column shows the number of simulations that ended up below the risk barrier set at 50 %.

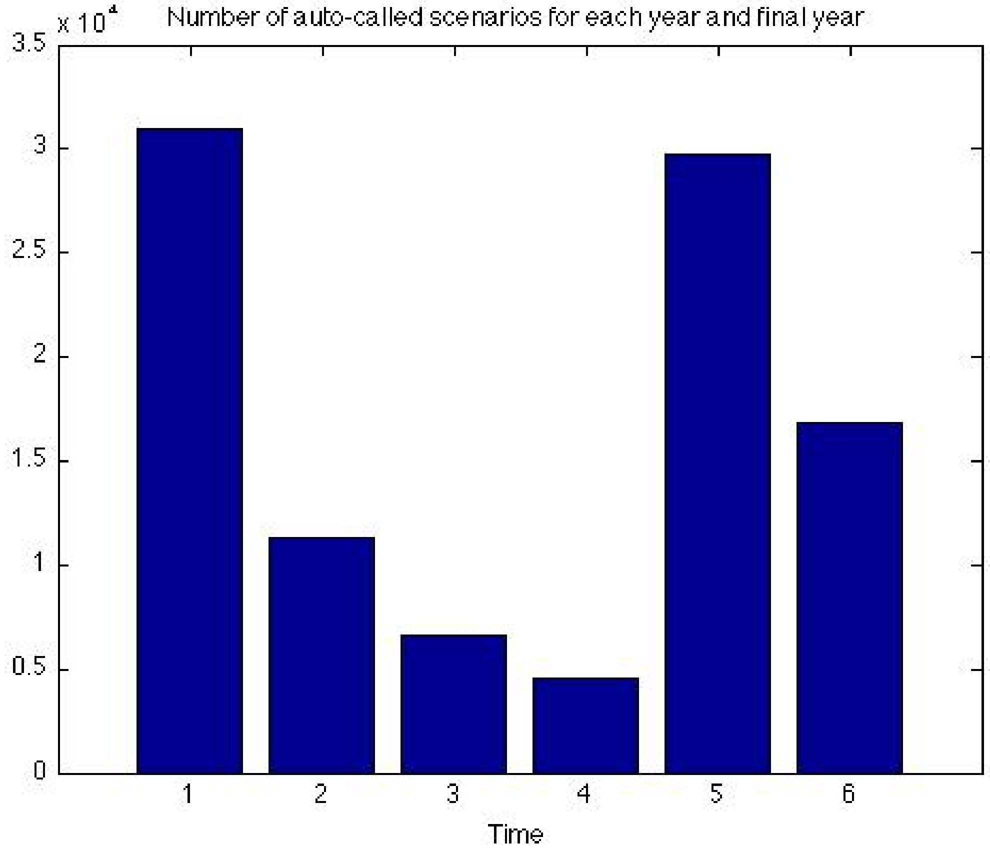


Figure 6: Bar graph indicating when/if each simulation got auto-called in advance

This method is also in line with what was suggested by Bouzoubaa and Osseuran (see [4]) where they present that one can price these Snowball and Worst-of put products by considering them as auto-callable digitals together with an sold put option.
It is then suggested that each price is evaluated by calculating the probability and how much this payout will give back to the investor, finally the price is discounted so that we get today's value.

This is also very much how the general method below is constructed to find the price for any auto-callable structure.

It is now tempting to increase the time steps for each simulation, making the model somewhat more representable a real asset assuming more than 250 trading days with several ups and downs each day as a result of market demands. This assumption has been proved unnecessary as this type of path generation only results in more computation time, without improved results.

When pricing a path-dependant product using Monte-Carlo simulation, the ambition is to reproduce a model that represents a scenario, that is as similar as possible to what we may expect from the real product.

This means that the model not only will be a fine-tuning of well-known mathematical instruments, but also, an adaptation for the modeler as how to best predict the future evolution of the underlying assets.

Before more precisely describing the algorithm, it should be mentioned that a model similar to what Fries and Joshi (see [7]) presented has been considered due to its effectiveness but in order to keep the model as general and adaptable as possible, it was logic to create the product in question and reproduce the scenario several times to then study the result and evaluate an reasonable price.

## 5.3 Algorithm

As for the algorithm itself, we start by defining given in-data parameters such as frequency and step-times, initial stock price, volatility, correlation between assets, payout function and expected drifts. (See [5])

These parameters are information that the sell-side present to the investor so that he/she can take the final decision whether or not to invest in the product. This is also the information we use trying to reproduce the product in order to price it.

This is uniquely the most challenging problem for the modeler as he/she has to take into consideration that the in-data parameters have to reflect the most likely scenario for the upcoming 5 years.

Then, with $n$ repetitions depending on the accuracy, we will regenerate the following procedure.

We generate paths for each asset according to Eq. (20), depending on the set up of the model, we either assume a constant drift and volatility or they could also be stochastic as seen above. Numerical results show that it is also very important that the model takes in to account the correct correlation between the different assets.

As described in the numerical results section, we will see that the difference between constant and stochastic drift will completely change the prediction of the model. Using the methods described above, we have generated the drift-and the volatility rates.

Shown by Xiao (see [16]), we generate the correlated paths by using that the stochastic part in the equation is given by the Cholesky factorized correlation matrix times the variance.
As we saw in the one-step survival and GHK Importance Sampling method, this variance equals $\sigma\sqrt{dt}Z_j$, with the sample from the standard normal distribution.

Cholesky factorization decomposes a symmetric matrix into a lower and an upper triangular matrix where triangular matrix with positive diagonal elements as in Eq. (30) below, L is here the Cholesky triangle

$$\rho = \begin{pmatrix} \rho_{1,1} & \cdots & \rho_{1,n} \\ \vdots & \ddots & \vdots \\ \rho_{n,1} & \cdots & \rho_{n,n} \end{pmatrix} \tag{30}$$
$$= \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} = LL^T.$$

Where $\rho_{ij}$=$\rho_{ji}$. The elements in the matrix are then calculated as

$$a_{jj} = \sqrt{\rho_{jj} - \sum_{k=1}^{j-1} a_{jk}^2}. \tag{31}$$

and

$$a_{ij} = \frac{(\rho_{ij} - \sum_{k=1}^{j-1} a_{ik}a_{jk})}{a_{jj}}. \tag{32}$$

For $j = 1, ..., n$ and $i = j + 1, ...n$.

When the different paths have been generated, we will evaluate, for each observation date, if the different asset values are above their respective risk barrier level.

If they are above this level, the product will be auto-called and no further cash-flow can be expected, if not, the observations will continue until maturity and, depending on the structure of the product, the investor will be reimbursed the final payment.

These conditions will have to be adjusted for each different product.

Finally, when this algorithm has been executed $n$ number of times, we will discount today's value of each simulation and divide the sum with $n$ in order to get the correct weight-adjusted value. This is also what we have seen before in Eq. (2).

The discount rate is simply the Stockholm STIBOR rate followed by the Swedish government bonds for each year plus the corresponding credit default swap rate for Royal Bank of Scotland, everything given by Bloomberg. (See [3])

The sum of these values will then reflect a risk neutral price for the product, not taken in to account the risk- and fee-price to the issuer and possibly the fee to the sale side of this product.

# 6 Other model assumptions

## 6.1 In-data parameters

To model these types of products it is important to gather correct information concerning initial properties such as: asset price $S_0$, drift $\mu$ and volatility $\sigma$ for each asset and the correlation between the different assets according to Eq. (33) below

$$\rho = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,1} & \rho_{2,2} & \cdots & \rho_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ \rho_{n,1} & \rho_{n,2} & \cdots & \rho_{n,n} \end{pmatrix}, \quad where \ \ \rho_{ij} = \rho_{ji}. \tag{33}$$

We will also have product parameters which will define the entire product such as risk barrier, $C$, coupon or auto-call payout $Q$ and the payout at maturity called $q$.

Many of these parameters can be provided by Bloomberg and are fairly straightforward, one that differs is the volatility and in this model, the implied volatility has been used whenever possible.

## 6.2 Implied Volatility

The most common use of volatility when pricing options includes a volatility that is reflected by the option market at the time. (See [11])

Calculating backwards, meaning that an observed options price, from the market is used to calculate the implied volatility, which is behind the fixing of that very price.

In finance, theoretical result is one thing, and what the market dictates is quite another. By using observed option prices, one takes into account the volatility for the underlier, as it is expected to be by the market. In contrast to taking a historical estimate on volatility, this is in some ways a forward looking estimate, predicated by the market.

To reproduce a model that is as reliable, describable and generalized as possible, we wish to construct an algorithm that can be applied whenever pricing an auto-callable product, not limited to a certain number of underlyings or the predetermined discrete call dates.

A perfect generalized algorithm is not possible as some products are tailor made for certain clients and we will cover some of them later on.

## 6.3 Monte-Carlo models

The "normal" simulation is the one that briefly has been described above, the Brownian motion that generates a normally distributed sample and, together with a deterministic drift, we create a path which will represent an underlying asset. We will now study the relevance of using a different type of simulation tool to improve the model.

It has been suggested that it is more effective, when generating paths for path dependent structures, to use quasi Monte-Carlo methods instead of ordinary Monte-Carlo. (See [13]) One type of Quasi Monte-Carlo is the "Quasi Sobol Sequence". Instead of sampling randomly over a specific interval, the sampling points are distributed over a specific space and using this technique we avoid clusters and misleading information, often caused by random sampling. Using this idea, we reduce the standard deviation for each price calculation and therefore, the price converges faster towards an expected value.

The Sobol sequence generation has an objective to fill the sampling interval $x_i \in [0, 1]$ with a low discrepancy so that our Monte-Carlo price approach is as effective as possible. (See [8]) Sobol's method for choosing a set of direction numbers starts by selecting a primitive polynomial with coefficients of either 0 or 1 over binary arithmetic. Together with a sequence of positive integers the model calculates so called direction numbers $(v_{1,i}, v_{2,i}...)$ and then each quasi-random number is calculated using these direction numbers and a binary number connected to the corresponding direction number. A more thorough presentation of this sequence presentation can be found in the reference above. An comparison of the effectiveness between the ordinary Monte-Carlo and the Quasi Sobol is presented in Figure 7 below. (See [12])

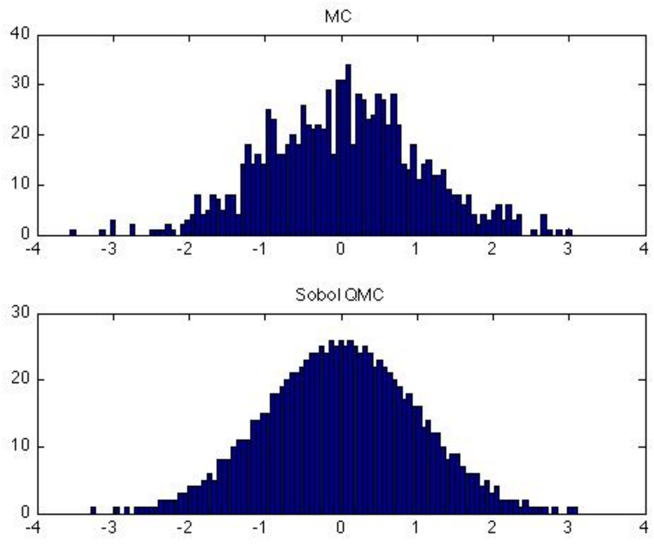


Figure 7: 1000 samplings using the MatLab functions randn and Sobol sampling

# 7 Special types of Auto-callable structured products

Earlier we have seen the ordinary auto-callabe structured product, often called an Snowball Effect and Worst-of put. This type of product has been priced according to the different models above but nowadays, more types of this very product enter the market because banks use innovation and meet investors' demands.

The generalized model above can be used as long as we correct the pricing and payout structure, which tends to differ. As follows, we will look especially at two types more and more commonly seen on the market. (See [14])

## 7.1 Accumulating coupon plus indicative value

This product is similar to the standard auto-call that we have seen before.

One important detail that is different is the fact that here, we will have an accumulating coupon that also will be auto-called given certain conditions. If these conditions not are met, then the coupon will continue and, as indicated by the name, accumulate until this level is reached.

At maturity, the last coupon plus possible accumulated coupons will be reimbursed to the investor assuming that every asset is above the risk barrier, as in the previous case. This is more clearly seen in Figure 8 below.

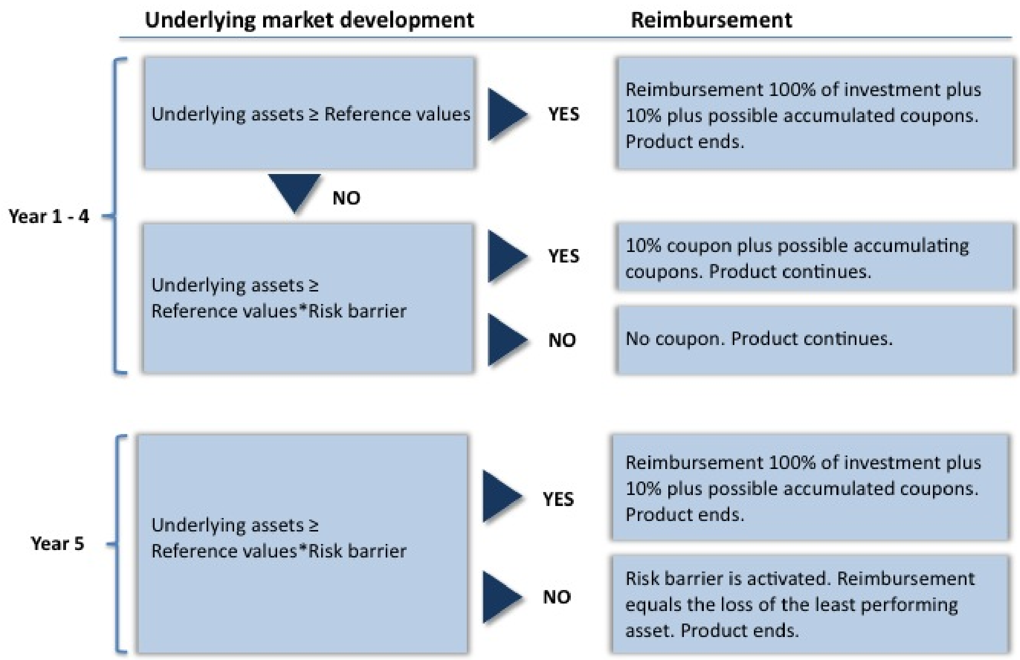


Figure 8: Product information: Accumulating coupon plus indicative value

In order to price this type of product, the observation algorithm used for the ordinary auto-call will be extremely time consuming as we will have to evaluate $3^T$ scenarios for all $n$ scenarios, $T$ indicating the number of time step e.g. $T = 5$ assumes one time step per year for a product with 5 years maturity, also the most commonly seen case.
What has been done instead is that we, for every time step, evaluate what has happened and how to treat the accumulating coupon. This method is somewhat similar to what we saw earlier in the one-step survival and GHK Importance Sampling method where we have some different possible actions for each time step, these possible actions are evaluated and the pricing function is based upon the outcome.

One such product that we will further study is an investment product called the Auto-call Asia offered by SIP Nordic, Stockholm.

The underlying assets are Hang Seng China Enterprise Index (Bloomberg: HSCEI Index), MSCI Taiwan Index (Bloomberg: TAMSCI Index), MSCI Singapore Cash Index (Bloomberg: SIMSCI Index) and Korean KOSPI 200 Index (Bloomberg: KOSPI2 Index).

We will see further examples of this product in the numerical results chapter.

## 7.2 Saviour certificate

This certificate is a modification of the standard product that we have seen before. Instead of the ordinary accumulating coupon, which is auto-called together with the nominal value, a feature of the product is that the investor also has a possibility of receiving a yearly coupon if at least all except one asset are above a coupon barrier.

If there is only one asset below this barrier, this asset will be replaced by an index and the observation will once again be evaluated, if now, the assets are above, the coupon will be paid out, this is more clearly described in Figure 9 below.

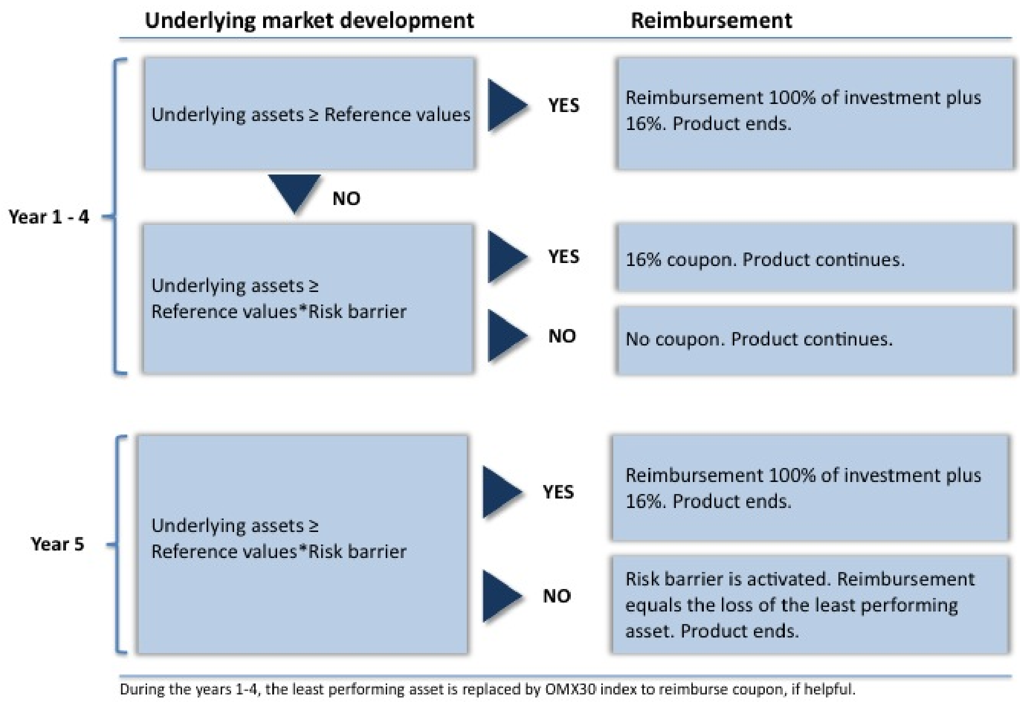


Figure 9: Product information: Saviour certificate

This product is priced in a similar way as before with the standard auto-call. We introduce the fact that the least performing asset in the portfolio might be replaced by the country index if this will help the investor to be paid the annual coupon.

One such product that we will further study is the Auto-call Index Saviour Sweden 2, offered by SIP Nordic Stockholm.

The underlying assets are here the following Swedish stocks: SEB A (Bloomberg: SEBA SS Equity), AB Electrolux B (Bloomberg: ELUXB SS Equity), SSAB A(Bloomberg: SSABA SS Equity), Boliden (Bloomberg: BOL SS Equity) and finally LM Ericsson AB B (Bloomberg: ERICB SS Equity).

We will see further examples of this product in the numerical results chapter.

# 8 Numerical results

Starting with the theoretical keystones, we will evaluate some of these important parts of the model build-up and then focus on the actual models themselves.

## 8.1 Quasi-Mote-Carlo vs. Mote-Carlo

A useful technique when evaluating efficiency while using Monte-Carlo simulations is to study the previously presented "Quati-Monte-Carlo". In Figure 7 we saw how the first method generates a higher standard deviation than the latter.

Using the Quasi technique, we could eliminate the extra time consuming run time to such a level that is well sufficient for the purpose of this thesis, namely to price certain new products for the sell-side of the structured products business.

This sufficiency is also the reason for why we do not need to further evaluate the efficiency and time consumption for the more generalized model.

## 8.2 The upsides/downsides using Stochastic volatility and stochastic drifts

To make the model as generalized as possible, we have evaluated the possibilities of implementing methods such as stochastic drifts and stochastic volatilities.

### 8.2.1 Stochastic drift

As for the drift, it seems as the Vasiček model is preferably used when having longer time horizons for the products, or perhaps for other products that are not prices using this very technique. As seen in Figure 10, the drifts or rate paths becomes much higher then what is probable, assuming in-data taken from Bloomberg.

This generates a too aggressive discount rate for the possible payouts and the product becomes unreasonably low priced.

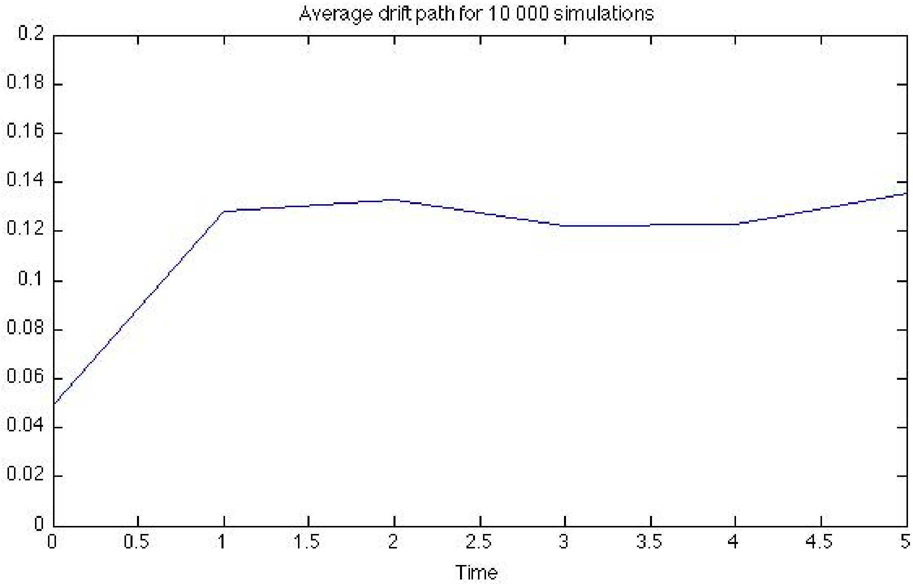


Figure 10: The average rate generates for the different assets when using the Vasiček model

### 8.2.2 Stochastic volatility

Analyzing the stochastic volatility has been more complex due to the fact that, in this case, we need to analyze the problem from a more "non-mathematical" point of view.

Looking at the implied volatilities, we can see by comparing with historical volatilities on Bloomberg, that those used when pricing for example the Saviour certificate presented above are very much in line with what we have seen before.

As for the other product, the Auto-call Asia, we can see that today's volatilities are at bottom levels. This means that the product will be priced using volatilities that perhaps does not present the underlying assets in a long-term representable way.

What could be done here was to introduce the stochastic volatility for this model with parameters that better estimate reasonable future values. As seen in Figure 11 and Figure 12, we can see how the volatility develops as time proceed where the first figure illustrates how one stochastic volatility can be generated, and the second illustrating an average for 10 000 simulations.

This argument that the pricing is more accurately done with a stochastic volatility is strengthened by the fact that the prices found are much in line with what is offered by SIP Nordic, fees included.

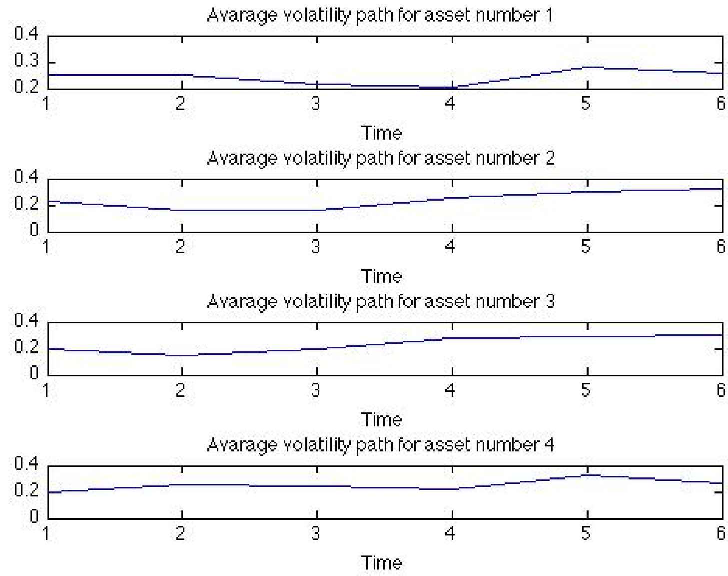


Figure 11: The stochastic volatilities for the underlying assets using 1 simulation

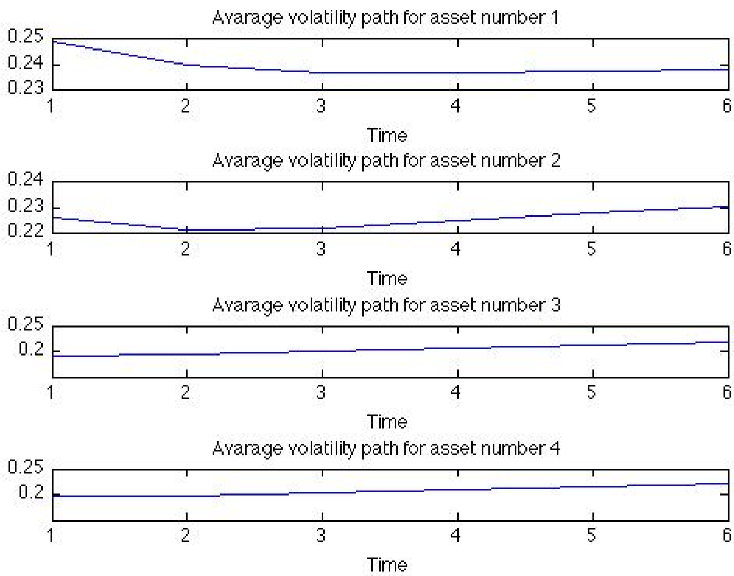


Figure 12: The average stochastic volatilities for the underlying assets using 10 000 simulations

## 8.3 The one-step survival and GHK Importance Sampling method

As presented earlier, the one-step survival and GHK Importance Sampling method have been derived to price auto-callable products using an innovative technique. Using this method, we can benchmark some results and compare with the more generalized model given certain assumptions.

First of all, the results given by this model are presented in Figure 13 - 16 below where the following parameters have been used: Auto-call levels $S^{1,2}_{ref} = 5000$, years to maturity $T = 5$, Investment = 10 000, $Q$, coupon payout is an accumulating 20 % per year, $\rho = 0.5$, $\sigma_{1,2} = 30\%$, $\mu_1 = 8\%$, $\mu_2 = 6\%$ and risk-free rate is set at 4 %, all in-data parameters are deterministic. $S^1_0$ and $S^2_0$ are varied between 0 and 10 000 with discrete steps every 100.

We can see how prices converge as we increase the number of simulations from 10 to 5 000.

First, with 10 simulations for each price calculation

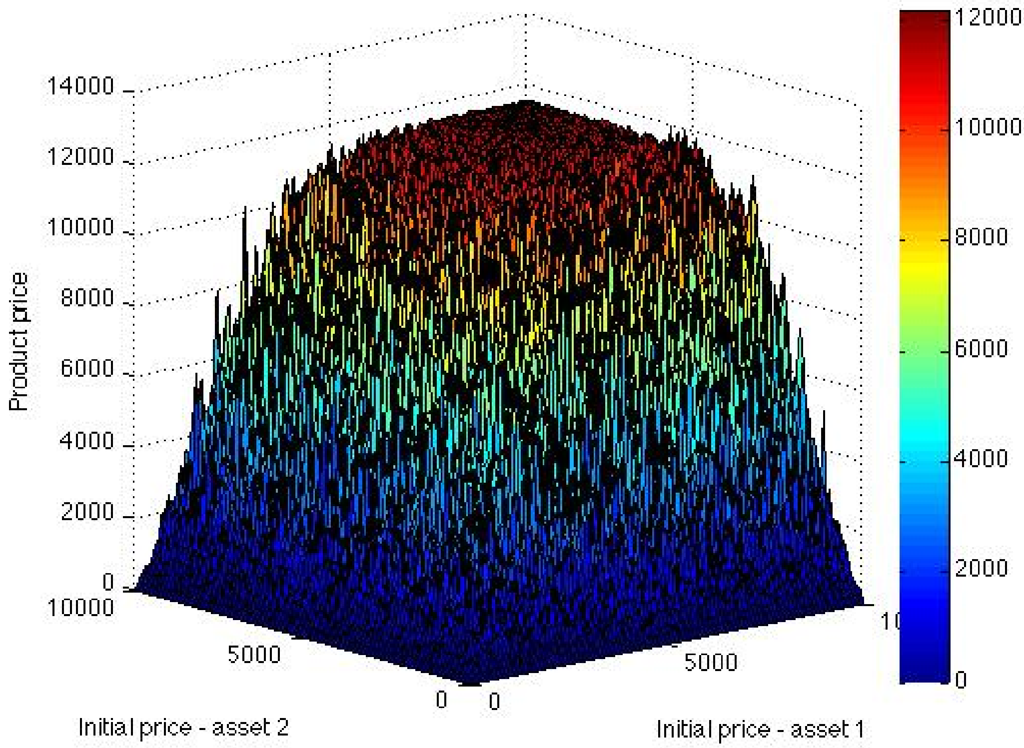


Figure 13: Price evolution with 10 simulations

100 simulations for each price calculation

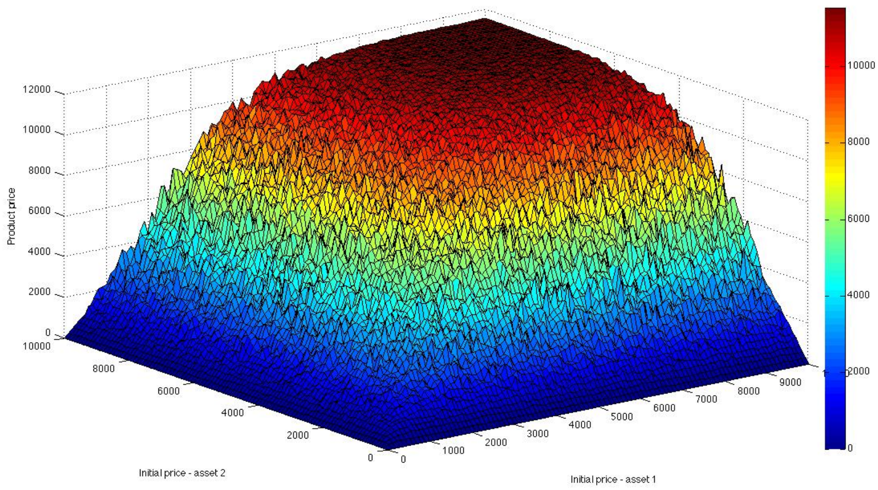


Figure 14: Price evolution with 100 simulations

1000 simulations for each price calculation

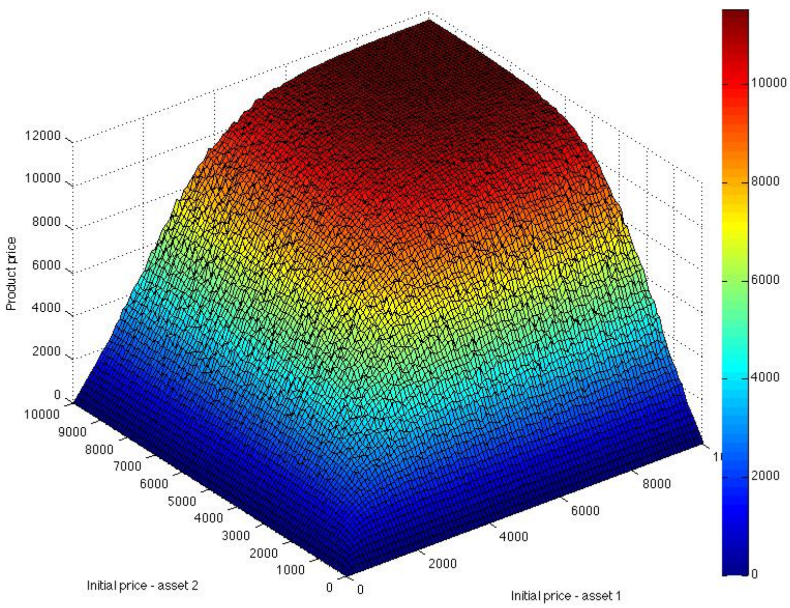


Figure 15: Price evolution with 1000 simulations

And then finally, using 5000 simulations for each price calculation

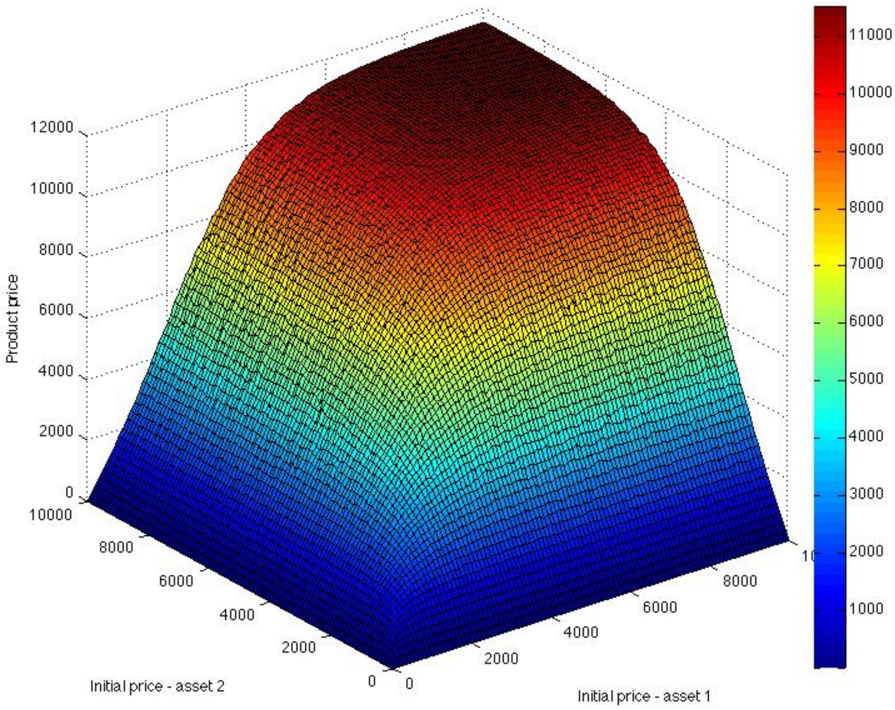


Figure 16: Price evolution with 5000 simulations

And we can see how fast the prices converge to its Monte-Carlo mean value, also representing the price for the product for all different initial asset prices.

To study the result similarities with the generalized model, we compare some prices from the to models, assuming the same in-data parameters as before and still changing the initial asset prices as in Figure 13 - 16 above.

| The one-step survival method | | The generalized model | | |
|---|---|---|---|---|
| $S_0^{1,2}$=4 000 | | $S_0^{1,2}$=4 000 | | |
| N simulations | Average Price | N simulations | Average Price | Difference |
| 10 | 9 325 | 10 | 9 330 | 0.054 % |
| 100 | 8 720 | 100 | 8 170 | 6.7 % |
| 1 000 | 8 191 | 1 000 | 8 055 | 1.7 % |
| 10 000 | 8 268 | 10 000 | 8 219 | 0.59 % |
| 100 000 | 8 248 | 100 000 | 8 256 | 0.097 % |
| $S_0^{1,2}$=5 000 | | $S_0^{1,2}$=5 000 | | |
| N simulations | Average Price | N simulations | Average Price | Difference |
| 10 | 9 266 | 10 | 11 777 | 21 % |
| 100 | 9 755 | 100 | 9 465 | 3.1 % |
| 1 000 | 9 823 | 1 000 | 9 932 | 1.1 % |
| 10 000 | 9 864 | 10 000 | 9 830 | 0.35 % |
| 100 000 | 9 853 | 100 000 | 9 841 | 0.12 % |
| $S_0^{1,2}$=6 000 | | $S_0^{1,2}$=6 000 | | |
| N simulations | Average Price | N simulations | Average Price | Difference |
| 10 | 10 892 | 10 | 11 483 | 5.4 % |
| 100 | 10 799 | 100 | 10 743 | 0.52 % |
| 1 000 | 10 802 | 1 000 | 11 068 | 2.4 % |
| 10 000 | 10 798 | 10 000 | 10 799 | 0.0 % |
| 100 000 | 10 780 | 100 000 | 10 781 | 0.0 % |

Table 1: Comparing prices between the one-step survival and GHK Importance Sampling- and the more generalized method.

One important drawback with this first model is of course the fact that we only have two underlyings, despite this problem, the model is useful so that we can compare results.
The most valuable property of this method is without doubt its effectiveness.

In Table 1 we can see some different prices using different number of Monte-Carlo simulations for the above mentioned model and the same using the more generalized model. We can see that the values converge towards the same answer but the first model is more efficient.

Having compared the one-step survival and GHK Importance Sampling method model with the more generalized model, it should be mentioned that it is only the price calculation, Monte-Carlo generation and the different approaches (for example whether or not use stochastic drift and volatilities etc.) that will be changing as the different products are priced.

Therefore, this comparison assures the validity of any of the different types of products mentioned in the next chapter.

## 8.4 The reconstruction of the products provided by SIP Nordic

### 8.4.1 The Auto-call Asia

As we saw in chapter 7.1, this "Auto-call Asia" is a product categorized under the group "accumulating coupon plus indicative value". Due to its special feature with the possible coupon payouts, this product can distribute an early percentage of the initial investment.

Assuming the same product as described last in chapter 7.1 above, we have reproduced using the generalized model. Using the stochastic volatility where the different volatilities approaches a higher value (as seen in chapter 8.2.2) than if we were to use a deterministic volatility for each asset, we see that the price of the product is very much in line with the price of 10 000 per certificate that is offered by SIP Nordic, fees included.

In an attempt to evaluate the potential up- and down-sides of this product, the constant drift $\mu$ has been varied and we can see in Figure 17 how the product has been auto-called, or not, for these different scenarios.

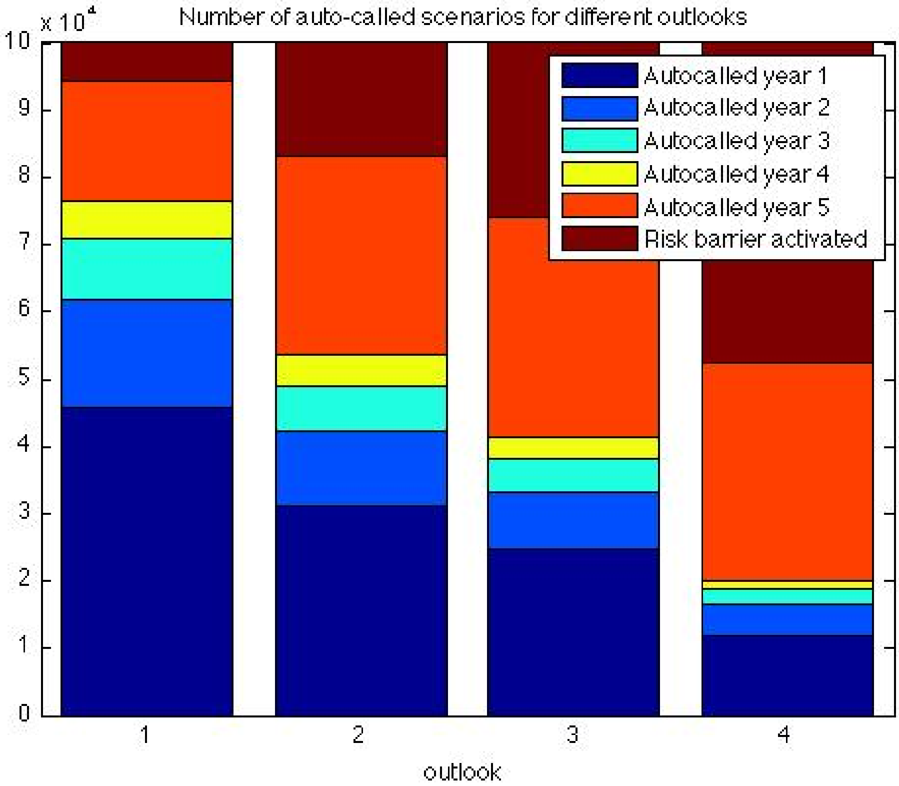


Figure 17: Four different scenarios, $\mu_{+positive}$ = 12 -15 %, $\mu_{positive}$ = 5 -12 % , $\mu_{negative}$ = 0 -5 %, $\mu_{+negative}$ = -7 -0 %

In this figure, we can clearly see how the product has been auto-called or not, as a complement to this figure, we have Table 2 indicating the percentage of the auto-calls for the scenarios together with the average internal rate of return for the investor, assuming the different scenarios. As we see in the figure, the generation was done with 100 000 simulations.

| | | | | | | |
|---|---|---|---|---|---|---|
| **Scenario 1 : IRR = 8.7 %** | | | | | | |
| % of scenarios auto-called Year : | 1 | 2 | 3 | 4 | 5 | Risk barrier |
| | 46.0 % | 16.5 % | 8.7 % | 5.5 % | 17.6 % | 5.8 % |
| **Scenario 2 : IRR = 6.1 %** | | | | | | |
| % of scenarios auto-called Year : | 1 | 2 | 3 | 4 | 5 | Risk barrier |
| | 30.5 % | 11.4 % | 6.7 % | 4.7 % | 29.5 % | 17.1 % |
| **Scenario 3 : IRR = 3.9 %** | | | | | | |
| % of scenarios auto-called Year : | 1 | 2 | 3 | 4 | 5 | Risk barrier |
| | 24.4 % | 8.3 % | 4.8 % | 3.4 % | 33.1 % | 26.0 % |
| **Scenario 4 : IRR = -1.7 %** | | | | | | |
| % of scenarios auto-called Year : | 1 | 2 | 3 | 4 | 5 | Risk barrier |
| | 11.9 % | 4.6 % | 2.3 % | 1.4 % | 32.4 % | 47.4 % |

Table 2: Asia: When scenarios were auto-called and average IRR.

Evaluating the performance, we can see that the product provided by SIP Nordic corresponds to scenario 2, this because of two things; according to Bloomberg, these are the drifts that we can assume for the underlying assets, secondly, we can see that it is when using these in-data parameters that we also find the corresponding price for the product.

Studying this scenario, we can see that the internal rate of return averages 6.1 %. Comparing to the possible yields given by savings account ect., we can see that this is considerably more interesting. The investor has to face a slightly higher risk as the risk barrier was hit 17.1 % of the scenarios generated.

As often when investing capital in order to generate a profitable return, the investor has to pay in risk. This risk, compared to the risk linked to other investment vehicles, can be considered fairly low, therefore this type of product would be classified as a medium-risk investment.

**Internal Rate of Return**

The internal rate of return indicated for every scenario is meant to calculate the discounting rate that makes net present value equal to zero. Eq. (34) below describes the methodology for this very calculation (See [15])

$$\sum_{n=1}^{N} \frac{F_n}{(1+r)^n} = V_0. \tag{34}$$

Where $F_n$ is the cash-flow, $r$ is the discount rate adjusted so that $V_0$, today's value, equals zero. This is then also the rate presented as the internal rate of return above.

As these values are based on a model that calculate values from path generated simulations, the conclusions are, to a certain extent, counterintuitive but they do help to determine the profitability of the product as we cannot present how the investor will be repaid assuming that the risk barrier is reached in the fifth year.

### 8.4.2 The Auto-call Index Saviour Sweden

Conducting the same analysis as in the previous chapter but replacing the "Auto-call Asia" with the "The Auto-call Index Saviour Sweden" described in chapter 7.2, we can see how we can expect a possible auto-call distribution for the same scenarios assumed above.

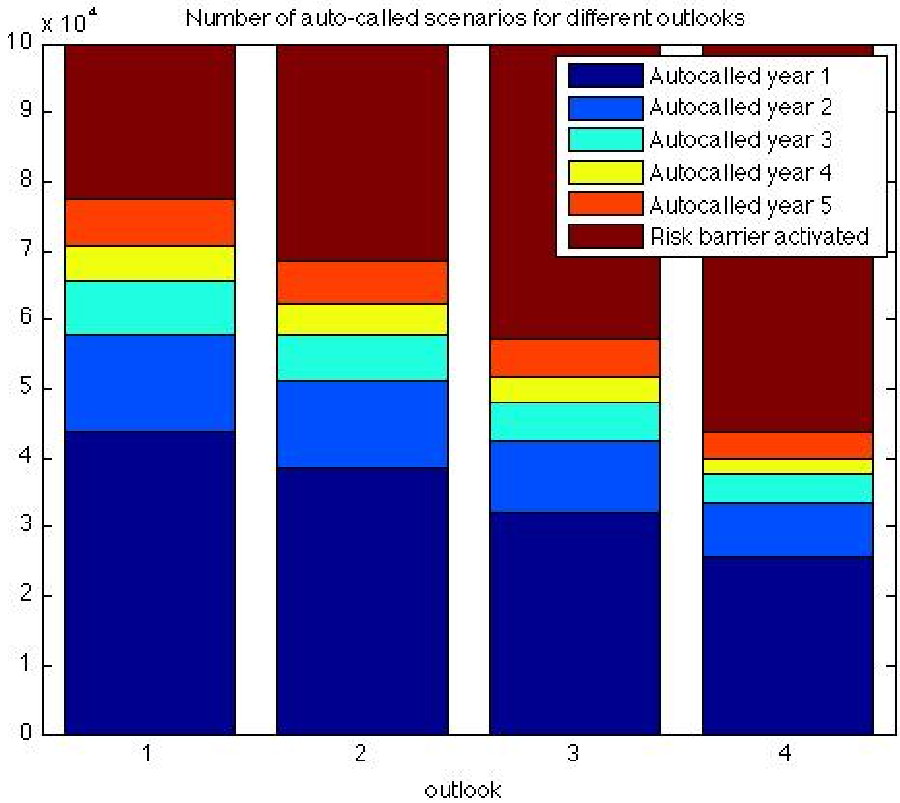


Figure 18: Four scenarios, $\mu_{+positive}$ = 12 -15 %, $\mu_{positive}$ = 5 -12 %, $\mu_{negative}$ = 0 -5 %, $\mu_{+negative}$ = -7 -0 %

In Figure 18, we can clearly see how the product has been auto-called or not, as a complement to this figure, we have Table 3 indicating the percentage of the auto-calls for the scenarios together with the average internal rate of return for the investor, assuming the different scenarios. As we see in the figure, the generation was done with 100 000 simulations.

| | | | | | | |
|---|---|---|---|---|---|---|
| **Scenario 1 : IRR = 8.5 %** | | | | | | |
| % of scenarios auto-called Year : | 1 | 2 | 3 | 4 | 5 | Risk barrier |
| | 43.6 % | 14.3 % | 7.8 % | 5.1 % | 6.9 % | 22.2 % |
| **Scenario 2 : IRR = 5.5 %** | | | | | | |
| % of scenarios auto-called Year : | 1 | 2 | 3 | 4 | 5 | Risk barrier |
| | 38.5 % | 12.5 % | 6.6 % | 4.5 % | 6.5 % | 31.3 % |
| **Scenario 3 : IRR = 1.0 %** | | | | | | |
| % of scenarios auto-called Year : | 1 | 2 | 3 | 4 | 5 | Risk barrier |
| | 31.9 % | 10.5 % | 5.6 % | 3.6 % | 5.6 % | 42.7 % |
| **Scenario 4 : IRR = -5.3 %** | | | | | | |
| % of scenarios auto-called Year : | 1 | 2 | 3 | 4 | 5 | Risk barrier |
| | 25.6 % | 8.0 % | 4.0 % | 2.4 % | 3.9 % | 56.1 % |

Table 3: Saviour: When scenarios were auto-called and average IRR.

At a first glance while comparing the results between the two models, it seems as if the Auto-call Asia product is more likely to be auto-called than the Auto-call Sweden certificate but, here one must take into consideration that, according to consensus given by Bloomberg, the estimated drift for the Asian product corresponds to the second case as for the other product, the consensus estimates a scenario similar to the first one.

If we add the stronger coupon for the second product to this performance analysis, we see that, performance-wise, they are fairly similar.

Evaluating the same performance as for the "Asian Auto-call", we can see that this product offers possibly an even higher internal rate of return. Once again comparing profitability against risk, we can see that one can expect a higher risk as in the first scenario, representing the actual product, 22.2 % of the simulations ended up below the risk barrier and therefore resulting in a loss.

As seen in this comparison, a constant question that investors should ask themselves is how much risk he/she is prepared to take in return of a higher yield.

## 8.5 Robustness tests of different parameters

In order to evaluate the robustness for these products, we have created a theoretical product with an estimated price of 10 000, without fees.

This product is characterized by the following in-data parameters:

- Investment = 10 000
- 4 underlying assets
- Risk barrier = 50 %
- $\mu_{1,2,3,4}$ = 12,10,8,6 %
- $\rho_{i,j}$ $(i \neq j) = 0.7$
- $\sigma_{1,2,3,4}$ =25,5 %
- Simulations: 1 000 000
- $r_f$ risk-free rate is given by Stockholm STIBOR rate followed by the Swedish government bonds plus RBS credit default swap rates. This rate structure is indicated in Table 4 below.

| **Discount rate** | T = 1 | T = 2 | T = 3 | T = 4 | T = 5 |
|---|---|---|---|---|---|
| | 4.9 % | 9.27 % | 7.32 % | 6.02 % | 6.89 % |

Table 4: Discount rate using Stockholm STIBOR, government bond coupons plus RBS CDS

In order to evaluate how different variations of in-data parameters affect the price, tests have been conducted where the vectors containing deterministic values for each parameter have been varied according to Table 5 below.

It below summarizes the price results while conducting these variations, each test has been done with 1 000 000 simulations and the price is then indicated for each case.

| **Sensitivity tests** | Volatility | Correlation | Risk-free rate | Coupon payout | Risk barrier |
|---|---|---|---|---|---|
| **15 %** | 9 494 | 10 211 | 9 737 | 11 211 | 9 855 |
| **12 %** | 9 602 | 10 171 | 9 789 | 10 972 | 9 878 |
| **9 %** | 9 696 | 10 129 | 9 841 | 10 729 | 9 912 |
| **6 %** | 9 800 | 10 080 | 9 891 | 10 489 | 9 938 |
| **3 %** | 9 898 | 10 041 | 9 941 | 10 239 | 9 965 |
| **0 %** | 9 999 | 9 999 | 10 000 | 10 000 | 10 000 |
| **-3 %** | 10 096 | 9 955 | 10 048 | 9 751 | 10 027 |
| **-6 %** | 10 191 | 9 918 | 10 107 | 9 507 | 10 059 |
| **-9 %** | 10 286 | 9 875 | 10 160 | 9 268 | 10 082 |
| **-12 %** | 10 393 | 9 837 | 10 209 | 9 022 | 10 116 |
| **-15 %** | 10 478 | 9 795 | 10 266 | 8 781 | 10 145 |

Table 5: Prices for the product while varying underlying asset prices and in-data parameters.

Same price values have been illustrated in Figure 19 below

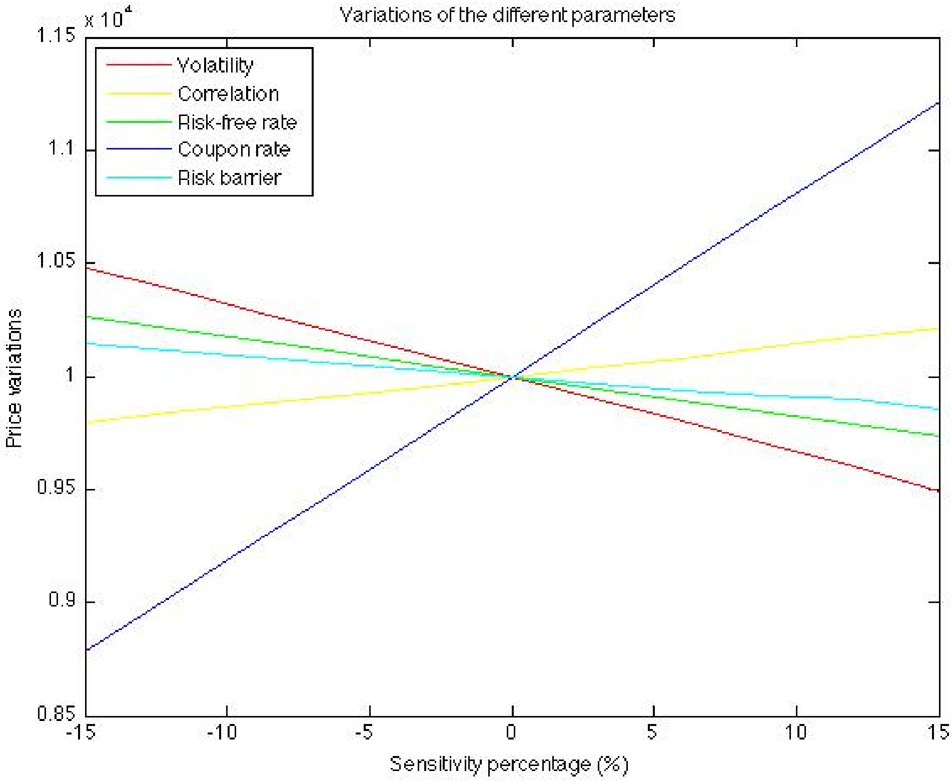


Figure 19: Prices for the product while varying underlying asset prices and in-data parameters.

Studying these results, we will analyze each parameter individually

### 8.5.1 Volatility

As proven before while implementing the stochastic volatility using the Heston model, we have seen that this parameter is highly decisive while pricing auto-called products.

Because of the extraordinary low volatilities for certain Asian markets seen in chapter 8.4.1, we used this stochastic volatility. This was though an exception and for the more general case, we will use a deterministic volatility.

Seen in Table 5 and Figure 19, the volatility is, after the coupon rate, the most price affecting in-data parameter. This seen, we conclude as earlier that, when seeking a more profitable product always comes a higher risk and vice versa.

### 8.5.2 Correlation

Concerning the correlation, we can see that when assets are more correlated, the price increases. This is also intuitively correct as the assets will more often pass the auto-call trigger level and the product will end with a profit.

This is one of the less affecting in-data parameters, as the volatility will be more decisive of whether or not the product will be auto-called.

### 8.5.3 Risk-free rate

The risk-free rate is an important aspect when pricing structured products. We can see that, when the rate increases, the price goes down. This because, for the "normal" structured product, the discount rate will offer a less expensive zero-coupon bond and more options can be bought, offering a more attractive product, still with an increased risk shown by the increased rate.

Even though the auto-called products not by definition are constructed with bonds, options and a sold put option, the same argument can be used for why the products are less expensive with an increasing rate.

### 8.5.4 Coupon payout

An issuers dilemma when offering higher coupon payouts is that the product will become more attractive and then also increase the price of the product. At the same time, the margins are lowered and the issuer face the risk of a future loss when pricing low. This parameter is solely the most important when marketing the product and also the most difficult to adjust as it will quickly generate a higher return for the investor and at the same time, a thinner margin for the issuer or the sell side.

### 8.5.5 Risk barrier

The last in-data parameter to be analyzed is the risk barrier. Due to the low probability of ending up in the risk barrier zone, this parameter is the least affecting in-data parameter. We can see that a decreasing risk barrier barely increases the price, this is therefore also an important parameter when trying to market these products as it can easily make the product more interesting for the non-risk taking investor.

## 8.6 Time dependency- Reinvest or keep auto-called structure?

To evaluate the time dependency for the different auto-calleable products, we have varied the maturities of the products and, at the same time, a sensitivity analysis on the underlying asset values was conducted. The results can be seen in Table 6 below. 1 000 000 simulations have been used for each price calculation together with the in-data as above for the theoretical product valued to 10 000, without fees.

This analysis calculates the price for a product as time goes by, this means that the accumulating coupon increases from 20 %, to 40, 60, 80 and 100 %, from $T = 5$ to $T = 1$(indicating the time left), for this theoretical product.

Reader should bear in mind that it is not the maturity that changes but rather the product that continues and not being auto-called, this meaning that the coupon is constantly increasing as time, $T$ years left to maturity, decreases.

The most interesting part is then the situation where the underlying assets are below 0 % i.e. just at and under their initial reference value. One should notice that we here assume that the underlying prices are all the same as time goes by, this in order to get an expected price as we study how these prices develop over time.

| **Time dependency** | T=5 Years | T=4 Years | T=3 Years | T=2 Years | T=1 Years |
|---|---|---|---|---|---|
| **15 %** | 10 682 | 12 132 | 13 350 | 14 368 | 14 780 |
| **12 %** | 10 572 | 11 972 | 13 103 | 14 025 | 14 570 |
| **9 %** | 10 454 | 11 792 | 12 827 | 13 659 | 14 347 |
| **6 %** | 10 316 | 11 587 | 12 545 | 13 269 | 14 110 |
| **3 %** | 10 162 | 11 372 | 12 229 | 12 860 | 13 830 |
| **0 %** | 9 999 | 11 135 | 11 907 | 12 429 | 13 535 |
| **-3 %** | 9 814 | 10 881 | 11 555 | 11 989 | 13 190 |
| **-6 %** | 9 616 | 10 614 | 11 198 | 11 542 | 12 830 |
| **-9 %** | 9 404 | 10 331 | 10 815 | 11 087 | 12 433 |
| **-12 %** | 9 177 | 10 036 | 10 427 | 10 629 | 11 999 |
| **-15 %** | 8 938 | 9 724 | 10 026 | 10 170 | 11 536 |

Table 6: Prices for the product while varying underlying asset prices and time left to maturity.

Same price values have been illustrated in Figure 20 below

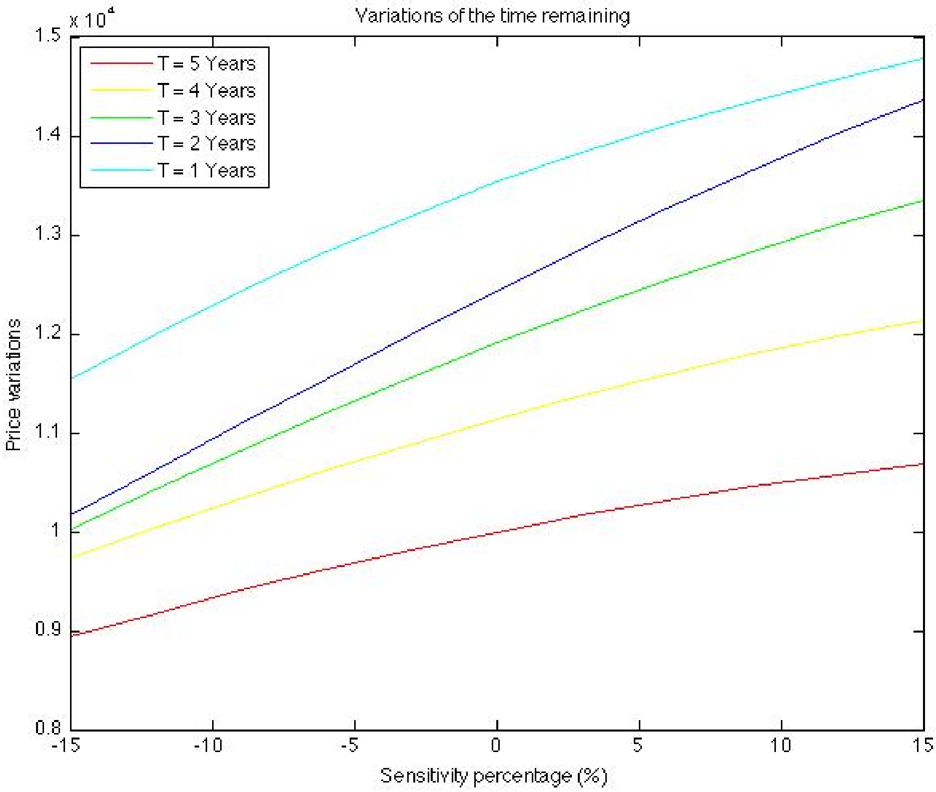


Figure 20: Prices for the product while varying underlying value and time left to maturity for a 5-year product.

These prices are discounted at their current time, meaning that the shown values are the calculated prices at each future time step.

One very important part of the product for the potential investor is of course the value if, in the future, he/she whishes to resell the once bought auto-callable structured product.

In order to evaluate this price, tests have been made where the underlying asset values have been varied and the model has stepped forward in time, assuming that the product not yet have been auto-called.

We can se in Figure 20 that, only when the product is issued (T = 5 Years left) and after 1 year, the product is worth less then its initial value. (This assuming underlying asset value variations of +/- 15 %)

These results show that, even with a negative asset development until a certain sensitivity limit, the values of the products on the market remain strong and this is of course an important fact for the non-risk taking investor.

Returning to the question whether or not to reinvest or keep the structured product assuming a bullish or bearish market, we once more analyze the results given by the time/sensitivity analysis.

We can see that when the markets are down, and the maturity of the product is getting closer, the value of the product remains high. This because the possibility of receiving a high coupon (for example 100 % when T = 1 i.e. product maturity is approaching), is attractive. An investor who faces this situation is once again dealing with a quantification of risk.

Therefore, a preliminary investment advice for the non-risk taking investor is, with a decreasing market and increased product value, an over-the-counter trade is preferable as the investor will face a higher probability of ending up under the risk barrier and therefore facing a negative return. This is also the positive argument for the more risk taking investor, as time proceeds, the accumulating coupon offers a potential extraordinary high return if the product not yet has been auto-called.

A last scenario that one could imagine while studying this time dependency is when the market is at extremely low levels and the product has activated the risk barrier before the maturity of the product. In this case, the auto-called product is similar to the situation where the investor has invested the nominal value in the worst-performing stock or index. In this situation, the most logical behavior for the investor is to compare his/hers anticipations to those of the market.

If the investor believes that the worst-performing stock/index has a potential come-back or even increase in value, which is higher that the anticipations of the market, then it is more profitable to keep the product as, he/she will otherwise loose value when selling to a buyers market i.e. a market where the buyer has the power to decrease prices.

## 8.7 Future possible products

As presented in the numerical results section, it was shown that, when constructing these products, the question of risk, closely connected to the volatility, is by far the most crucial parameter.

Because of the last years with unusually volatile markets, investors tend to take less risk and therefore, many structured products have been constructed thereafter.

To meet this increasing demand, future products should focus on a relative high yield with a measurable risk. As we could see in the robustness tests, a high coupon rate affect the prices much more than for example risk barrier or correlation.

Knowing that the demand for non-high risk products increases, it would be wise to create products with highly correlated assets and perhaps a capital protected auto-callable product where the risk barrier is set at 0 %.

Of course, future product development should always be done in line with what investors want and therefore, a close contact between issuer/sell-side and investor is always appreciated.

# 9 Conclusion

This report presents a generalized, adaptable and functioning pricing algorithm for auto-callable structured products. Comparing three different models, one generalized model is constructed. Together with this algorithm presentation is a theory background that present the main keystones that the model is built upon. The model is compared to a previous published model and it shows that this program is somewhat less efficient but more adjustable, more suitable for the ambitions of this project.

Certain products offered by SIP Nordic Fondkommission AB, Stockholm, are reproduced and a performance- and time dependency analysis are conducted with the results indicating that these products offer the investor a significant higher return but also with an increased risk. This increased risk is well correlated with the return and finally, it is for the investor to decide how much risk he/she is willing to take.

Concerning the properties of the different parameters connected to these products, we see that they vary from one to another and for the future, a recommendation is to always analyze the demands in order to know how much risk the investors would like to face and then construct the products thereafter.

An interesting further study would be to evaluate whether or not it would be possible to extend the one-step survival and GHK Importance Sampling method to more than two dimensions. This would generalize the model and offer the modeler an effective alternative to price the auto-callable structure.
If possible, one could extend this model with stochastic rates and volatilities in order to create a as adjustable and as general model as possible, similar to what was done with the generalized model described in this thesis.

Also, one important part when pricing these types of products is to further study the volatilities. In order to price different auto-callable products as reliable as possible, one should consider the fact that each volatility for each underlying asset has an volatility surface that has to be considered when pricing.

Such an implementation might be a alternative solution instead of using stochastic volatilities, as this method would differently sample the volatilities and then also the calculated prices.

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