KTH
NASDAQ OMX

2012-08-01

Selecting the Worst-Case Portfolio

A proposed pre-trade risk validation algorithm of SPAN

By
Gustav Montgomerie-Neilson

Supervisors
Tobias Rydén, KTH
Jörgen Brodersen, Nasdaq OMX
Johan Olsson, Nasdaq OMX

Abstract

This thesis outlines a possible pre-trade risk validation algorithm for portfolios of commodities futures and options. A method is proposed that, given an order book of unmatched orders, determines the particular order selection, or portfolio, with the maximum margin requirement, as calculated by the risk analysis methodology SPAN. The method consists of a selection algorithm, where all orders in the order book are either included or discarded according to a specified criterion. The selection criterion approach reduces the problem from exponential to linear time complexity, complying with pre-trade risk validation requirements. Further, three different selection criteria are proposed and evaluated by accuracy and time performance. Simulations indicate that one of the criteria has considerable accuracy in determining the worst-case portfolio in linear time, without relying on approximations of the orders it includes. This makes it a particularly suitable candidate for pre-trade risk validation.

Acknowledgements

I should like to begin by thanking Jörgen Brodersen, one of my supervisors at Nasdaq OMX, for offering me the opportunity to do my thesis work at Nasdaq OMX and proposing the thesis topic. I have benefited greatly from his guidance, and without him this thesis surely would have floundered. I should also like to thank my other supervisors at Nasdaq OMX, Johan Olsson, for sharing with me his knowledge and advice, and pointing me in the right direction one more than one occasion. A very special thanks is extended to Johan Bergenudd, whose ideas and suggestions are the basis for most of this paper. The results presented here are directly attributable to the meetings I had with him.

Lastly, I wish to express my gratitude to my supervisor at KTH, Tobias Rydén, who through repeated readings advised me on the form and content of this paper. If anyone finds the following pages pleasant to read, please direct your regards to him.

Gustav Montgomerie-Neilson, July 2012

Frequently Used Terms

Future A contract between a buyer and a seller for an asset of specified quantity and price today, for delivery and payment at a future delivery date.

Call option A contract in which the buyer has the right, but not the obligation, to buy a specified financial instrument from the seller of the option at a future date (the maturity) for a price agreed upon today (the strike price).

Put option A contract in which the buyer has the right, but not the obligation, to sell a specified financial instrument to the seller of the option at a future date (the maturity) for a price agreed upon today (the strike price).

Black Scholes option pricing

A mathematical formula to determine the price of European call and put options, given the price of the underlying asset, the strike price, the time to maturity, the implied volatility of the option, and the risk-free interest rate. A European-style option is an option that can be exercised only at the maturity date, and not before.

Option delta The rate at which the price of an option changes compared to price changes in its underlying asset. The expression for the option delta of European options is derived analytically in the Appendix.

SPAN The Standard Portfolio Analysis of Risk is a risk analysis methodology for determining the initial margin requirement of a given portfolio.

Combined commodity

A term in SPAN that signifies the group of all orders of financial instruments with the same underlying asset in a portfolio.

Initial margin requirement

The initial margin requirement is the amount that a holder of a portfolio must post in collateral to a clearing house to guard against the possibility of default.

Table of Contents

Introduction

Central counterparty clearing and the performance bond

In a bilateral financial transaction, where there exists a buyer and a seller of an agreed upon financial instrument, both parties assume a so-called counterparty risk: the non-negligible risk of default by the opposing party in the transaction. An alternative arrangement, called central counterparty clearing, enables this counterparty risk to be assumed by a third party, the clearing house, also called the exchange, which acts as a middle-man in the transaction. This arrangement enables the clearing house to broker financial transactions between large numbers of buyers of sellers, lower the risk exposure of its members, and generate fees. In exchange, the clearing house requires its members to post a performance bond requirement, which is meant to act as collateral to cover the potential losses incurred by the clearing house in case of member default. This performance bond requirement, also called a margin requirement, is normally tied to the size of the trades a member has on its books. As such, from the perspective of the clearing house, the margin requirement has to be low enough to attract trading customers but high enough to discourage excessive risk taking.

In practice, this margin requirement is calculated daily at the end of trading by the clearing house, and members have to post collateral before the trading starts the following day. In addition, the clearing house may demand of its members an instantaneous increase in collateral if their trading results in a margin change severe enough to warrant it. To monitor this, margin calculations are regularly performed during the day.

HFT and algorithmic trading

Changes in current trading practices have made this risk management strategy untenable. High-frequency trading (HFT) and algorithmic trading now account for a significant portion of trades in exchanges worldwide. These are trading strategies executed by proprietary computer algorithms, meant to exploit minute market movements to secure small but guaranteed profits. They are performed extremely fast, with large volumes of orders of financial instruments changing hands in mere microseconds. In 2010, Tabb Group estimated that HFT accounted for 56 percent of equity trades in the United States and 38 percent in Europe (Grant, 2012). In this environment, exchanges constantly focus on lowering latency in their systems to attract the types of customers that employ HFT strategies. However, this emphasis must be weighed against the increased need for risk monitoring of these customers, and the high volume of trades they execute daily. With trading speeds as high as 1500 orders per second, the clearing house now requires margin calculations to be performed in near real time with as little added latency as possible.

The increased prevalence of sponsored access has also created a demand by members of an exchange for pre-trade risk monitoring systems. Sponsored access, the ability of an actor to trade in the name of an established member of an exchange, enables trading by actors that do not comply with the criteria set by the exchange for regular membership. With the sponsor member taking responsibility of all the trades performed by the sponsored actor, there is an incentive by the member to closely monitor the portfolio and cap the amount of exposure the sponsored actor is allowed to take.

The exposure from the currently matched orders in the portfolio is the purview of the risk management at the exchange, but the orders in the order book that have yet to be matched is not. In other words, the member wants the ability to constantly monitor not the current portfolio risk, but rather the worst possible portfolio that can be formed by the currently unmatched orders in the order book. If a sponsored actor is found to have an order book whose worst case portfolio risk exceeds the allowed exposure, the member instantly cuts off access before the trade can be matched and transferred to the exchange. As exchange members wants to attract HTF actors with sponsored access, this monitoring needs to be performed without significant latency.

The aim

There is a dual demand for real time risk monitoring systems, both by exchanges and by exchange members, to counter the high speeds at which trading is currently performed. While the risk monitoring by the exchange has a linear input, the currently matched orders in the portfolio, the pre-trade risk monitoring calculation is combinatorial in nature. With an order flow of 1500 orders per second, certain approximations have to be made for such a risk calculation to be feasible in real time.

The aim of this paper is to explore the possibilities of such a risk monitoring system, keeping in mind the latency requirements put on exchanges and exchange members by algorithmic trading. The main focus is the adaptation of an existing industry-standard performance bond calculation, the Standard Portfolio Analysis of Risk (SPAN), to suit the demands of pre-trade risk monitoring in a high frequency trading environment. The investigation will be performed by applying an adapted SPAN algorithm to randomized portfolios of futures contracts and options on futures contracts, and will include accuracy and time performance analysis.

1. The Standard Portfolio Analysis of Risk (SPAN)

One prevalent method of calculating the margin requirement of a given portfolio of financial contracts is SPAN, the Standard Portfolio Analysis of Risk. It is a system developed by economists at the Chicago Mercantile Exchange in 1988 and is widely used to determine the performance bond requirement at registered stock exchanges, clearing organizations and regulatory agencies all over the world (CME Group). The SPAN methodology consists of a series of calculations, including portfolio stress testing and commodity price correlations, that yields the worst possible performance loss a portfolio can suffer over a given time period. To do this, SPAN leverages an extensive set of parameters set by the clearing house, reflecting the market conditions of traded commodities, and allowing the clearing house to choose its desired degree of coverage.

At its heart, SPAN is based on the division of orders of financial instruments into so-called *combined commodities*, groupings of orders that share the same underlying asset. In other words, a portfolio containing futures contracts and options on futures contracts is segmented into different bins (combined commodities), where each bin only contains contracts of one specific asset, such as steel or copper. SPAN then performs a number of calculations, some on each combined commodity separately, and some on all combined commodities in the portfolio.

SPAN yields as a final result an *initial margin requirement*, which is the loss of value of the portfolio in a worst-case risk scenario. The steps performed in order to obtain the margin requirement are detailed in this section.

The formula for the initial margin requirement is

$$\begin{aligned} &Initial\ Margin\ Requirement \\ &\quad = \max(Scanning\ Risk + Intermonth\ Spread\ Charge + Delivery\ Month\ Charge \\ &\quad - Intercommodity\ Spread\ Credit, Short\ Option\ Minimum\ Charge) \\ &\quad -\ Net\ Option\ Value \end{aligned}$$

Each of the six terms in the formula requires a separate calculation, and is performed in different ways on the portfolio. Some are applied to each combined commodity, and others on individual orders. In the hopes of better describing the SPAN methodology, the following example portfolio will be used to illustrate each of the steps.

The example portfolio consists of four orders with the same underlying asset: steel. The types of instruments and maturities have been consciously selected to provide a balance of complexity and clarity in calculations. In portfolios with larger amounts of orders and with different underlying assets, calculations by hand quickly become unwieldy. Throughout this section, each step of SPAN will be described and demonstrated to yield a full initial margin requirement for the example portfolio.

Example Portfolio

Underlying Asset	Steel			
Instrument	**Future**	**Call**	**Future**	**Future**
Position	10	-5	15	-5
Maturity (days)	90	60	25	150
Price (USD)	1200	31	1100	1300
Underlying Price (USD)	-	1200	-	-
Strike (USD)	-	1250	-	-
Implied Volatility	-	20%	-	-

1.1 The Scanning Risk

The first calculation in SPAN is the *Scanning Risk,* and it is performed on a combined commodity level. Each bin of orders in the portfolio with the same underlying asset is subjected to a series of 16 different risk scenarios, where two parameters are used: the *price scan range* and *volatility scan range*.

Price Scan Range

This is a measure of the likely price movements of a futures contract on a particular underlying asset. The price scan range is calculated like so:

$$Price\ Scan\ Range = Price * Volatility * \sqrt{Time\ Horizon} * Quantile$$

where

Price is the market price of the futures contract

Volatility is the annual volatility of the price of the futures contract.

Time horizon, also called *lead days*, is the time it takes for the clearing house to get out of a position. It is expressed in the same time scale as volatility. It is typically set to 2 working days, which equals 2/252 years (where the year only comprises trading days)

Quantile, measured in standard deviations, is the parameter used to choose the likelihood of given price movements. Assuming a normal distribution, a quantile of 2 standard deviations would statistically cover roughly 95% of likely price movements.

Combined commodities usually contain several orders with the same underlying asset, but with different maturities and prices. The price scan range can in these cases be determined by using the mean price of the assets in the formula above.

Volatility Scan Range

The volatility scan range measures the range in which the implied volatility is likely to fall in a worst-case scenario over the time horizon used above (typically 2 days). By convention, this measure is rarely changed and can be set to a fixed number, such as 10%. This means that the implied volatility of a given order can at worst increase or decrease by 10 percentage points.

The Risk Array

The *risk array* is set by the exchange and is the regime by which 16 risk scenarios are generated using the twin scan ranges above. A typical risk scenario can look like in table 1.1:

Table 1.1 The Risk Array

	Price Change, fraction of Price Scan Range	Volatility Change, fraction of Volatility Scan Range	Weight
1	0	1	100%
2	0	-1	100%
3	+1/3	1	100%
4	+1/3	-1	100%
5	-1/3	1	100%
6	-1/3	-1	100%
7	+2/3	1	100%
8	+2/3	-1	100%
9	-2/3	1	100%
10	-2/3	-1	100%
11	+3/3	1	100%
12	+3/3	-1	100%
13	-3/3	1	100%
14	-3/3	-1	100%
15	2	0	35%
16	-2	0	35%

The 16 risk scenarios are all different combinations of movements in price and implied volatility of the futures contracts, with applied weights to vary probabilities for these movements. The two extreme scenarios, scenarios 15 and 16, consist of drastic price movements, but their low probabilities of occuring are reflected in the lower weights placed on them. When applied to a futures contract or option, each risk scenario will yield the value loss for that order at the given price and volatility movements. For instance, a long futures contract under risk scenario 10 will experience a value loss of two thirds its price scan range, whereas a short futures contract in the same scenario would experience a value gain of the same amount, indicated by a negative value loss.

The value loss on an option is calculated by comparing its market value to its theoretical price in the different risk scenarios, taking into account the changing price of the underlying futures contract and implied volatility. The theoretical price is calculated using the Black-Scholes formula for European options. This, however, only applies to *short* options. Long options are not considered at all in this step, and do not contribute to the final scanning risk.

Calculating the Scanning Risk

Each combined commodity can consist of several futures contracts and options, each with a different position. To calculate the Scanning Risk for the combined commodity, each order has its associated risk array multiplied by its position, and then the value changes of all order in each risk scenario are summed together. The risk scenario with the highest value, indicating the conditions under which the combined commodity will experience the highest possible loss, is then chosen as the Active Scenario, and the associated loss is set as the Scanning Risk.

The Scanning Risk, in other words, is just the worst case outcome of the stress tests in the risk array. The Active Scenario is used at a later stage in SPAN to calculate the Intercommodity Spread Credit. To illustrate the Scanning Risk calculation, it will now be performed on the example portfolio. The price scan range for the example portfolio is calculated using the mean price of the orders. All the parameters required to perform the Scanning Risk calculation are given below.

Underlying Asset	Steel
Annual Volatility	30%
Time Horizon	2 days
Quantile	3 standard deviations

$$Price\ Scan\ Range = 1200 * 0.30 * \sqrt{\frac{2}{252}} * 3 = 96\ USD$$

Price Scan Range (USD)	96
Volatility Scan Range	10%
Interest Rate	3%

Example Portfolio

Instrument	**Future**	**Call**	**Future**	**Future**
Position	10	-5	15	-5
Maturity (days)	90	60	25	150
Price (USD)	1200	31	1100	1300
Underlying Price (USD)	-	1200	-	-
Strike (USD)	-	1250	-	-
Implied Volatility	-	20%	-	-

Apart from position, the calculation for the three different futures contracts are identical.

Table 1.3 Steel future risk array

	PC	VC	W	Loss	Position Losses	10	15	-5
1	0	1	100%	0		0	0	0
2	0	-1	100%	0		0	0	0
3	+1/3	1	100%	-32		-320	-480	160
4	+1/3	-1	100%	-32		-320	-480	160
5	- 1/3	1	100%	32		320	480	-160
6	- 1/3	-1	100%	32		320	480	-160
7	+2/3	1	100%	-64		-640	-960	320
8	+2/3	-1	100%	-64		-640	-960	320
9	- 2/3	1	100%	64		640	960	-320
10	- 2/3	-1	100%	64		640	960	-320
11	+3/3	1	100%	-96		-960	-1440	480
12	+3/3	-1	100%	-96		-960	-1440	480
13	-3/3	1	100%	96		960	1440	-480
14	-3/3	-1	100%	96		960	1440	-480
15	2	0	35%	-67		-670	-1005	335
16	-2	0	35%	67		670	1005	-335

To calculate the option price under different market conditions, the Black-Scholes options pricing model is used.

$$c(S,t) = \Phi(d_1)S - \Phi(d_2)Ke^{-r(T-t)}$$

$$p(S,t) = \Phi(-d_2)Ke^{-r(T-t)} - \Phi(-d_1)S$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

where

c and *p* are the prices of call and put options, respectively
Φ is the cumulative normal distribution function
T-t is the time to maturity of the option
S is the price of the underlying asset
K is the strike price
r is the risk free interest rate
σ is the volatility of price returns of the underlying asset

The formula above is used to populate the Option Price column in table 1.4. The loss is then obtained by subtracting this price from the current market price of the call, which for this option was 31 USD.

Table 1.4 5 Short 2 month steel future call risk array

	PC	VC	W	Price	Implied Volatility	Option Price	Loss	Position Loss
1	0	1	100%	1200	30%	54.4	-23.4	117.1
2	0	-1	100%	1200	10%	9.2	21.8	-108.9
3	+1/3	1	100%	1232	30%	69.7	-38.7	193.3
4	+1/3	-1	100%	1232	10%	20.6	10.4	-52.1
5	-1/3	1	100%	1168	30%	41.4	-10.4	52
6	-1/3	-1	100%	1168	10%	3.3	27.7	-138.4
7	+2/3	1	100%	1264	30%	87.1	-56.1	280.1
8	+2/3	-1	100%	1264	10%	38.4	-7.4	36.8
9	-2/3	1	100%	1136	30%	30.6	0.4	-2
10	-2/3	-1	100%	1136	10%	0.9	30.1	-150.4
11	+3/3	1	100%	1296	30%	106.7	-75.7	378.4
12	+3/3	-1	100%	1296	10%	62	-31	155.2
13	-3/3	1	100%	1104	30%	21.9	9.1	-45.5
14	-3/3	-1	100%	1104	10%	0.2	30.8	-154
15	2	0	35%	1267	20%	64.3	-33.3	166.3
16	-2	0	35%	1133	20%	11.7	19.3	-96.3

Table 1.5 Summarized portfolio Risk Array

	Future 10	Future 15	Future -5	Call -5	Total
1	0	0	0	117.1	117.1
2	0	0	0	-108.9	-108.9
3	-320	-480	160	193.3	-446.7
4	-320	-480	160	-52.1	-692.1
5	320	480	-160	52	692
6	320	480	-160	-138.4	501.6
7	-640	-960	320	280.1	-999.9
8	-640	-960	320	36.8	-1243.2
9	640	960	-320	-2	1278
10	640	960	-320	-150.4	1129.6
11	-960	-1440	480	378.4	-1541.6
12	-960	-1440	480	155.2	-1764.8
13	960	1440	-480	-45.5	**1874.5**
14	960	1440	-480	-154	1766
15	-670	-1005	335	166.3	-1173.7
16	670	1005	-335	-96.3	1243.7

The Scanning Risk calculation for the above portfolio has resulted in

$$Scanning\ Risk = 1874.5\ USD$$

This corresponds to risk scenario 13, representing the worst likely portfolio loss using the current risk parameters. Thus, the Active Scenario for this portfolio is 13. If two risk scenarios results in an equal loss, the lower numbered scenario is chosen. If all losses are negative, indicating no possible loss of the portfolio in any risk scenario, the Scanning Risk is set to 0 and the active scenario is set to 1.

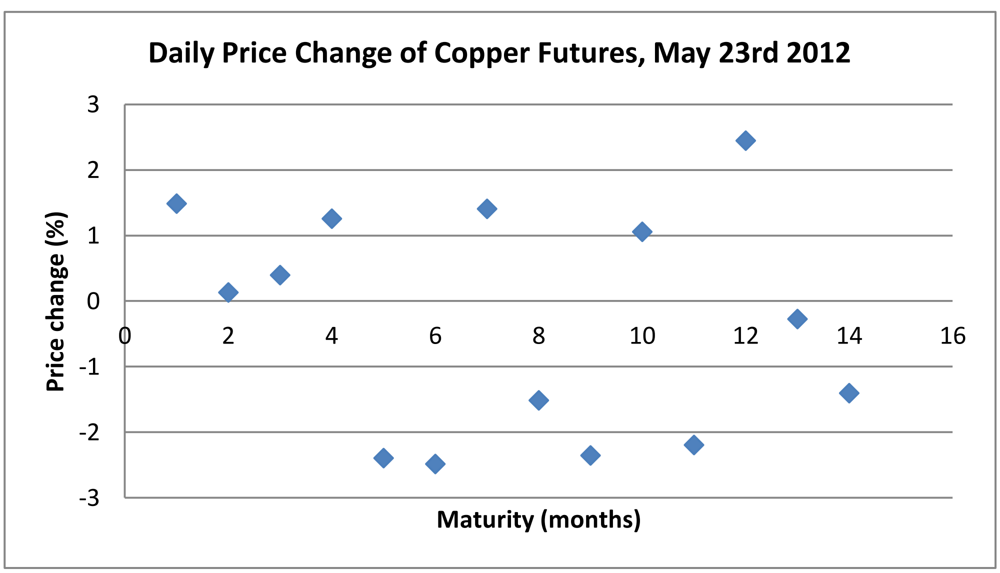
At this stage, the formula for the initial margin requirement looks like this:

$$\begin{aligned} &Initial\ Margin\ Requirement \\ &\qquad = \max(1874.5 + Intermonth\ Spread\ Charge + Delivery\ Month\ Charge \\ &\qquad - Intercommodity\ Spread\ Credit, Short\ Option\ Minimum\ Charge) \\ &\qquad - \ Net\ Option\ Value \end{aligned}$$

1.2 The Intermonth Spread Charge

While the portfolio under consideration consists of two orders with different maturities - 3 months for the futures contract and 2 months until the exercise date of the option - the price movements of these orders are considered to be perfectly correlated in the Scanning Risk step. In each risk scenario, all prices move in the same direction and by the same amount simultaneously. In other words, the Scanning Risk calculation does not account for the fact that prices of orders with different maturities respond differently to changing market conditions. The price changes for a copper futures chain, as listed on the COMEX market on May 23rd 2012 (Yahoo Finance), is shown in Figure 1.1. The Intermonth Spread Charge compensates for this by adding a spread charge to combined commodities containing orders of many different maturities.

Figure 1.1 Daily price change of copper futures



Tiers

The combined commodity is first divided into *tiers*, where each tier contains orders with a preset range of maturities. An example of a tier division is given in Table 1.6.

Table 1.6 The Tier Division

Tier	Maturity Range
1	1-2 months
2	3-4 months
3	5-6 months

The Tier Spread Table then sets the fixed costs of having spreads between different tiers in the combined commodity. A Tier Spread Table might look like table 1.7. These charges are typically set by the exchange and are dependent on the underlying asset of the combined commodity. To decide which spreads get what charge applied to them and in what order, a Spread Priority Table is also required.

Table 1.7 The Tier Spread Table

Tier	Maturity Range	Tier 1	Tier 2	Tier 3
1	1-2 months	50 USD	-	-
2	3-4 months	80 USD	60 USD	-
3	5-6 months	90 USD	100 USD	70 USD

Table 1.8 The Spread Priority Table

Priority	1	2	3	4	5	6
Tier Spread	1 to 1	2 to 2	3 to 3	1 to 2	1 to 3	2 to 3

The portfolio is then organized by tier and position to form maturity spreads. Recall the example portfolio.

Example Portfolio

Instrument	**Future**	**Call**	**Future**	**Future**
Position	10	-5	15	-5
Maturity (days)	90	60	25	150

It contains orders with different maturities, but also of different instruments. To properly form spreads between a future and a call option, the concept of *position delta* is introduced to relate the price movements of different instruments to each other.

Position and Composite Delta

The delta of a future or an option on a future is the sensitivity its price has to a price change of its underlying asset. Mathematically, it is defined as:

$$\Delta = \frac{\partial P}{\partial S}$$

where

Δ is delta
P is the price of the order
S is the price of the underlying asset

It is clear from the definition that the delta of a futures contract is 1. For an option, the calculation is more involved. As the scenario option prices calculated in the risk arrays use the Black-Scholes pricing model, the deltas for calls and puts are derived analytically from this model. The explicit derivation is given in the Appendix. The expression for the deltas of the two options are:

$$\Delta_c = N(d_1)$$

$$\Delta_p = -N(-d_1) = N(d_1) - 1$$

SPAN makes use of the position delta to relate it to different instruments in the combined commodity. The position delta is defined as:

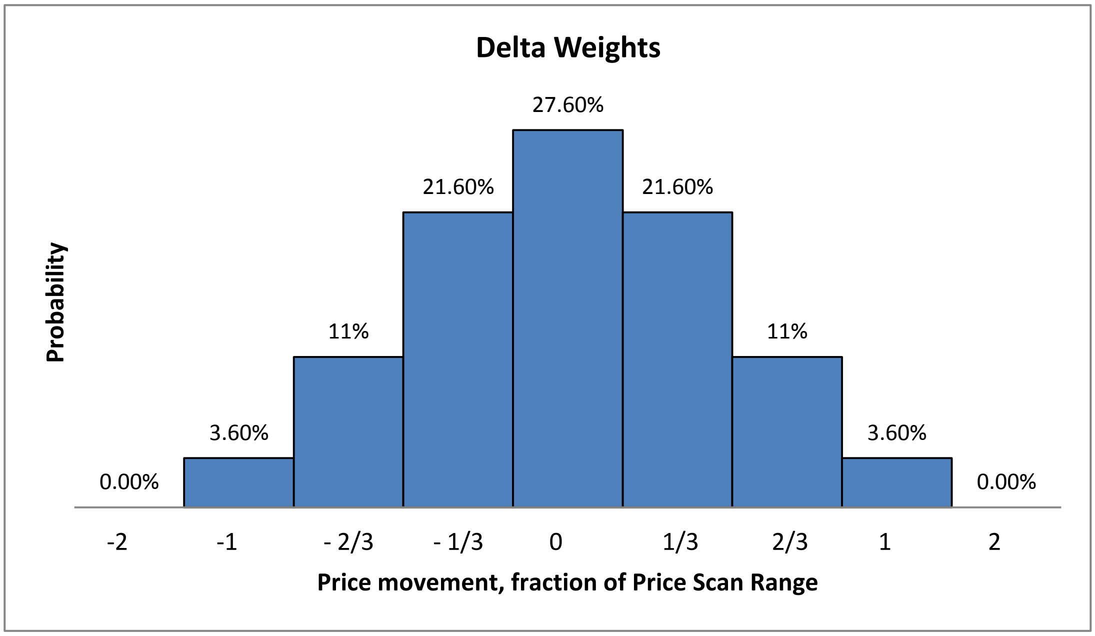
$$Position\ delta = Position * Composite\ delta$$

It is an indicator of how the value of the position of an order is affected by a price change in its underlying asset. The *composite delta* is in turn calculated using the risk array of the order. The composite delta of a future is always 1 since its price is determined directly by the price of its underlying asset. For options, however, the composite delta ranges from -1 to 0 for puts and 0 to 1 for calls.

Instrument	**Composite Delta**
Future	$\delta = 1$
Call	$0 < \delta < 1$
Put	$-1 < \delta < 0$

The composite delta multiplies the probability of a given risk scenario occurring with the option delta at that risk scenario and sums this product for all 16 risk scenarios. The probabilities for the risk scenarios, called delta weights, are preconfigured and can look like figure 1.2.

Figure 1.2 Delta Weights



The composite delta for the call option of the portfolio is calculated as in table 1.9:

Table 1.9 Short 2 month steel future call composite delta

	PC	VC	W	Price	Implied Volatility	Option Price	Delta	Delta Weight	Weighted Delta
1	0	1	100%	1200	30%	54.4	0.44	13.8%	0.06
2	0	-1	100%	1200	10%	9.2	0.26	13.8%	0.04
3	+1/3	1	100%	1232	30%	69.7	0.51	10.8%	0.06
4	+1/3	-1	100%	1232	10%	20.6	0.45	10.8%	0.05
5	-1/3	1	100%	1168	30%	41.4	0.37	10.8%	0.04
6	-1/3	-1	100%	1168	10%	3.3	0.12	10.8%	0.01
7	+2/3	1	100%	1264	30%	87.1	0.58	5.5%	0.03
8	+2/3	-1	100%	1264	10%	38.4	0.65	5.5%	0.04
9	-2/3	1	100%	1136	30%	30.6	0.30	5.5%	0.02
10	-2/3	-1	100%	1136	10%	0.9	0.04	5.5%	0
11	+3/3	1	100%	1296	30%	106.7	0.64	1.8%	0.01
12	+3/3	-1	100%	1296	10%	62	0.82	1.8%	0.01
13	-3/3	1	100%	1104	30%	21.9	0.24	1.8%	0
14	-3/3	-1	100%	1104	10%	0.2	0.01	1.8%	0
15	2	0	35%	1267	20%	64.3	0.60	0%	0
16	-2	0	35%	1133	20%	11.7	0.20	0%	0
								Composite Delta	**0.37**

Its corresponding position delta is:

$$Position\ delta = -5 * 0.37 = -1.9$$

Recalling that the composite delta for futures is 1, the example portfolio summary can be extended to include the position deltas of all orders:

Example Portfolio

Instrument	**Future**	**Call**	**Future**	**Future**
Position	10	-5	15	-5
Maturity (days)	90	60	25	150
Position Delta	10	-1.9	15	-5

The Delta Spread Table

The portfolio is now assigned spreads by constructing a Delta Spread Table, using the divisions in table 1.6. Table 1.10 shows the corresponding table for the portfolio.

Table 1.10 Delta Spread Table

Tier	Long	Short
1	15	-1.9
2	10	0
3	0	-5

Consulting the Priority Spread Table, any spreads within tier 1 are handled first. Within tier 1, 1.9 spreads can be formed. The Tier Spread Table sets the charge for this spread to 50 USD. The first spread charge is thus:

$$Tier\ spread\ charge = 1.9 * 50 = 95\ USD$$

The Delta Spread Table is updated to reflect that the spreads within tier 1 have been consumed and the corresponding charge recorded.

Table 1.11 Updated Delta Spread Table

Tier	Long	Short
1	13.1	0
2	10	0
3	0	-5

Next, spreads within tiers 2 and 3 are considered. Since no spreads can be formed, these steps are skipped. The next priority spread is between tiers 1 and 2. Again, since both position deltas are positive, this step is also skipped. Between tiers 1 and 3, however, 5 spreads can be formed for a unit charge of 90 USD.

$$Tier\ spread\ charge = 5 * 90 = 450\ USD$$

The updated Delta Spread Table is given in table 1.9. No more spreads can be formed within the combined commodity at this point, and the total Intermonth Spread Charge is the sum of the contributions from each tier spread:

$$Intermonth\ Spread\ Charge = 95 + 450 = 545\ USD$$

Table 1.12 Final Delta Spread Table

Tier	Long	Short
1	8.1	0
2	10	0
3	0	0

This completes the calculation for the Intermonth Spread Charge. For larger combined commodities that contain orders with longer maturities, more tiers are formed to accommodate these orders, but the general method of calculation remains the same. The margin requirement formula can now be further populated:

$$\begin{aligned} &Initial\ Margin\ Requirement \\ &\quad = \max(1874.5 + 545 + Delivery\ Month\ Charge \\ &\quad - Intercommodity\ Spread\ Credit, Short\ Option\ Minimum\ Charge) \\ &\quad - \ Net\ Option\ Value \end{aligned}$$

1.3 The Delivery Month Charge

The Delivery Month Charge is closely related to the Intermonth Spread Charge in that it adds charges to combined commodities based on the maturities of its containing orders. However, whereas the Intermonth Spread Charge divides orders into different tiers and assigns charges to spreads formed within and between these tiers, the Delivery Month Charge is only applicable to orders whose maturity is in the delivery month. In the case of a futures contract of a commodity with delivery on a specific date, SPAN assigns an additional charge when delivery is less than one month from today. This step accounts for risks associated with the actual delivery process, such as transportation and storage.

Specifically, the Delivery Month Charge assigns one charge to each spread formed using deltas from an order with maturity less than one month, called the *spread charge*. In addition, it adds an *outright charge* to deltas of orders in the delivery month that remain unconsumed. An example of such charges are given in table 1.13.

Table 1.13 Delivery Month Charges

Charge	
Spread	25 USD
Outright	50 USD

Consider the example portfolio again.

Example Portfolio

Instrument	**Future**	**Call**	**Future**	**Future**
Position	10	-5	15	-5
Maturity (days)	90	60	25	150
Position Delta	10	-1.9	15	-5

If a table similar to Delta Spread Table is formed, but expanding the tiers into individual maturity months instead, the following table is obtained:

Table 1.14 Expanded Delta Spread Table

Month	Long	Short
1	15	0
2	0	-1,9
3	10	0
4	0	0
5	0	-5

The Delivery Month Charge for this combined commodity would be calculated as follows. First, any spreads within the delivery month are considered. In this case, none can be formed. Between months 1 and 2, 1.9 spreads can be formed. Consulting table 1.13, this translates to a charge of:

$$Delivery\ month\ spread\ charge = 1.9 * 25 = 47.5\ USD$$

Table 1.14 is updated to reflect that these deltas have been consumed.

Table 1.15 Expanded Delta Spread Table

Month	Long	Short
1	13.1	0
2	0	0
3	10	0
4	0	0
5	0	-5

The only remaining spreads to be formed are between months 1 and 5.

$$Delivery\ month\ spread\ charge = 5 * 25 = 125\ USD$$

Table 1.16 Expanded Delta Spread Table

Month	Long	Short
1	8.1	0
2	0	0
3	10	0
4	0	0
5	0	0

The remaining 8.1 unconsumed deltas in the delivery month are assigned an outright charge according to table 1.13:

$$Delivery\ month\ outright\ charge = 8.1 * 50 = 405\ USD$$

This sums to a combined Delivery Month Charge of:

$$Delivery\ Month\ Charge = 47.5 + 125 + 405 = 577.5\ USD$$

The initial margin requirement formula is updated once more:

$$\begin{aligned} &Initial\ Margin\ Requirement \\ &\qquad = \max(1874.5 + 545 + 577.5 \\ &\qquad - Intercommodity\ Spread\ Credit, Short\ Option\ Minimum\ Charge) \\ &\qquad - \ Net\ Option\ Value \end{aligned}$$

1.4 The Intercommodity Spread Credit

Correlations and offsetting effects

The fourth step in SPAN is unique in that it is not a charge but a credit; it reduces the overall margin requirement for a portfolio. The reasoning behind this that positions in two different assets which the exchange considers to have a correlation between them can have an offsetting effect to the overall risk exposure of the portfolio. In these cases, a credit rate is assigned to such positions. For example, if an exchange considers the price of silver to be positively correlated with the price of gold, a credit rate on opposing positions in these assets is set. This represents the belief that losses in a gold long position, due to a decrease in gold price, is partially offset by gains in a short position on silver, due to a accompanying decrease in silver price. A portfolio with a long position in silver and a short position in gold would thus have its overall margin requirement reduced.

The Intercommodity Spread Credit is the sum of all such credit assigned to pairs of combined commodities. The partial credit is calculated using this formula:

$$Partial\ Spread\ Credit = Credit\ Rate * Spreads * Weighted\ Future\ Price\ Risk$$

Here, the *credit rate* is the rate set by the exchange between a specific pair of assets mentioned above. The *spreads* are formed between the Combined Commodities set by the credit rate. To do this, all the position deltas in each combined commodity are added up to form the so-called *net position delta*, and then spreads are formed exactly like in the Intermonth Spread Charge and Delivery Month Charge steps. For a credit rate on positions of steel and copper, the combined commodities containing instruments with these underlying assets are considered and spreads are formed from the net position deltas of both.

The Weighted Future Price Risk

The *Weighted Future Price Risk* is calculated like so:

$$Weighted\ Future\ Price\ Risk = \frac{Future\ Price\ Risk}{|Net\ Position\ Delta|}$$

where

$$Future\ Price\ Risk = Volatility\ Adjusted\ Scanning\ Risk - Time\ Risk$$

Note that in the case where the net position delta is zero, the Weighted Future Price Risk is set to zero. The *Volatility Adjusted Scanning Risk* and *Time Risk* are both components extracted from the total risk array of the combined commodity. Recall that for the example portfolio, this risk array is given in table 1.5.

The Volatility Adjusted Scanning Risk is the Scanning Risk of the combined commodity but with the risk component due to variations in volatility removed. This is done using the Active Scenario that is assigned to the combined commodity when the Scanning Risk is calculated. Each scenario in the risk array has a unique *Paired Scenario*, where the change in volatility is equal but opposite. To adjust the Scanning Risk for variations in volatility, SPAN takes the Scanning Risk of the Active Scenario and its Paired Scenario and calculates the average.

$$Volatility\ Adjusted\ Scanning\ Risk = \frac{Active\ Scenario + Paired\ Scenario}{2}$$

Recall that for the example portfolio, the Active Scenario is 13. Consulting table 1.1, scenario 13 corresponds to a volatility change of +1 Volatility Scan Range with a price change of -1 Price Scan Range. Its Paired Scenario is thus scenario 14, with equal price change and equal but opposite volatility change. Taking the Scanning Risks of both scenarios, listed in the total risk array in table 1.5, yields:

$$Volatility\ Adjusted\ Scanning\ Risk = \frac{1874.5 + 1766}{2} = 1820.25\ USD$$

The Time Risk is calculated similarly, but only concerns scenarios 1 and 2 of the risk array of the portfolio.

$$Time\ Risk = \frac{Scenario\ 1 + Scenario\ 2}{2}$$

Yet again referring to table 1.5, the following Time Risk is calculated:

$$Time\ Risk = \frac{117.1 - 108.9}{2} = 4.1\ USD$$

Thus:

$$Future\ Price\ Risk = 1820.25 - 4.1 = 1816.15\ USD$$

The only component remaining is the net position delta for the example portfolio.

Example Portfolio

Instrument	**Future**	**Call**	**Future**	**Future**
Position Delta	10	-1.9	15	-5

Adding up the position deltas yields:

$$Net\ Position\ Delta = 10 - 1.9 + 15 - 5 = 18.1$$

Thus, the Weighted Future Price Risk (WFPR) is:

$$Weighted\ Future\ Price\ Risk = \frac{1816.15}{|18.1|} = 100.34$$

Now, the Intercommodity Spread Credit can be calculated. For the attentive reader, it has been evident throughout that the Intercommodity Spread Credit will be zero for the example portfolio, since it only contains one combined commodity, and as such, no spreads can be formed. However, for illustrative purposes, the calculation will be performed here using an imaginary second combined commodity, with copper as the underlying asset.

Imaginary Example Portfolio

Combined commodity	Steel	Copper
Net Position Delta	18.1	-15
WFPR	100.34	70

Credit Rate

Assets	Steel : Copper
Ratio	1 : -1
Credit	40%

Using the imaginary second combined commodity, 15 spreads can be formed. The credit rate is for opposing positions in these assets and is set to 40 percent. For the steel combined commodity, the partial spread credit comes to:

$$Steel\ Partial\ Spread\ Credit = 40\% * 15 * 100.34 = 602.04\ USD$$

Similarly, for the copper combined commodity the result is:

$$Copper\ Partial\ Spread\ Credit = 40\% * 15 * 70 = 420\ USD$$

The imaginary Intercommodity Spread Credit would thus be:

$$Intercommodity\ Spread\ Credit = 602.04 + 420 = 1022.04\ USD$$

However, due to there being only a single combined commodity present in the example portfolio, the actual Intercommodity Spread Credit is zero.

$$\begin{aligned} &Initial\ Margin\ Requirement \\ &\qquad = \max(1874.5 + 545 + 577.5 - 0, Short\ Option\ Minimum\ Charge) \\ &\qquad - \ Net\ Option\ Value \end{aligned}$$

1.5 The Short Option Minimum Charge

Option Risk

A long option is an instrument with a clearly defined downside, namely the price of the option. Consider a long call option on a stock with stock price S_t, strike price K, and option price C_0. The payoff at maturity of the option is:

$$payoff = \max(S_T - K, 0) - C_0 = \begin{matrix} S_T - K - C_0 & if\ S_T > K \\ -C_0 & if\ S_T \leq K \end{matrix}$$

In other words, the largest possible loss is $-C_0$. Similarly for a long put option with price P_0:

$$payoff = \max(K - S_T, 0) - P_0 = \begin{matrix} S_T - K - P_0 & if\ S_T \leq K \\ -P_0 & if\ S_T > K \end{matrix}$$

Clearly, the risks associated with long call and put options are low, since the potential downside has a clearly defined lower bound, whereas the potential upside does not. However, short options reverse these conditions completely, and ensure a bounded upside but an unbounded downside. This is why SPAN dedicates an entire step towards assigning extra charges to all short options in a portfolio.

Portfolio Options Counting

Any market movement in the underlying asset of a short option can lead to large losses, and the Short Option Minimum Charge accounts for this by setting a fixed charge on each short option in a portfolio. SPAN has two modes of determining the Short Option Minimum Charge. The first is to count all short options in the portfolio, regardless of type, and use the total sum as the amount of options to be charged. Alternatively, the larger of the separate sums of put and call options can be chosen instead. The latter has been selected for use in this paper. The options are then multiplied by their corresponding fixed short option charge and summed to produce the Short Option Minimum Charge.

$$Short\ Options = max(\sum calls, \sum puts)$$

$$Short\ Option\ Minimum\ Charge = \sum_{Short\ Options} |Short\ Option\ Position| * \ Short\ Option\ Charge$$

The fixed charge can be derived in different ways, but the one used here is based on the price scan range. The *short option charge* is set to be:

$$Short\ Option\ Charge = 0.05 * Price\ Scan\ Range$$

This parameter can be changed to reflect the attitude of the exchange towards short options trading. Since the price scan range is determined on a combined commodity basis, the short option charge is also different for each combined commodity. For the example portfolio, the calculation is rather simple.

Example Portfolio

Price Scan Range	96			
Instrument	**Future**	**Call**	**Future**	**Future**
Position	10	-5	15	-5

$$Short\ Option\ Charge = 0.05 * 96 = 4.8\ USD$$

$$Short\ Option\ Minimum\ Charge = 4.8 * 5 = 24\ USD$$

It is clear from the formula that the Short Option Minimum Charge is the lower bound of the initial margin requirement.

$$Initial\ Margin\ Requirement = \max(1874.5 + 545 + 577.5 - 0, 24) - \ Net\ Option\ Value$$

1.6 The Net Option Value

The final step in SPAN is to calculate the Net Option Value of the portfolio. This is a trivial step, as the Net Option Value is obtained by multiplying each option price with its position and summing over the entire portfolio.

$$Net\ Option\ Value = \sum (Option\ Price * Option\ Position)$$

As short options thus have a negative contribution, the Net Option Value can be either positive or negative.

Applying this to the example portfolio is simple.

Example Portfolio

Instrument	**Future**	**Call**	**Future**	**Future**
Position	10	-5	15	-5
Price (USD)	1200	31	1100	1300

$$Net\ Option\ Value = 31 * -5 = -155\ USD$$

With this, the initial margin requirement for the example portfolio can now be calculated.

$$\begin{aligned} Initial\ Margin\ Requirement &= \max(1874.5 + 545 + 577.5 - 0, 24) - \ (-155) \\ &= max(2997{,}24) + 155 = 3152\ USD \end{aligned}$$

With this, the demonstration of the SPAN methodology of assigning initial margin requirements to portfolios is complete. The next section will detail the problem formulation mentioned in the introduction.

2. Problem Formulation

With the relevant method of risk analysis thoroughly presented and demonstrated on a simple example portfolio, it is now time to properly introduce the central problem this essay is meant to address and explore possible means of solving it.

The SPAN methodology is a fairly extensive analysis, but can with the help of modern computing capacity be performed on rather large portfolios. If a portfolio of, say, 100 matched orders was to be subjected to the SPAN analysis to calculate an initial margin requirement, the overall calculation can be handled rather quickly. This is a relevant analysis for an exchange that wishes to perform a check on client accounts to ensure compliance with current margin requirements. Here, the accounts are static, consisting only of orders made by the client that have been matched in the exchange system; that is to say, a seller has been matched with a buyer. A rough estimate of the complexity of this analysis is on the order of $O(16N)$, with N being the number of matched orders in the account. The multiplier 16 comes from the Scanning Risk step, detailed above, where each order is subjected to the 16 risk scenarios for analysis to determine the worst price movement given current market data.

This analysis, however, is inadequate for exchange members which provides customers sponsored access to the exchange. By letting these customers, who are often trading firms which employs high frequency trading technology, perform trades in its name, the exchange member requires a means of controlling these trades, and ensuring that they do not result in undue increase in the margin requirement set on it by the exchange. As such, they need to perform a similar risk analysis as the exchange, but on potential portfolios of unmatched orders rather than on portfolios of orders already in the system. Simply performing a risk analysis of matched orders that have already been placed is to be one step too late from a risk management stand point.

Take the portfolio of 100 matched orders mentioned above. Say that this portfolio is the result of trades made in the name of an exchange member by a sponsored party. This portfolio of trades might give rise to an adjustment of the margin requirement of the relevant exchange member. Without a risk management system in place to perform a pre-trade analysis, this adjustment is completely unknown to the exchange member until the orders are matched. In other words, it has no way of determining the level of risk it commits to until it is already committed. What is required is a risk analysis that does not consider a static portfolio of matched orders already in the exchange system, but rather the set of potential portfolios given an order book of unmatched orders, when such trades can still be cancelled if deemed necessary.

Such a pre-trade risk analysis would in its simplest form have as its input not the static number of matched orders N as above, but rather all the possible combinations of unmatched orders 2^M, where M is the size of the order book of unmatched orders. To put it in concrete terms, consider the portfolio of 100 matched orders. Say that these orders were taken from an order book of a client with sponsored access, which initially consisted of 150 unmatched orders. The pre-trade risk analysis would take as input the order book and consider the 2^{150} different possible combinations of orders to find a portfolio configuration of potentially matched orders whose margin requirement is the highest. In this manner, the level of risk being committed to by sponsored clients in the name of the exchange member can be controlled before the trade is performed. The effect of the portfolio of 100 matched orders above would be known beforehand, since it would necessarily be less than the worst-case portfolio yielded by the pre-trade risk analysis. This is the brute force method, where every possible outcome is considered and an exact worst-case outcome will always be found.

However, given the high-frequency trading environment of today, where order books are constantly adjusted and in miniscule time scales, this pre-trade risk analysis would have to be performed in real time without noticeably affecting latency. A brief look at the example above illuminates the difficulties inherent in the brute force method. Even with the relatively modest size of 150 orders, the number of potential portfolios are:

$$number\ of\ portfolios = 2^{150} \approx 1.43 * 10^{45}$$

Each of these potential portfolios would have to be subjected to a full SPAN analysis and have its initial margin requirement recorded to find the worst-case portfolio. This simply cannot be done within the time frames demanded by exchange members and customers with sponsored access. What is needed is a method of finding the worst-case portfolio, or a reasonable approximation of it, without a computational complexity of $O(2^M)$ but rather $O(M)$, linear to the size of the order book. This is the central problem.

2.1 A possible avenue: Dynamic Programming

There is a large class of mathematical problems where a solution is readily available through an exhaustive search, but where such a search is not feasible. A notable example is the traveling salesman problem: the salesman needs to visit each of a given set of destinations exactly once and then return to the origin. He wants to find the shortest possible path that achieves this. The problem is easy to understand but computationally hard to solve for larger sets of destinations. Another example is the knapsack problem, where given a set of items of weights and values, find the collection of items where the total weight is less than a set limit and where the total value is as high as possible. This particular problem can be solved by dynamic programming. A closer look at this solution is illuminating.

A brute force solution of the knapsack problem uses the same approach as the brute force method described above: try out all possible combinations of the items and pick the combination with the highest value that complies with the weight requirement. However, this analysis does more work than necessary. Dynamic programming, on the other hand, solves simpler sub-problems of the initial problem, leverages these computations to solve incrementally more complex sub-problems to finally arrive at a solution for the full problem. A novel dynamic programming solution to the knapsack problem will be demonstrated below (Otten).

An example

Consider a set of three items with weights and values:

$$\boldsymbol{w} = (1,2,3)$$

$$\boldsymbol{V} = (5,6,9)$$

The weight limit is set to be

$$w_{max} = 4$$

The exhaustive search simply tries each and every combination of items to include in the knapsack, discards the combinations that violates the weight limit, and picks the one out of the remaining combinations that has the highest total value.

$$\boldsymbol{t_1} = (0,0,0): \quad \boldsymbol{t_1} \cdot \boldsymbol{w} = 0 \leq w_{max}, \boldsymbol{t_1} \cdot \boldsymbol{V} = 0$$

$$\boldsymbol{t_2} = (0,0,1): \quad \boldsymbol{t_2} \cdot \boldsymbol{w} = 2 \leq w_{max}, \boldsymbol{t_2} \cdot \boldsymbol{V} = 9$$

$$\boldsymbol{t_3} = (0,1,0): \quad \boldsymbol{t_3} \cdot \boldsymbol{w} = 3 \leq w_{max}, \boldsymbol{t_3} \cdot \boldsymbol{V} = 6$$

$$\boldsymbol{t_4} = (0,1,1): \quad \boldsymbol{t_4} \cdot \boldsymbol{w} = 5 > w_{max}, \boldsymbol{t_4} \cdot \boldsymbol{V} = 15$$

$$\boldsymbol{t_5} = (1,0,0): \quad \boldsymbol{t_5} \cdot \boldsymbol{w} = 1 \leq w_{max}, \boldsymbol{t_5} \cdot \boldsymbol{V} = 5$$

$$\boldsymbol{t_6} = (1,0,1): \quad \boldsymbol{t_6} \cdot \boldsymbol{w} = 4 \leq w_{max}, \boldsymbol{t_6} \cdot \boldsymbol{V} = 14$$

$$\boldsymbol{t_7} = (1,1,0): \quad \boldsymbol{t_7} \cdot \boldsymbol{w} = 3 \leq w_{max}, \boldsymbol{t_7} \cdot \boldsymbol{V} = 11$$

$$\boldsymbol{t_8} = (1,1,1): \quad \boldsymbol{t_8} \cdot \boldsymbol{w} = 6 > w_{max}, \boldsymbol{t_8} \cdot \boldsymbol{V} = 20$$

For a limited number of items, the combinations are not too many; in this case, only $2^3 = 8$. For a larger number of items *n*, however, this approach becomes cumbersome, and the computation time grows proportionally with 2^n.

A dynamic approach instead constructs and fills a table consisting of solutions to sub-problems to this problem, to be used in more complex solutions. Such a table consists of the maximum possible total value $V[i,w]$ of items {0, 1,..., *i*} subject to weight limit *w*, where $0 \leq i \leq 3$ and $0 \leq w \leq 4$. The cells are filled bottoms-up using the following formulae:

$$V[0,w] = 0 \; for \; all \; 0 \leq w \leq w_{max} \quad (1)$$

$$V[i,w] = \begin{matrix} V[i-1,w] \quad if \quad w_i > w \\ \max(V[i-1,w], v_i + V[i-1,w-w_i]) \quad if \quad w_i \leq w \end{matrix} \quad (2)$$

Formula (1) states that with no included items, the maximum possible value is 0. Formula (2) moves row-wise in the table along the *w*-axis, and for each cell the following determination is made to calculate $V[i,w]$: either item *i* is is discarded because it singlehandedly violates the weight limit *w, or* it is included in the solution. If the item is discarded, the solution is equal to $V[i\text{-}1,w]$, which is the solution to the sub-problem where item *i* was not considered.

If item i is included, the new solution now consists of the value v_i, contributed directly from the newly included item i, and the best possible solution to the sub-problem $V[i\text{-}1,w\text{-}w_i]$, with remaining items {0, 1, ..., i-1} subject to remaining weight limit w -w_i. Moving row-wise from left to right in the table, this solution has already been calculated and can be fetched from the correct table cell. In other words, the solution to increasingly more complex sub-problems make use of solutions to more simple ones that have already been calculated. This new solution is only used if it exceeds the solution where item i was not included. When the table is filled, cell V[3,4] is the solution to the initial problem. The process will now be illustrated.

The initial table is only partly populated as per the first formula.

$V[i,w]$	w = 0	1	2	3	4
i = 0	0	0	0	0	0
1					
2					
3					

The first cell under consideration is V[1,0]. Item 1 has weight w_1 = 1 and value v_1= 5, and cannot be included in the solution, since

$$w_1 > w$$

Thus, formula (2) gives

$$V[1,0] = \mathrm{V}[0,0] = 0$$

$V[i,w]$	w = 0	1	2	3	4
i = 0	0	0	0	0	0
1	0				
2					
3					

In cell V[1,1], item 1 can be included. The solution is

$$V[1,1] = \max(V[0,1], v_1 + V[0,1 - w_1]) = \max(0,5 + V[0,0]) = \max(0,5) = 5$$

Evidently, including item 1 is the best choice.

$V[i,w]$	w = 0	1	2	3	4
i = 0	0	0	0	0	0
1	0	5			
2					
3					

The rest of the cells in the row are found to have the same best solution: to include item 1. Cells V[2,0] and V[2,1] are also determined as before.

V[i,w]	w = 0	1	2	3	4
i = 0	0	0	0	0	0
1	0	5	5	5	5
2	0	5			
3					

The interesting calculation comes at cell V[2,2]. With weight w_2 = 2 and value v_2 = 6,

$$w_2 \leq w$$

and so item 2 can safely be included in the solution. Formula 2 now gives

$$V[2,2] = \max(V[1,2], v_2 + V[1,2 - w_2]) = \max(5,6 + V[1,0]) = \max(5,6) = 6$$

V[i,w]	w = 0	1	2	3	4
i = 0	0	0	0	0	0
1	0	5	5	5	5
2	0	5	6		
3					

The next cell is:

$$V[2,3] = \max(V[1,3], v_2 + V[1,3 - w_2]) = \max(5,6 + V[1,1]) = \max(5,11) = 11$$

Continuing this, the rest of the table is promptly filled.

V[i,w]	w = 0	1	2	3	4
i = 0	0	0	0	0	0
1	0	5	5	5	5
2	0	5	6	11	11
3	0	5	6	11	14

The formula for the solution to the initial problem is thus:

$$\begin{aligned} V[3,4] &= \max(V[2,4], v_3 + V[2,4 - w_3]) = \max(11,9 + V[2,4 - 3]) = \max(11,9 + V[2,1]) \\ &= \max(11,14) = 14 \end{aligned}$$

The dynamic programming approach elegantly makes use of computations of sub-problems to arrive at solutions to increasingly more complex problems. For a number of items n, the dynamic programming approach would in this example grow in computation time proportionally with $4n$. For the general knapsack problem formulation with weight limit w_{max}, it would grow proportional to $w_{max}n$.

Possible application to SPAN

The dynamic programming approach provides a solution in linear time to a combinatorial problem. The example above immediately invites a possible application to a SPAN pre-trade risk analysis for order books of unmatched orders: the brute force approach has prohibitive time complexity whereas the dynamic programming solution is solved in linear time. It is easy to envision the solution to the worst-case portfolio problem being constructed in an analogous manner to the simple knapsack example.

For a given set of unmatched orders, solutions to increasingly larger order books are computed, using previously computed solutions for smaller order books. In this case, the solutions are simply the largest possible initial margin requirement. An algorithm using this approach would have a linear time complexity and thus be suitable as a SPAN pre-trade risk analysis.

However, this approach is fundamentally flawed in one important aspect, and this highlights the complexity of the SPAN risk analysis. By using solutions to sub-problems in solving more complex problems, the dynamic programming approach assumes that the items included in the solution are independent. Inclusion of one item does not alter the contributions to the solution of the other items. In the simple knapsack problem above, the interpretation is obvious: including item 3 in the knapsack does not alter the weight or value of any other items in the solution.

In SPAN, there is only a single step that conforms to the notion of independence: the Net Option Value. In every other step, the inclusion of orders in a portfolio has a collective impact on the final result that simply renders simpler solutions irrelevant.

In the Scanning Risk, each order is subjected to risk scenarios independently, but in summing the results of all risk scenarios together to find the worst outcome, the contributions of each are dependent. The case is similar for the Intermonth Spread Charge, Delivery Month Charge and Intercommodity Spread Credit. By relying on tiers and forming spreads in a specific pattern, no single order can be viewed as having an independent contribution. Finally, the Short Option Minimum Charge is an input to a max() function and as such is not always present in the final solution. Clearly, dynamic programming does not offer a simple answer to the problem.

2.2 An alternate approach: The marginal contribution

The dynamic programming approach requires that each possible combination of orders in a portfolio are independent. For three orders, including the third to a portfolio of two cannot have an effect on the contributions of the first two orders. SPAN clearly does not exhibit this characteristic. It is very difficult to determine beforehand the effect on the margin requirement of including or excluding an order in a portfolio. If a marginal contribution of a single order to the final margin requirement could somehow be found, this would be a solid basis for inclusion in or exclusion from the worst-case portfolio: if the marginal contribution is positive, thus increasing the margin requirement, the order is included; otherwise, it is excluded.

The difficulty in finding this comes in large part from the fact that SPAN does not consist of a single calculation but rather a collection of calculations, each treating a single order in a different way. An obvious way to side-step this quandary, however, is to focus solely on a single step in SPAN, and determine how the inclusion of an order to the portfolio affects the contribution from that step only.

Considering the steps in SPAN, the steps that most lend themselves to such an analysis are the Net Option Value and the Scanning Risk. While the Net Option Value is very straight-forward in this regard, it is limited to only options and as such loses its relevance to a large portion of potential orders. Thus, the Scanning Risk is more appropriate. The following section will explore the possibility of finding a marginal contribution of a single order to the Scanning Risk, and how such a contribution might be used to find the worst-case portfolio.

Finding the marginal Scanning Risk

The problem put in simple terms is this: given an order slated for inclusion to a portfolio, can the new Scanning Risk be easily determined? In other words: can the marginal contribution to the Scanning Risk from a single order be determined independently? The difficulty of this has already been discussed. The marginal contribution of an order to the total sum for each of the 16 risk scenarios is well-defined, but since the Scanning Risk is set to be the largest of these only, the matter is less clear.

There is, however, one approach that proves to be very useful in determining the marginal contribution, and it is the central component of the proposed algorithm that will be outlined next. The difficulty presented above essentially boils down to the fact that the prevailing risk scenario that determines the Scanning Risk, the Active Scenario, cannot be known by looking at only a single order. That single order can have both negative and positive contributions to the Scanning Risk, depending on which risk scenario is ultimately chosen. Therefore, all risk scenarios have to be considered beforehand.

A suitable criterion

If it is known which particular risk scenario is chosen as Active Scenario, the marginal contribution of an order is easily determined. This then provides a suitable criterion for inclusion of an order to the worst-case portfolio:

On each order and for all 16 risk scenarios, the following check is performed:

Given risk scenario X, does the current order have a positive contribution to the Scanning Risk of the portfolio?

- For a positive or neutral contribution, the order is added to the worst case portfolio for the given risk scenario.
- For a negative contribution, the order is excluded from the worst case portfolio for the given risk scenario.

This gives a set of 16 specific combinations of orders particular to each risk scenario where all orders contribute non-negatively to the Scanning Risk of the portfolio. An order that is included in the portfolio given one certain risk scenario might not be included given another risk scenario, and so on. Now, the total Scanning Risks of these combinations are compared to find the largest, thereby determining the Active Scenario in the normal way.

The virtue of this approach is that the resulting Active Scenario is irrelevant, since each risk scenario has an associated combination of orders that maximizes the Scanning Risk for that particular risk scenario. This approach is promising enough to warrant further investigation and means of improvement and will be the central focus of the next section. An algorithmic implementation of the idea introduced here is also presented.

3. The Algorithm

3.1 Outlining the initial criterion

For an algorithm tasked with selecting a set of orders out of an order book that maximizes the initial margin requirement, the so-called worst-case portfolio, the initially proposed criterion for inclusion of an order is formulated as:

Criterion 1: Given risk scenario X, Scanning Risk ≥ 0

where each order is checked 16 times, once for each risk scenario. Each order is to be included in the worst-case portfolio for that particular risk scenario if its contribution to the Scanning Risk is positive. This gives 16 different configurations of orders that each maximizes the Scanning Risk component for its respective risk scenario.

The configuration which produces the largest Scanning Risk by summing the contributions of its included orders is set as the worst-case portfolio, and the associated risk scenario is set as the Active Scenario. This worst-case portfolio is then subjected to the remaining steps in SPAN to yield the initial margin requirement.

Underlying assumptions

The underlying assumptions are

a) a worst-case portfolio is attained by maximizing the Scanning Risk component in SPAN

b) the maximum Scanning Risk is attained by ensuring that each order in the portfolio contributes positively to it.

Assumption b) is straight-forward. Suppose there exists an order that is not included in a worst-case portfolio that purportedly maximizes the Scanning Risk, and that this order has a non-negative contribution to the Scanning Risk. It is obvious that including this order would increase the overall Scanning Risk to produce a "better" worst-case portfolio.

Assumption a), however, is less clear. Is it warranted to claim that a maximum Scanning Risk necessarily implies a maximum overall initial margin requirement? It is not obvious that this is necessarily the case. However, any algorithm need not be analytically exact but rather produce results with acceptable accuracy. In this light, the question can be put another way: for a given order book, is the portfolio selection yielded by Criterion 1 close enough to the actual worst-case portfolio? How might such an accuracy be evaluated?

A demonstration

The best way to answer this is through simulation. For a lower number of orders, the accuracy of any algorithm can be effectively evaluated by comparing its results to that of the brute force method. The procedure is simple: let the brute force method run through all possible combinations of orders in an order book to arrive at the configuration that yields the highest initial margin, apply the algorithm to the same order book to select a worst-case portfolio, and compare results. To achieve this, a JAVA implementation of the SPAN methodology as outlined above has been written for purposes of simulation.

To better get a grasp of the selection process, Criterion 1 is easily applied to the example portfolio used previously. For the purposes of this illustration, the order book here is said consists of the four orders included in the example portfolio. Would all four orders be selected for inclusion in the worst-case portfolio? As an initial indication, it is clear that Criterion 1 will, depending on the Active Scenario, select exclusively long or short futures for its worst-case portfolio, but never both in conjunction.

Example Portfolio

Instrument	**Future**	**Call**	**Future**	**Future**
Position	10	-5	15	-5
Maturity (days)	90	60	25	150
Price (USD)	1200	31	1100	1300
Position Delta	10	-1.9	15	-5

Referring to table 1.5 gives all the necessary information. It is given again below, but with a slight modification. The first time the table was used, each line in the table was static, indicating that the order configuration of the portfolio did not change with the risk scenario. This time, the algorithm evaluates each cell to determine if the order under a specific risk scenario is to be included in the worst-case portfolio. Cells that do not comply with Criterion 1 and contain negative contributions are marked in table 3.1.

Table 3.1 Portfolio Risk Array

	Future 10	Future 15	Future -5	Call -5	Total
1	0	0	0	117.1	117.1
2	0	0	0	-108.9	-108.9
3	-320	-480	160	193.3	-446.7
4	-320	-480	160	-52.1	-692.1
5	320	480	-160	52	692
6	320	480	-160	-138.4	501.6
7	-640	-960	320	280.1	-999.9
8	-640	-960	320	36.8	-1243.2
9	640	960	-320	-2	1278
10	640	960	-320	-150.4	1129.6
11	-960	-1440	480	378.4	-1541.6
12	-960	-1440	480	155.2	-1764.8
13	960	1440	-480	-45.5	1874.5
14	960	1440	-480	-154	1766
15	-670	-1005	335	166.3	-1173.7
16	670	1005	-335	-96.3	1243.7

Table 3.2 outlines the results of applying Criterion 1 to the example portfolio, where each risk scenario now has a worst-case portfolio configuration. Checking the totals of all risk scenarios, the largest is found under risk scenario 13.

Table 3.2 Worst-case Portfolio Risk Array, Criterion 1

	Future 10	Future 15	Future -5	Call -5	Total
1	0	0	0	117.1	117,1
2	0	0	0	-	0
3	-	-	160	193.3	353.3
4	-	-	160	-	160
5	320	480	-	52	852
6	320	480	-	-	800
7		-	320	280.1	600.1
8	-	-	320	36.8	356.8
9	640	960	-	-	1600
10	640	960	-	-	1600
11	-	-	480	378.4	858.4
12	-	-	480	155.2	635.2
13	960	1440	-	-	**2400**
14	960	1440	-	-	2400
15	-	-	335	166.3	501.3
16	670	1005	-	-	1675

The worst-case portfolio under Criterion 1 is thus determined to include the two long futures orders and exclude both the short future and call orders.

Worst-case of the Example Portfolio, Criterion 1

Instrument	**Future**	**Future**
Position	10	15
Maturity (days)	90	25
Price (USD)	1200	1100
Position Delta	10	15

The Active Scenario is set to 13, and the Scanning Risk is found to be

$$Scanning\ Risk_{crit\ 1} = 2400\ USD$$

Going through the remaining steps in SPAN, the following result is obtained for the worst-case portfolio under Criterion 1.

$$Intermonth\ Spread\ Charge_{crit\ 1} = 0$$

$$Delivery\ Month\ Charge_{crit\ 1} = 750$$

$$Intercommodity\ Spread\ Credit_{crit\ 1} = 0$$

$$Short\ Option\ Minimum\ Charge_{crit\ 1} = 0$$

$$Net\ Option\ Value_{crit\ 1} = 0$$

The results are expected. Since both orders are long positions in futures, no spreads can be formed within the combined commodity. The Delivery Month Charge thus consists solely of outright charges on the 15 deltas in the delivery month. The Intercommodity Spread Credit, Short Option Minimum Charge and Net Option Value are zero for obvious reasons: there is only a single combined commodity in the portfolio, and it contains no options.

This translates to

$$Initial\ Margin\ Requirement_{crit\ 1} = \max(2400 + 0 + 750 - 0, 0) - 0 = 3150\ USD$$

Recall that the result for the example portfolio when including all orders was

$$Initial\ Margin\ Requirement = 3152\ USD$$

which is higher than the supposedly worst-case result yielded by Criterion 1. Clearly, Criterion 1 is insufficient to select the worst-case configuration of orders to produce the highest possible margin requirement. This damning result notwithstanding, the worst-case portfolio selected by Criterion 1 will now be compared to the real worst-case portfolio, obtained using the brute force method.

The JAVA implementation of SPAN was configured to run through all the $2^4 = 16$ configurations of the orders that comprised the original example portfolio. The portfolio that produced the highest initial margin requirement is given below.

Worst-case of the Example Portfolio, Brute Force Method

Instrument	**Future**	**Call**	**Future**
Position	10	-5	15
Maturity (days)	90	60	25
Price (USD)	1200	31	1100
Underlying Price (USD)	-	1200	-
Strike (USD)	-	1250	-
Implied Volatility	-	20%	-

$$Scanning\ Risk_{brute} = 2347.88\ USD$$

$$Intermonth\ Spread\ Charge_{brute} = 91.84$$

$$Delivery\ Month\ Charge_{brute} = 704.08$$

$$Intercommodity\ Spread\ Credit_{brute} = 0$$

$$Short\ Option\ Minimum\ Charge_{brute} = 24$$

$$Net\ Option\ Value_{brute} = -155$$

$$Initial\ Margin\ Requirement_{brute} = \max(2347.88\ + 91.84 + 704.08\ - 0, 24) -\ (-155)$$
$$= \max(3143.8\ ,24) + 155 = 3298.8\ USD$$

To verify the result of Criterion 1 and gauge its accuracy, the following metric is used:

$$Margin\ Ratio_{crit\ 1} = \frac{Initial\ Margin\ Requirement_{crit\ 1}}{Initial\ Margin\ Requirement_{brute}} = \frac{3150}{3298.8} = 0.955$$

An algorithm that yields a result this close to that of brute force method is promising, given that the computational time of the brute method is orders of magnitudes larger for larger order books. To properly evaluate the performance of Criterion 1, this same "accuracy" check will be carried out multiple times for randomized order books of slightly larger size. Note that this simulation is severely inhibited by the verification step, and the computation time of the brute force method makes thorough simulations for order books of even moderate size unfeasible.

Before this simulation is carried out, a cursory overview of the JAVA implementation will be provided, where the randomization of orders and fixed parameters are detailed.

3.2 The JAVA implementation of SPAN

The JAVA application that models SPAN works exactly as has been detailed in the SPAN description section, with one significant exception: the Intercommodity Spread Credit is skipped entirely. Neither the algorithms nor the brute force method include that step in their SPAN calculation due to its nature as a credit rather than a charge. As such, none of the pertinent parameters relating to the Intercommodity Spread Credit are included here.

For simulation purposes, a large number of orders have to be generated quickly to check the accuracy of any algorithm against the brute force method. To this end, a set of fixed parameters and variable parameters are used. The fixed parameters remain the same for all simulations, whereas the variable parameters are perturbed for each generated order to produce a randomized portfolio. The parameters given below are all mock parameters, and should not be taken to reflect realistic market conditions. For the purposes of testing the algorithm, however, they are sufficient.

Fixed parameters

Time Horizon	2 days
Quantile	3 standard deviations
Volatility Scan Range	10%
Interest Rate	3%

Spread Priority Table

Priority	1	2	3	4	5	6	7	8
Tier Spread	1 to 1	2 to 2	3 to 3	4 to 4	5 to 5	1 to 2	1 to 3	1 to 4
Priority	**9**	**10**	**11**	**12**	**13**	**14**	**15**	
Tier Spread	1 to 5	2 to 3	2 to 4	2 to 5	3 to 4	3 to 5	4 to 5	

Tier Spread Table

Tier	Maturity Range	Tier 1	Tier 2	Tier 3	Tier 4	Tier 5
1	1-2 months	100 USD	-	-	-	-
2	3-4 months	110 USD	100 USD	-	-	-
3	5-6 months	120 USD	120 USD	100 USD	-	-
4	7-8 months	130 USD	140 USD	130 USD	100 USD	-
5	9-10 months	120 USD	150 USD	140 USD	150 USD	100 USD

Delivery Month Charges

Charge	
Spread	25 USD
Outright	50 USD

Variable parameters

Order Types Futures, put and call options

Underlying Asset	Oil	Steel	Copper	Silver	Gold	Zinc	Beef	Gas	Helium	Wheat
Price Baseline (USD)	8400	3000	1500	15000	25000	1000	4500	7500	10000	1500
Tier Scale Factor	1	0.36	0.18	1.79	2.98	0.12	0.54	0.89	1.19	0.18
Daily Volatility Baseline	1.75%	1.85%	1.5%	1.85%	2.25%	1.5%	2.5%	3%	4%	1.5%
Annual Implied Volatility	20%	15%	10%	25%	28%	10%	20%	25%	30%	10%

For each order, the randomization constraints are as follows:

Price Up to 5% deviation from price baseline
Maturity Interval From 1 to 120 days
Strike Price Up to 5% deviation from price baseline
Volatility Up to 50 % deviation from baseline

Implied Volatility	Up to 50 % deviation from baseline
Position	From -10 to 10
Option Price	Calculated using Black-Scholes

Here, deviation is meant to signify

$$Deviation = \frac{|Randomized\ Value - Baseline|}{Baseline}$$

The *Tier Scale Factors* are used to scale the tier spread charges and delivery month charges, so that each charge is proportional to the baseline price of the underlying asset. This is simplified from the real SPAN model, where the tier charges are set on individual asset basis.

Note that for option pricing and composite delta calculations, an approximated normal distribution is used. For a full description of the approximated normal distribution and error, see the appendix.

When generating a single order, the JAVA application makes use of a uniform random distribution to select an order type and all the subsequent values as described above. This is repeated until an order book of predetermined size has been populated. This order book is then subjected to the same procedure as was demonstrated above, to give a comparison of results between the algorithm and the brute force method.

3.3 Evaluation of Criterion 1

For the simulation, the JAVA implementation of SPAN has been configured to generate randomized order books, given a set of fixed inputs and parameters. The size of the order books are increased incrementally to check if the results of Criterion 1 diverges from the brute force method. The output is given in figure 3.3. The left column contains the size of the order book, and the right side the margin ratio of Criterion 1 and the brute force method on that order book.

```
CCs: 4, Criterion 1

Size of
order book                Margin Ratio
1                         1
2                         -0,2733
3                         0,9068
4                         0,6731
5                         0,535
6                         -0,242
7                         0,5293
8                         -0,1099
9                         0,8288
10                        0,3388
11                        0,7958
12                        -0,7817
13                        0,9294
14                        0,9083
15                        0,6471
16                        -0,1166
17                        0,9596
18                        0,7254
19                        0,2829
20                        0,4876
```

Figure 3.1 JAVA output: Margin Ratios of Criterion 1, 20 randomized order books of increasing size.

The simulation was set to generate randomized order books from size 1 to 20, containing futures, put and call orders based on the first four preconfigured underlying assets, namely oil, steel, copper, and silver. The orders were generated with slight variations in price, volatility, strike price, implied volatility and position as detailed above.

It is immediately apparent that the results of Criterion 1 when applied to the example portfolio are not representative of its general performance on randomized portfolios. It manages to select the worst-case portfolio for the smallest order book, but the margin ratio varies wildly as the order book grows larger, displaying no noticeable trend at all. Aside from the initial selection, no perfect worst-case portfolio is found.

One point of alarm is the apparent ability of Criterion 1 to select portfolios that in reality have a negative initial margin requirement. The only way this is possible is by having the Net Option Value exceed the previous steps in magnitude, since it is negative in the summary calculation. This highlights a glaring weakness of Criterion 1: its inability to account for the effects of long options.

As mentioned previously, a long option does not contribute to the Scanning Risk whatsoever, as SPAN assigns no associated risk to holding a long option in any market condition. Criterion 1, however, still includes a long option to its worst-case portfolio since it has potential positive contributions to the Intermonth Spread Charge and Delivery Month Charge that are as yet unknown. What is not accounted for is that a long option has a necessarily negative contribution to the initial margin requirement through the Net Option Value step. An example portfolio that illustrates this is given below.

The output given in figures 3.2 and 3.3 are the result of a calculation on a randomized order book of 8 orders, randomly drawn from the first four underlying instruments in the JAVA implementation.

Figure 3.2 JAVA output: The brute force method applied to an order book of size 8. The selected orders and accompanying results for each combined commodity and overall portfolio are included.

```
****Brute Force Method, 256 runs****

Combined Commodities in Portfolio: 2

copper: Price Scan Range = 95,46
1       -4.0    f       S = 1479.0      T = 61.0

        Scanning Risk = 381,84  Active Scenario: 11
        Intermonth Spread Charge = 0
        Delivery Month Charge = 0

steel: Price Scan Range = 235,47
1       5.0     f       S = 3031.0      T = 60.0
2       10.0    f       S = 3067.0      T = 17.0

        Scanning Risk = 3532    Active Scenario: 13
        Intermonth Spread Charge = 0
        Delivery Month Charge = 180

Margin Calculations

---- Portfolio Scanning Risk = 3913,84
---- Portfolio Intermonth Spread Charge = 0
---- Portfolio Delivery Month Charge = 180

-- Combined Portfolio Risk = 4093,84
-- Short Option Minimum Charge = 0       (Shorts: 0 calls, 0 puts, 0.05 percent of Scanning Range)

- Total Risk = 4093,84
- Net Option Value = 0  (Number of Options = 0.0)

Initial Margin Requirement = 4093,84
```

Figure 3.3 JAVA output: An algorithm using Criterion 1 applied to the same order book. The output details the number of total generated orders in each combined commodity, and the number of those orders selected for inclusion in the worst-case portfolio. The results are given in the same format as in figure 3.2, with orders included by Criterion 1 but not the brute force method marked in bold green.

```
***Algorithm: Criterion 2, 1 run***

Combined Commodities in Portfolio: 4

copper: 1 contract(s)/1 total    Price Scan Range = 95,46
1       -4.0    f       S = 1479.0      T = 61.0

        Scanning Risk = 381,84   Active Scenario: 11
        Intermonth Spread Charge = 0
        Delivery Month Charge = 0

silver: 3 contract(s)/3 total    Price Scan Range = 1177,33
1       9.0     c       S = 15322.0     T = 39.0        K = 14625.0     c = 1158,3      ImSigma = 0,3
2       7.0     c       S = 15240.0     T = 68.0        K = 15012.0     c = 1089,89     ImSigma = 0,29
3       4.0     p       S = 15112.0     T = 77.0        K = 14664.0     p = 617,3       ImSigma = 0,27

        Scanning Risk = 0        Active Scenario: 1
        Intermonth Spread Charge = 258,16
        Delivery Month Charge = 0

steel: 2 contract(s)/2 total     Price Scan Range = 235,47
1       5.0     f       S = 3031.0      T = 60.0
2       10.0    f       S = 3067.0      T = 17.0

        Scanning Risk = 3532     Active Scenario: 13
        Intermonth Spread Charge = 0
        Delivery Month Charge = 180

oil: 2 contract(s)/2 total       Price Scan Range = 623,67
1       9.0     c       S = 8575.0      T = 21.0        K = 8243.0      c = 451,12      ImSigma = 0,24
2       6.0     p       S = 8505.0      T = 48.0        K = 8204.0      p = 181,57      ImSigma = 0,23

        Scanning Risk = 0        Active Scenario: 1
        Intermonth Spread Charge = 185,21
        Delivery Month Charge = 281,39

Margin Calculations (8/8 contracts included)

---- Portfolio Scanning Risk = 3913,84
---- Portfolio Intermonth Spread Charge = 443,37
---- Portfolio Delivery Month Charge = 461,39

-- Combined Portfolio Risk = 4818,6
-- Short Option Minimum Charge = 0       (Shorts: 0 calls, 0 puts, 0.05 percent of Scanning Range)

- Total Risk = 4818,6
- Net Option Value = 25672,71    (Number of Options = 35.0)

Initial Margin Requirement = -20854,12
```

The margin ratio here is

$$Margin\ Ratio_{crit\ 1} = \frac{Initial\ Margin\ Requirement_{crit\ 1}}{Initial\ Margin\ Requirement_{brute}} = \frac{-20\,854.12}{4093.84} = -5.09$$

This astonishing result is altogether due to the fact that the randomized order book is exclusively made up of long orders, and that the Net Option Value dwarfs the rest of the components in the SPAN calculation. The Scanning Risk components are equal for Criterion 1 and the brute force method, so Criterion 1 succeeds in maximizing that part of the calculation. However, the results here prove that the Scanning Risk contribution is too narrow a scope for the algorithm to reliably select a worst-case portfolio; assumption a) mentioned at the start of this section has to be discarded.

The next section proposes an extension to Criterion 1 to specifically avoid the disastrous results in the example above.

3.4 Extending the selection criterion

The simplest way to deal with the deficiencies of Criterion 1 is to account for the marginal contribution of an order to the Net Option Value of the portfolio as well. This approach is appealing since it requires no approximations at all and specifically targets orders of options. As such, the effects of such an extension are more easily predicted.

Criterion 2: Given risk scenario X, Scanning Risk - Net Option Value ≥ 0

Like in Criterion 1, each risk scenario is assigned a specific configuration of orders in the order book, where each order in each scenario complies with the above criterion. This is a simple extension where virtually no calculation complexity is added, but where several of the drawbacks of Criterion 1 are addressed.

Consider a long option:

- By construction, its marginal Scanning Risk is zero.
- It has a positive Net Option Value.

Thus, a long option will *never* be included in a worst-case portfolio, since the difference of the two terms in Criterion 2 will always be negative. This is as it should be: a long option is a risk-reducing instrument, with a limited down-side in any market conditions.

A short option, on the other hand, is handled differently:

- It contributes positively or negatively to the Scanning Risk depending on the risk scenario.
- It has a negative Net Option Value independent of risk scenario.

A short options is thus more likely to be selected for a worst-case portfolio, since any small negative contribution to Scanning Risk is trumped by its always-positive contribution to the initial margin requirement through the Net Option Value.

The anatomy of a worst-case portfolio

At this stage, it is possible to draw certain conclusions as to the nature of a worst-case portfolio under Criterion 2. Given bearish market conditions - that is, when the price of assets decrease - long futures contribute positively to Scanning Risk whereas short futures reduce total risk. Conversely, in bullish market conditions, when prices go up, the opposite occurs. Thus, a typical portfolio selected in compliance with Criterion 2 includes either long or short futures, but never both, depending on the Active Scenario. Given also the fact that long options are always discarded while short options are favored, there are only two sets of orders to be found in any worst-case portfolio selected in compliance with Criterion 2.

Bull Market

Instrument	**Position**
Future	Short
Call	Short
Put	Short

Bear Market

Instrument	**Position**
Future	Long
Call	Short
Put	Short

Similarly to Criterion 1, the example portfolio will be used to illustrate the selection process under Criterion 2. For the example portfolio, which includes only one short option, using table 1.5 with slight modifications is sufficient. In the short call option column, each cell has its value subtracted by the marginal Net Option Value of the call:

$$Net\ Option\ Value = Option\ Price * Option\ Position = 31 * -5 = -155$$

Now, every cell with a negative value is removed. The remaining cells are given in table 3.3.

Table 3.3 Worst-case Portfolio "Risk Array", Criterion 2

	Future 10	Future 15	Future -5	Call -5	Total
1	0	0	0	272.1	272.1
2	0	0	0	46.1	46.1
3	-	-	160	348.3	508.3
4	-	-	160	102.9	262.9
5	320	480	-	207	1007
6	320	480	-	16.6	816.6
7		-	320	435.1	755.1
8	-	-	320	191.8	511.8
9	640	960	-	153	1753
10	640	960	-	4.6	1604.6
11	-	-	480	533.4	1013.4
12	-	-	480	310.2	790.2
13	960	1440	-	109.5	2509.5
14	960	1440	-	1	2401
15	-	-	335	321.3	656.3
16	670	1005	-	58.7	1733.7

Since each cell in the option column contains a non-negative value, they are all included in the worst-case configuration. Before the Scanning Risk calculation is performed, however, each cell must first have their respective Net Option Value contributions removed. Table 3.4 gives the actual risk array, with associated Scanning Risk and Active Scenario.

Table 3.4 Worst-case Portfolio Risk Array, Criterion 2

	Future 10	Future 15	Future -5	Call -5	Total
1	0	0	0	117.1	117.1
2	0	0	0	-108.9	-108.9
3	-	-	160	193.3	353.3
4	-	-	160	-52.1	107.9
5	320	480	-	52	852
6	320	480	-	-138.4	661.6
7		-	320	280.1	600.1
8	-	-	320	36.8	356.8
9	640	960	-	-2	1598
10	640	960	-	-150.4	1449.6
11	-	-	480	378.4	858.4
12	-	-	480	155.2	635.2
13	960	1440	-	-45.5	**2354.5**
14	960	1440	-	-154	2246
15	-	-	335	166.3	501.3
16	670	1005	-	-96.3	1578.7

As a result, even though it has a negative contribution to the Scanning Risk, the short call option is included in the worst-case portfolio. This is the same portfolio selection as that yielded by the brute force method.

Worst-case of the Example Portfolio, Criterion 2

Instrument	**Future**	**Call**	**Future**
Position	10	-5	15
Maturity (days)	90	60	25
Price (USD)	1200	31	1100
Underlying Price (USD)	-	1200	-
Strike (USD)	-	1250	-
Implied Volatility	-	20%	-

$$Margin\ Ratio_{crit\ 2} = \frac{Initial\ Margin\ Requirement_{crit\ 2}}{Initial\ Margin\ Requirement_{brute}} = 1$$

Criterion 2 thus manages to select the worst-case portfolio of the example order book in one run, where the brute force method required $2^8 = 256$ runs. Note that the disparity in the Scanning Risk between

Criterion 2 and the brute force method is due to the fact that the JAVA application uses exact numbers, whereas the calculations done by hand are rounded.

3.5 Evaluation of Criterion 2

This initial result is auspicious, but will be further verified through simulations on multiple randomized order books. Figure 3.4 gives the results. As before, order books ranging from size 1 to 20 are generated, containing randomized orders of instruments with the first four preset commodities as underlying assets. The right column lists the margin ratio for each order book, with a 1 indicating that the worst-case portfolios selected by Criterion 2 and the brute force method have equal initial margin requirements.

Note that the actual order selections are not listed, and are not required in this instance. While two equal initial margin requirements do not necessarily equate to an identical selection of orders, this is very likely the case due to the randomized nature of the individual orders. Second, the initial margin requirement is the relevant result, and not the actual order selection used to calculate it. Should a given order book have two different selections of orders that both produce the highest possible initial margin requirement for that order book, either of them will do as a worst-case portfolio. The margin ratio is thus sufficient as a performance metric of the algorithm.

Figure 3.4 JAVA output: Margin Ratios of Criterion 2,
20 randomized order books of increasing size.

```
CCs: 4, Criterion 2

Size of
order book              Margin Ratio
1                       1
2                       1
3                       1
4                       1
5                       1
6                       1
7                       1
8                       1
9                       1
10                      1
11                      1
12                      1
13                      1
14                      1
15                      0,9851
16                      0,9986
17                      1
18                      1
19                      0,9963
20                      0,9988
```

This is a vastly improved result compared to that of Criterion 1. The worst-case portfolio was successfully selected for nearly all the generated order books of randomized orders. In the cases where the worst-case portfolio of Criterion 2 had an initial margin requirement less than that of the brute force method, the largest deviation was about 1.5 percent.

The data set given here is very limited, and no definite determinations can be made as to the accuracy of Criterion 2, but the margin ratios do not exhibit a noticeable trend as order book size grows. Further simulations and performance testing is warranted.

First, however, the nature of the error in Criterion 2 must be investigated. Similar to Criterion 1, the full output of a simulation on an order book of size 12, comprised of randomly generated orders of instruments with the first four underlying assets as detailed in the JAVA implementation, is given in figures 3.5 and 3.6.

Figure 3.5 JAVA output: The brute force method applied to an order book of size 12. The selected orders and accompanying results for each combined commodity and overall portfolio are included.

```
***Brute force: 4096 runs***

Combined Commodities in Portfolio: 4

copper: Price Scan Range = 95,46
1       -4.0    c       S = 1477.0      T = 22.0        K = 1464.0      c = 24,67       ImSigma = 0,09

        Scanning Risk = 363,93   Active Scenario: 11
        Intermonth Spread Charge = 0
        Delivery Month Charge = 16,93

silver: Price Scan Range = 1177,33
1       3.0     f       S = 15091.0     T = 98.0
2       -6.0    p       S = 14706.0     T = 23.0        K = 14885.0     p = 431,33      ImSigma = 0,2

        Scanning Risk = 9456,97  Active Scenario: 13
        Intermonth Spread Charge = 0
        Delivery Month Charge = 294,78

steel:  Price Scan Range = 235,47
1       -6.0    f       S = 2942.0      T = 3.0
2       -7.0    p       S = 2935.0      T = 40.0        K = 2984.0      p = 74,48       ImSigma = 0,12

        Scanning Risk = 1134,68  Active Scenario: 11
        Intermonth Spread Charge = 146,79
        Delivery Month Charge = 218,09

oil:    Price Scan Range = 623,67
1       -8.0    c       S = 8204.0      T = 43.0        K = 8461.0      c = 120,42      ImSigma = 0,15

        Scanning Risk = 3848,6   Active Scenario: 11
        Intermonth Spread Charge = 0
        Delivery Month Charge = 0

Margin Calculations

---- Portfolio Scanning Risk = 14804,17
---- Portfolio Intermonth Spread Charge = 146,79
---- Portfolio Delivery Month Charge = 529,81

-- Combined Portfolio Risk = 15480,77
-- Short Option Minimum Charge = 435,61  (Shorts: 13 puts, 12 calls, 0.05 percent of Scanning Range)

- Total Risk = 15480,77
- Net Option Value = -4171,37

Initial Margin Requirement = 19652,15
```

Figure 3.6 JAVA output: Criterion 2 applied to the same order book. Orders discarded by Criterion 2 but included by the brute force method are marked in bold red.

```
***Algorithm: Criterion 2, 1 run***

Combined Commodities in Portfolio: 4

copper: 1 contract(s)/2 total           Price Scan Range = 95,46
1       -4.0    c       S = 1477.0      T = 22.0        K = 1464.0      c = 24,67       ImSigma = 0,09
*** Discarded Instruments ***
2       1.0     c       S = 1497.0      T = 77.0        K = 1471.0      c = 55,6        ImSigma = 0,1

        Scanning Risk = 363,93   Active Scenario: 11
        Intermonth Spread Charge = 0
        Delivery Month Charge = 16,93
```

```
silver: 2 contract(s)/4 total           Price Scan Range = 1177,33
1       3.0     f       S = 15091.0     T = 98.0
2       -6.0    p       S = 14706.0     T = 23.0        K = 14885.0     p = 431,33      ImSigma = 0,2
*** Discarded Instruments ***
3       1.0     p       S = 14976.0     T = 58.0        K = 14673.0     p = 511,88      ImSigma = 0,25
4       7.0     p       S = 15231.0     T = 22.0        K = 14697.0     p = 272,31      ImSigma = 0,29

        Scanning Risk = 9456,97  Active Scenario: 13
        Intermonth Spread Charge = 0
        Delivery Month Charge = 294,78

steel: 1 contract(s)/4 total            Price Scan Range = 235,47
1       -7.0    p       S = 2935.0      T = 40.0        K = 2984.0      p = 74,48       ImSigma = 0,12
*** Discarded Instruments ***
2       -6.0    f       S = 2942.0      T = 3.0
3       3.0     c       S = 3019.0      T = 55.0        K = 3011.0      c = 104,02      ImSigma = 0,16
4       10.0    c       S = 3071.0      T = 87.0        K = 2991.0      c = 194,02      ImSigma = 0,19

        Scanning Risk = 1488,92  Active Scenario: 13
        Intermonth Spread Charge = 0
        Delivery Month Charge = 0

oil: 1 contract(s)/2 total              Price Scan Range = 623,67
1       -8.0    c       S = 8204.0      T = 43.0        K = 8461.0      c = 120,42      ImSigma = 0,15
*** Discarded Instruments ***
2       3.0     f       S = 8461.0      T = 59.0

        Scanning Risk = 3848,6   Active Scenario: 11
        Intermonth Spread Charge = 0
        Delivery Month Charge = 0

Margin Calculations (5/12 contracts included)

---- Portfolio Scanning Risk = 15158,42
---- Portfolio Intermonth Spread Charge = 0
---- Portfolio Delivery Month Charge = 311,71

-- Combined Portfolio Risk = 15470,13
-- Short Option Minimum Charge = 435,61  (Shorts: 13 puts, 12 calls, 0.05 percent of Scanning Range)

- Total Risk = 15470,13
- Net Option Value = -4171,37

Initial Margin Requirement = 19641,5
```

The margin ratio here is

$$Margin\ Ratio_{crit\ 2} = \frac{Initial\ Margin\ Requirement_{crit\ 2}}{Initial\ Margin\ Requirement_{brute}} = \frac{19\,641.5}{19\,652.15} = 0.9995$$

The small difference in initial margin requirements comes from the fact that Criterion 2 discards an order from the worst-case portfolio that should be included to form the true worst-case portfolio. The order in question is a short futures order, which under the bearish Active Scenario 13 set by Criterion 2 has a negative Scanning Risk contribution, and zero Net Option Value contribution. It is therefore discarded, since the marginal Scanning Risk - Net Option Value for that order under risk scenario 13 is negative.

However, the brute force method not only included this order, but also set the Active Scenario for that combined commodity to 11 rather than 13. Note that under risk scenario 11, Criterion 2 also includes the short futures order. How, then, is the Active Scenario set to 13 rather than 11, as the brute force method indicates is the better choice? The answer is given by looking at the respective Scanning Risks for that combined commodity:

$$CC\ level: Scanning\ Risk_{brute} = 1134.68\ USD$$

$$CC\ level: Scanning\ Risk_{crit\ 2} = 1488.92\ USD$$

The best configuration is always determined to be the one that maximizes the Scanning Risk, after taking into account the contributions to Net Option Value. Based on this, Criterion 2 sets the Active Scenario to 13, as the order selection for that risk scenario yields the highest overall Scanning Risk. However, look at the rest of the SPAN components for the combined commodity:

$$CC\ level: Intermonth\ Spread\ Charge_{brute} + Delivery\ Month\ Charge_{brute} = 146.79 + 218.09 = 364.88\ USD$$

$$CC\ level:\ Intermonth\ Spread\ Charge_{crit\ 2} + Delivery\ Month\ Charge_{crit\ 2} = 0 + 0 = 0\ USD$$

The difference in Scanning Risk that is achieved by selecting risk scenario 13 rather than 11 as Active Scenario is:

$$CC\ level: \Delta Scanning\ Risk = 1488.92 - 1134.68 = 354.24\ USD$$

The difference in combined spread charges is:

$$CC\ level: \Delta Spread\ Charges = 0 - 364.88 = -364.88\ USD$$

All else being equal, the net difference between the two initial margins are thus:

$$\Delta Scanning\ Risk + \Delta Spread\ Charges = 354.24\ - 364.88 = -10.64\ USD$$

which accounts for the disparity observed above.

What the example highlights is the fact that Criterion 2 does not take into account the potential contributions of an order through the formations of spreads. In fact, the type of worst-case portfolio being selected by Criterion 2 has an overall negative impact on spread formations in a portfolio. Consider the typical worst-case portfolio under Criterion 2, mentioned above, but this time including the composite and position deltas.

Bull Market

Instrument	**Position**	**Composite Delta**	**Position Delta**
Future	Short	$\delta = 1$	Negative
Call	Short	$0 < \delta < 1$	Negative
Put	Short	$-1 < \delta < 0$	Positive

Bear Market

Instrument	**Position**	**Composite Delta**	**Position Delta**
Future	Long	$\delta = 1$	Positive
Call	Short	$0 < \delta < 1$	Negative
Put	Short	$-1 < \delta < 0$	Positive

Recall that spreads are formed by matching positive and negative position deltas of two orders against each other. In these two worst-case portfolios, given an equal number of all possible types of orders, there is a considerable imbalance in positive and negative position deltas. The consequences of this are:

1. The Intermonth Spread Charge tends to decrease, since less spreads are formed when positions are not balanced.
2. The Delivery Month Charge tends to increase, since it consists more broadly of outright position charges than spread charges.

Evidently, an extension of the selection criterion is necessary so that consideration as to the contributions of an orders on the Intermonth Spread Charge and the Delivery Month Charge is also made. This way, those orders whose overall net contribution to the initial margin requirement is positive, but whose marginal Scanning Risk - Net Option Value is negative, are also included. This refined criterion can then be evaluated alongside Criterion 2 to gauge differences in performance.

3.6 Refining the extended criterion

To properly evaluate the extent to which a particular order contributes to the initial margin requirement of a portfolio through spread charges, it is impossible to consider only that single order. As outlined in the SPAN description section, the Intermonth Spread Charge is assigned to a combined commodity by performing calculations on the tier structure of all orders included in that combined commodity. Adding a single order, or even moving an order from one tier to another, has an impact on the final result that is impossible to predict without taking into account the existing tier structure in its entirety.

To evaluate the marginal Intermonth Spread Charge of a single order, an approximation is made that does not require consideration of the orders already in the portfolio. Rather than using the tier structure of the relevant combined commodity, of which all existing orders are part, the so-called Charge Impact is a measure of the spread charges when all orders are put in one single tier with one single spread charge. This sets the stage for Criterion 3:

<u>Criterion 3: Scanning Risk + Charge Impact - Net Option Value > 0</u>

Here, the Charge Impact is:

$$Charge\ Impact = \ Marginal\ Intermonth\ Spread\ Charge + Marginal\ Delivery\ Month\ Charge$$

As will be seen, only the first component in the Charge Impact needs to be approximated. This approximation is outlined below.

Collapsing the tier structure

To best illustrate the virtue of using a single tier to calculate the marginal spread charges of an order, consider the delta spread table of the example portfolio used in previous sections. It consists of four orders located in three different tiers, with same spread charges and spread priority order as before.

Table 3.5 The Delta Spread Table of the Example Portfolio

Tier	Long	Short
1	15	-1.9
2	10	0
3	0	-5

Table 3.6 The Tier Spread Table of the Example Portfolio

Tier	Maturity Range	Tier 1	Tier 2	Tier 3
1	1-2 months	50 USD	-	-
2	3-4 months	80 USD	60 USD	-
3	5-6 months	90 USD	100 USD	70 USD

Table 3.7 The Spread Priority Table of the Example Portfolio

Priority	1	2	3	4	5	6
Tier Spread	1 to 1	2 to 2	3 to 3	1 to 2	1 to 3	2 to 3

Table 3.8 The Delivery Month Charges of the Example Portfolio

Charge	
Spread	25 USD
Outright	50 USD

$$Intermonth\ Spread\ Charge = 1.9 * 50 + 5 * 90 = 95 + 450 = 545\ USD$$

Consider Case 1: a fifth order is added to the portfolio, with position delta -4 and maturity within the range of tier 3. It affects the Intermonth Spread Charge and Delivery Month Charge like so:

Table 3.9 The Amended Delta Spread Table of the Example Portfolio, Case 1

Tier	Long	Short
1	15	-1.9
2	10	0
3	0	-9

→

Tier	Long	Short
1	4.1	0
2	10	0
3	0	0

$$Intermonth\ Spread\ Charge_{Case\ 1} = 1.9 * 50 + 9 * 90 = 95 + 810 = 1005\ USD$$

$$\Delta Intermonth\ Spread\ Charge_{Case\ 1} = 1005 - 545 = 460\ USD$$

Now consider Case 2: the fifth order instead has position delta -4 and maturity 1 month, putting it in tier 1.

Table 3.10 The Amended Delta Spread Table of the Example Portfolio, Case 2

Tier	Long	Short
1	15	-5.9
2	10	0
3	0	-5

→

Tier	Long	Short
1	4.1	0
2	10	0
3	0	0

$$Intermonth\ Spread\ Charge_{Case\ 2} = 5.9 * 50 + 5 * 90 = 295 + 450 = 745\ USD$$

$$\Delta Intermonth\ Spread\ Charge_{Case\ 2} = 745 - 545 = 200\ USD$$

These exact results are impossible to deduce without knowledge of the entire tier structure. The end calculation is entirely dependent on all the position deltas in all the tiers, and not the single order added. In Criterion 3, the Charge Impact approximation is used instead. First, the total long and short positions of all tiers are summed and placed in a single tier.

Table 3.11 The Collapsed Tier Spread Table of the Example Portfolio

Tier	Long	Short
1	25	-6.9

→

Tier	Long	Short
1	18.1	0

The spread charge used is the average of all the spread charges for all the tiers:

$$Tier\ Spread\ Charge = \frac{50 + 60 + 70 + 80 + 90 + 100}{6} = 75\ USD$$

The approximation is rather crude, but has reasonable accuracy in this instance.

$$Intermonth\ Spread\ Charge_{Collapsed} = 6.9 * 75 = 517.5\ USD$$

For cases 1 and 2, the effects on the approximated Intermonth Spread Charge are known beforehand by simply consulting table 3.10, and using the total net position of the combined commodity. For the example portfolio, which is comprised of only one combined commodity, this is

$$Combined\ Commodity\ Net\ Position = 18.1$$

In case 1, an order with position delta -18.1 enables 14.1 new spreads to be formed.

Table 3.12 The Collapsed Amended Tier Spread Table of the Example Portfolio, Case 1

Tier	Long	Short
1	25	-10.9

→

Tier	Long	Short
1	14.1	0

Thus, the marginal Intermonth Spread Charge of the order is

$$Marginal\ Intermonth\ Spread\ Charge_{Case\ 1} = 4 * 75 = 300\ USD$$

$$Combined\ Commodity\ Net\ Position_{Case\ 1} = 14.1$$

In case 2, the order position delta is equal.

Table 3.13 The Collapsed Amended Tier Spread Table of the Example Portfolio, Case 2

Tier	Long	Short
1	25	-10.9

→

Tier	Long	Short
1	14.1	0

$$Marginal\ Intermonth\ Spread\ Charge_{\ Case\ 2} = 4 * 75 = 300\ USD$$

$$Combined\ Commodity\ Net\ Position_{Case\ 2} = 14.1$$

By only considering the net position of the combined commodity and the order position delta, a rough estimate of the change to the Intermonth Spread Charge is obtained. In Case 1, the estimate is smaller than the exact value, and in Case 2 the estimate is larger. An alternate approach would be to assign the exact tier spread charge to the spreads formed by an added order, since its maturity and associated tier is known beforehand. This does not account for the fact that new spreads can replace existing spreads within the combined commodity. Using an average spread charge, while in no way exact, at least recognizes this effect.

Yet a third approach is to use the lowest tier spread charge in the approximation, thus calculating the lower bound of the marginal Intermonth Spread Charge of an order. This ensures that all orders have *at least* the same marginal effect as the approximation indicates, and possibly higher. However, the underestimation also leads to a higher rate of erroneous rejection of orders. Similarly, if the highest tier spread charge is used, an upper bound of the marginal Intermonth Spread Charge is calculated. Orders are now more likely to be included under Criterion 3, but they now have *at most* the same marginal effect as the approximation, leading to a higher rate of erroneous inclusion. Using the average tier spread charge strikes a balance between these two extremes.

The marginal Delivery Month Charge

The Delivery Month Charge does not have to be approximated, since it already makes use of a single "tier": the delivery month. If the number of existing outright deltas of a portfolio is known, the effect on the Delivery Month Charge of a single order can easily be evaluated.

The example portfolio had 6.9 spread charges and 8.1 outright charges assigned to it.

$$Delivery\ Month\ Charge = 6.9 * 25 + 8.1 * 50 = 577.5\ USD$$

Using the outright deltas, which in this case is 8.1, the marginal impact in cases 1 and 2 to the Delivery Month Charge is simple to calculate. For case 1, the order position delta is -4, but the maturity is not within the delivery month, and so the order has no impact here.

$$Marginal\ Delivery\ Month\ Charge_{Case\ 1} = 0\ USD$$

In case 2, the order has position delta -4 and has maturity within the delivery month, thus converting four of the outright deltas in the portfolio into spreads. The net effect is:

$$Marginal\ Delivery\ Month\ Charge_{Case\ 2} = 4 * 25 - 4 * 50 = -100\ USD$$

where the decrease in outright charges is subtracted from the added spread charges.

The Charge Impact

The Charge Impact for the two cases are:

$$Charge\ Impact_{Case\ 1} = 300 + (-100) = 200\ USD$$

$$Charge\ Impact_{Case\ 2} = 0 + 200 = 200\ USD$$

The contributions of the orders in cases 1 and 2 to the Intermonth Spread Charge and Delivery Month Charge are then used in Criterion 3 as additional basis for inclusion to or rejection from the worst-case portfolio.

The Charge Impact can theoretically be negative, but it requires the delivery month charges to be larger than the tier spread charges. This is not the case in the simulations discussed in this paper, and the Charge Impact is thus always positive. As such, Criterion 3 is more inclusive than Criterion 2: all the orders selected by Criterion 2 are also selected by Criterion 3, but Criterion 3 is also able to catch those few extra orders whose spread charge contributions merit their inclusion into the worst-case portfolio.

The major drawback of the Charge Impact, however, is that it is dependent on the order in which the orders in an order book are selected. The decision to include an order in the worst-case portfolio based on its Charge Impact rests on the current net position of its combined commodity. If, for example, five orders with short positions and a common underlying asset are evaluated for inclusion in the worst-case portfolio, all of them will be assigned a marginal Intermonth Charge of zero due to a lack of spreads being formed. If, then, five orders with long positions are considered afterwards, they will all be able to have spreads formed, and thus their marginal Intermonth Spread Charge increases. If the order was reversed, the short positions would have their marginal Intermonth Spread Charge increase instead of the long positions.

This effect is counter-balanced for orders in the delivery month, as less spreads mean more outright charges. Thus, while the marginal Intermonth Charge is lower, the marginal Delivery Month Charge is higher.

3.7 Evaluation of Criterion 3

Figure 3.7 JAVA output: Margin Ratios of Criterion 3, 20 randomized order books of increasing size.

```
CCs: 4, Criterion 3

Size of
order book                Margin Ratio
1                         1
2                         1
3                         1
4                         1
5                         1
6                         1
7                         1
8                         1
9                         0,9998
10                        1
11                        1
12                        1
13                        1
14                        1
15                        0,99995
16                        0,993986
17                        0,999886
18                        1
19                        1
20                        0,999806
```

Applying the same simulation regime on Criterion 3 as on the previous criteria, it is immediately apparent that the accuracy is very high. Note that the margin ratios listed in figure 3.7 are given with six significant digits, meaning that margin ratios equal to 1 are either exactly 1 or as close to it as not to matter for all practical purposes. The results are similar to those of Criterion 2, with the majority of randomized order books having its worst-case portfolio successfully selected. For all cases where Criterion 3 underperformed, the error was less than 1 percent.

Clearly, Criteria 2 and 3 perform very well on smaller randomized portfolios, whereas Criterion 1 simply does not account for enough factors of the SPAN calculation to be relevant. The next section will be include further simulations of Criteria 2 and 3, including time performance analysis.

4. Accuracy and Time Performance

Two candidate criteria have been proposed and detailed in the previous section, each capable of selecting, on an order-by-order basis, a worst-case portfolio of a given order book. They can be summarized operationally as:

For all orders:

For risk scenarios 1-16:

Criterion 2: Select order if (Scanning Risk - Net Option Value) ≥ 0

Criterion 3: Select order if (Scanning Risk + Charge Impact - Net Option Value) ≥ 0

For selections 1-16:

Criteria 2 and 3: Set the selection with maximum Scanning Risk as worst-case portfolio.

This section will investigate the difference in performance of these two criteria, first in terms of selection accuracy, but also in terms of computation time. Does the added check in Criterion 3 translate into noticeably improved selection performance, and if so, does it merit the added computational burden?

4.1 Absolute accuracy comparisons

In previous evaluations, both candidate criteria were individually subjected to a randomized order book performance test. The worst-case portfolios selected from generated order books were compared to the exact solution, as determined by the brute force approach. While a rough indicator of accuracy of Criteria 2 and 3, the evaluations give no hint as to which criterion performs better. In this context, performance is meant to signify the ability of a criterion to select the correct worst-case portfolio out of a given order book. Previous results, given in tables 3.4 and 3.7, hint at a similar general performance of both Criteria 2 and 3, with the majority of smaller-sized order books having their proper worst-case portfolio selected, and a maximum error in the range of single percentage points. At this stage, additional simulations will be performed where both criteria are applied to the same randomized set of order books.

Simulation 1: Increasing order book size, 4 Combined Commodities

The first test is the same as before but slightly extended to larger order books. Recall that the limited order book sizes used in the simulation is wholly due to the computational burden of the brute force approach. Where an order book of size 5 requires $2^5 = 32$ runs for an exact solution, an order book of size 25 requires $2^{25} = 33,554,432$ runs. The value of increasing the order book size is quickly outweighed by the computation time required for an exact solution.

In the simulation, the cap is put at 30 orders in an order book. Figure 4.1 gives the JAVA output of the simulation, with additional comparison data at the end.

Figure 4.1 JAVA output: Margin Ratio comparison of Criteria 2 and 3, 30 randomized order books of increasing size. Additional comparison statistics are given at the bottom.

```
CCs: 4, Criteria 2 and 3

Order book              Margin Ratio            Margin Ratio            Computational time:
size                    Criterion 2             Criterion 3             brute force (µs)
1                       1                       1                       264,72
2                       1                       1                       195
3                       1                       1                       565,18
4                       1                       1                       1414,89
5                       1                       1                       3236,39
6                       1                       1                       6261,52
7                       1                       1                       6520,21
8                       1                       1                       9318,48
9                       1                       1                       13379,11
10                      1                       1                       30709,14
11                      1                       1                       19367,08
12                      1                       1                       36257,22
13                      1                       1                       44859,73
14                      1                       1                       103332,52
15                      1                       0,998246                211589,56
16                      0,996647                1                       293393,33
17                      1                       1                       574996,2
18                      1                       1                       1123178,65
19                      1                       0,975891                2433098,24
20                      1                       1                       4937669,01
21                      1                       0,99883                 10256618,87
22                      1                       0,98362                 20944765,88
23                      0,998893                0,998893                41434664,5
24                      1                       1                       87890277,83
25                      1                       1                       177151037,39
26                      0,998409                0,998409                346673404,79
27                      1                       1                       742144151,32
28                      0,999101                1                       1488301151,82
29                      1                       1                       3054643879,44
30                      0,99942                 0,99942                 6038765718,55

***

Criterion 2: 25 hits, accuracy: 83,33 percent
Criterion 3: 23 hits, accuracy: 76,67 percent

Same result:            24 times (80 percent)

Different results:      6 times (20 percent)
Better performance:     Criterion 2             Criterion 3
                        4 times                 2 times
```

The data is similar to before in that both criteria have rather high accuracy in selecting the worst-case portfolio. In this simulation, Criterion 2 had better performance in this regard, with 25 out of 30 randomized order books having the correct worst-case portfolio selected. Criterion 3 only succeeded 23 times. This is more or less in line with previous results.

As for relative statistics, Criterion 2 and 3 selected identical worst-case portfolios 24 times, 21 of which were the same as the brute force portfolio. In the three remaining cases, the maximum error was less than 0.5 percent.

The six cases where Criteria 2 and 3 differed are of the most interest. The simulation parameters and nature of Criterion 3 have been discussed above, but to reiterate: Criterion 3 tends to include more orders than Criterion 2. Each order is given a second look, so to speak, by having its always positive Charge Impact evaluated on top of the Scanning Risk and Net Option Value. Therefore, in each case where the results of Criterion 2 and 3 are different, one or more orders that were rejected by Criterion 2 have been included by Criterion 3. The data in figure 4.1 indicate that in four of these cases, the inclusion of extra orders was erroneous, whereas in two cases it proved correct.

No conclusions can be drawn from such a limited data set, which is why the next simulation will focus on a large number of randomized order books of equal size.

Simulation 2: Fixed order book size, 4 combined commodities

All other parameters being equal, this simulation subjects both criteria to 10,000 randomized order books, each of size 10. The summary of the results are given in figure 4.2.

Figure 4.2 JAVA output: Summary of results, Margin Ratio comparison of Criteria 2 and 3, 10,000 randomized order books of size 10.

```
CCs: 4, Criteria 2 and 3

Simulation runs: 10000
Order book size: 10

***

Criterion 2: 8270 hits, accuracy: 82,7 percent
Criterion 3: 8182 hits, accuracy: 81,82 percent

Same result:            9371 times (93,71 percent)

Different results:      629 times (6,29 percent)
Better performance:     Criterion 2             Criterion 3
                        362 times               267 times
```

The results are revealing. While not in consensus every time, Criteria 2 and 3 selects the same worst-case portfolio in the vast majority of cases. The general accuracy of both at around 80% correct selections is the same as in previous simulations. The striking statistic is the rather low figure of 6.29% of cases where the Charge Impact is significant enough to affect the outcome. In all other cases, the order selections of Criteria 2 and 3 are identical, which means that the Charge Impact for an order is never large enough to warrant an inclusion by Criterion 3 when it was rejected by Criterion 2. Also, in the limited number of cases where such an inclusion is made, it is made in error more often than not. This result, of course, is highly dependent on the number of orders in each combined commodity in the portfolio that is available to form spreads. In this simulation, a mere 10 orders are spread evenly over 4 different combined commodities, with an average order per combined commodity in the worst-case portfolio of 2.5*P(order included)*. The performance of Criterion 3 might improve if this number is increased. The next simulation is adjusted to account for this fact.

Simulation 3: Fixed order book size, single combined commodity

In this simulation, the order book size is slightly increased, from 10 to 12, and the number of combined commodities in the portfolio is decreased from four to one. This effectively increases the average number of included orders in a combined commodity almost five-fold:

$$\frac{Average\ included\ orders\ per\ CC_{sim\ 3}}{Average\ included\ orders\ per\ CC_{sim\ 2}} = \frac{\frac{12\ orders}{1\ CCs} P(order\ included)}{\frac{10\ orders}{4\ CCs} P(order\ included)} = 4.8$$

where *P(order included)* is the average rate of inclusion of a random order into the worst-case portfolio. The number of runs is set to 10,000. Figure 4.3 gives the summary of results.

Figure 4.3 JAVA output: Summary of results, margin ratio comparison of Criteria 2 and 3, 10,000 randomized order books of size 12, single combined commodity.

```
CCs: 1, Criteria 2 and 3

Simulation runs: 10000
Order book size: 12

***

Criterion 2: 9346 hits, accuracy: 93,46 percent
Criterion 3: 9214 hits, accuracy: 92,14 percent

Same result:             9415 times (94,15 percent)

Different results:       585 times (5,85 percent)
Better performance:      Criterion 2              Criterion 3
                         360 times                225 times
```

While general accuracy is significantly improved, up from around 80% to above 90% for both criteria, the relative statistics tell the same story. In roughly the same proportion of cases, this time 5.85%, Criterion 2 and 3 select different worst-case portfolios, with Criterion 2 more likely to have a better selection. Criterion 3, more likely than not, commits a sin of commission by including orders that Criterion 2 rejects. To round off this stage of simulations, simulation 4 increases yet again the order book size to check any differences in results.

Simulation 4: Fixed order book size, single combined commodity

In this simulation, the order book size is set to 16, so the brute force approach requires 16 times the computation time per order book as before. As such, the runs have been decreased to 1,000. All other parameters are as in simulation 3.

Figure 4.4 JAVA output: Summary of results, margin ratio comparison of Criteria 2 and 3, 1,000 randomized order books of size 16, a single combined commodity.

```
CCs: 1, Criteria 2 and 3

Simulation runs: 1000
Order book size: 16

***

Criterion 2: 933 hits, accuracy: 93,3 percent
Criterion 3: 910 hits, accuracy: 91,0 percent

Same result:            912 times (91,2 percent)

Different results:      88 times (8,8 percent)
Better performance:     Criterion 2             Criterion 3
                        56 times                32 times
```

These results are in accordance with those of simulations 1 through 3: that Criterion 3 has overall better performance in selecting worst-case portfolios. Criterion 2 has consistently outperformed Criterion 3, both when both are compared to the brute force method, and in relative terms, when compared to each other. These simulations, although repeated a large number of times, all share one basic flaw: the very limited size of the randomized order books.

Referring back to figure 4.1, the margin ratios for increasing sizes of order books do not seem to diverge notably for either criterion. While exact worst-case portfolio selection is less common for order books above size 20, the relative errors do not seem to follow an increasing trend. Indeed, for the order book of size 30, the relative error is the lowest out of all non-exact results. It is upon this foundation that the following rather bold assumption is made: *that the absolute accuracy performance of both Criteria 2 and 3 is independent of order book size*. That is to say, for any given order book size, the criteria will perform similarly to the results indicated in simulations 1 through 4. This assumption enables coming simulations to forgo the brute force check, and thus increase the size of randomized order books without incurring an exponential increase in computation time. While the general accuracy of Criteria 1 and 2 is only assumed in these simulations, the relative data will still be of interest.

4.2 Relative accuracy comparisons

The simulations in this stage are similar to those performed above, but excludes the brute force calculation. This enables the size of the randomized order book to be increased dramatically.

Simulation 5: Fixed order book size, maximum Combined Commodities

Simulation 5 is set to run 1,000 times, with a fixed randomized order book size of 10,000. The results are listed in figure 4.5. In addition to the relative data given as in previous simulations, the average percentage breakdown of the initial margin requirement by step is included as well.

Figure 4.5 JAVA output: Summary of results, margin ratio comparison of Criteria 2 and 3, 1,000 randomized order books of size 10,000, ten combined commodities.

```
CCs: 10, Criteria 2 and 3

Simulation runs: 1000
Order book size: 10000

***

Same result:             0 times (0 percent)
Different results:       1000 times (100 percent)
Better performance:      Criterion 2              Criterion 3
                         773 times                227 times

SPAN Breakdown (Average Percentage)
                         Criterion 2              Criterion 3
Scanning Risk:           74,65    74,65
Net Option Value:        22,6     22,58
Spread Charges:          2,75     2,77
```

These results are slightly different from previous simulations. The fact that the criteria do not once agree on a selection is noteworthy, but a simple calculation indicates why. Even given a very optimistic 99% rate of consensus for Criteria 2 and 3 (that is to say, the probability that a random order is treated the same by Criterion 2 and 3), the probability of the criteria arriving at different order selections out of an order book of 10,000 orders is:

$$P(different\ selection) = 1 - P(same\ selection) = 1 - 0.99^{\ 10{,}000}$$

Now, the probability that Criteria 2 and 3 yield the same selection once or more during the 1,000 runs in the simulation is:

$$P(any\ selection\ same) = 1 - P(all\ selections\ different) = 1 - (1 - 0.99^{\ 10{,}000})^{1{,}000}$$
$$= 2.25 * 10^{-41} \approx 0$$

Even with an overestimated rate of consensus - previous simulations suggest a rate around 90% - the calculation shows that the result above is highly plausible.

The second result of note is that the rate of better performance by Criterion 2 is still better than that of Criterion 3, but this time by a larger margin. This result, taken together with those of simulations 2 through 4, points to a dependence of the rate of better performance on the size of the order book or the amount of combined commodities used in the simulation, or both. This will be investigated next.

Simulations 6-9: Fixed order book size, maximum combined commodities

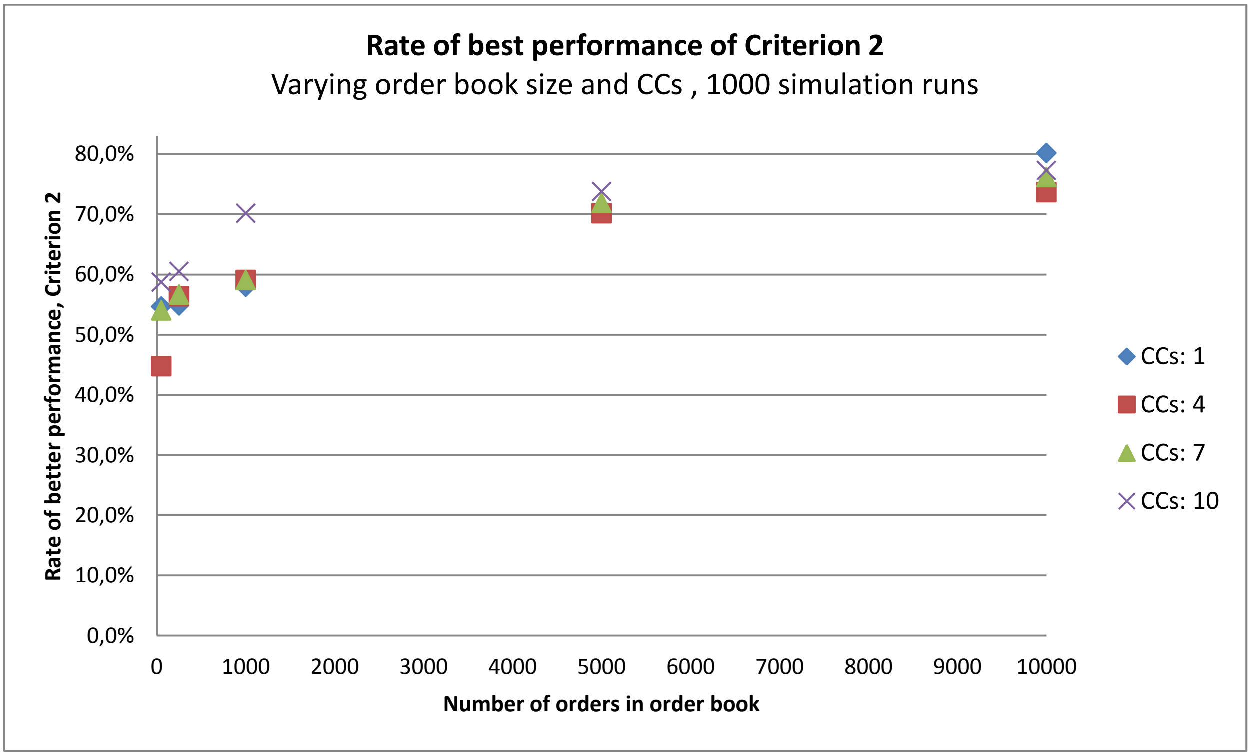
Figure 4.6 shows the results of four simulations, where the size of the order book is successively decreased down from 5,000 to 50. Two things are clear from the results. First, the rate at which Criteria 2 and 3 select the same worst-case portfolio shrinks as the size of the order book grows. This is consistent with the calculation made above. By simply shrinking the size of the order book, the probability that Criteria 2 and 3 select the same portfolio at least once approaches 1 very quickly.

Figure 4.6 JAVA output: Summary of results, margin ratio comparisons of Criteria 2 and 3, multiple simulations, each with ten combined commodities.

```
CCs: 10, Criteria 2 and 3

Simulation runs: 1000

***

Order book size: 5000

Same result: 0 times (0 percent)
Different results: 1000 times (100 percent)
Better performance:      Criterion 2             Criterion 3
                         738 times               262 times

SPAN Breakdown (Average Percentage)
                         Criterion 2             Criterion 3
Scanning Risk:           74,67                   74,68
Net Option Value:        22,58                   22,56
Spread Charges:          2,74                    2,76

***

Order book size: 1000

Same result: 0 times (0 percent)
Different results: 1000 times (100 percent)
Better performance:      Criterion 2             Criterion 3
                         702 times               298 times

SPAN Breakdown (Average Percentage)
                         Criterion 2             Criterion 3
Scanning Risk:           75,22                   75,24
Net Option Value:        22,15                   22,12
Spread Charges:          2,63                    2,64

***

Order book size: 250

Same result: 88 times (8,8 percent)
Different results: 912 times (91,2 percent)
Better performance:      Criterion 2             Criterion 3
                         552 times               360 times

SPAN Breakdown (Average Percentage)
                         Criterion 2             Criterion 3
Scanning Risk:           76,81                   76,84
Net Option Value:        20,77                   20,73
Spread Charges:          2,42                    2,43

***

Order book size: 50

Same result: 651 times (65,1 percent)
Different results: 349 times (34,9 percent)
Better performance:      Criterion 2             Criterion 3
                         205 times               144 times

SPAN Breakdown (Average Percentage)
                         Criterion 2             Criterion 3
Scanning Risk:           80,53                   80,61
Net Option Value:        17,52                   17,45
Spread Charges:          1,95                    1,94
```

Second, the rate of better performance of Criterion 2 is higher than that of Criterion 3 in all cases, which lines up qualitatively with previous findings. The rate also seems to grow with order book size, reaching its maximum of 77.3% better portfolio selections for simulation 5, using order books of size 10,000. In simulations with other numbers of combined commodities, this result is also found, as plot 4.1 shows.

Plot 4.1 The diagram shows results of simulations 6-9, and equal simulations for numbers of combined commodities 1, 4, and 7. The data points indicate the percentage of simulation runs where Criterion 2 performed better than Criterion 3.



The trend shown in the diagram is clear: Criterion 2 performs better than Criterion 3, and the disparity in performance is more notable as the size of the order book grows. There is negligible effect in varying the number of combined commodities in the portfolio.

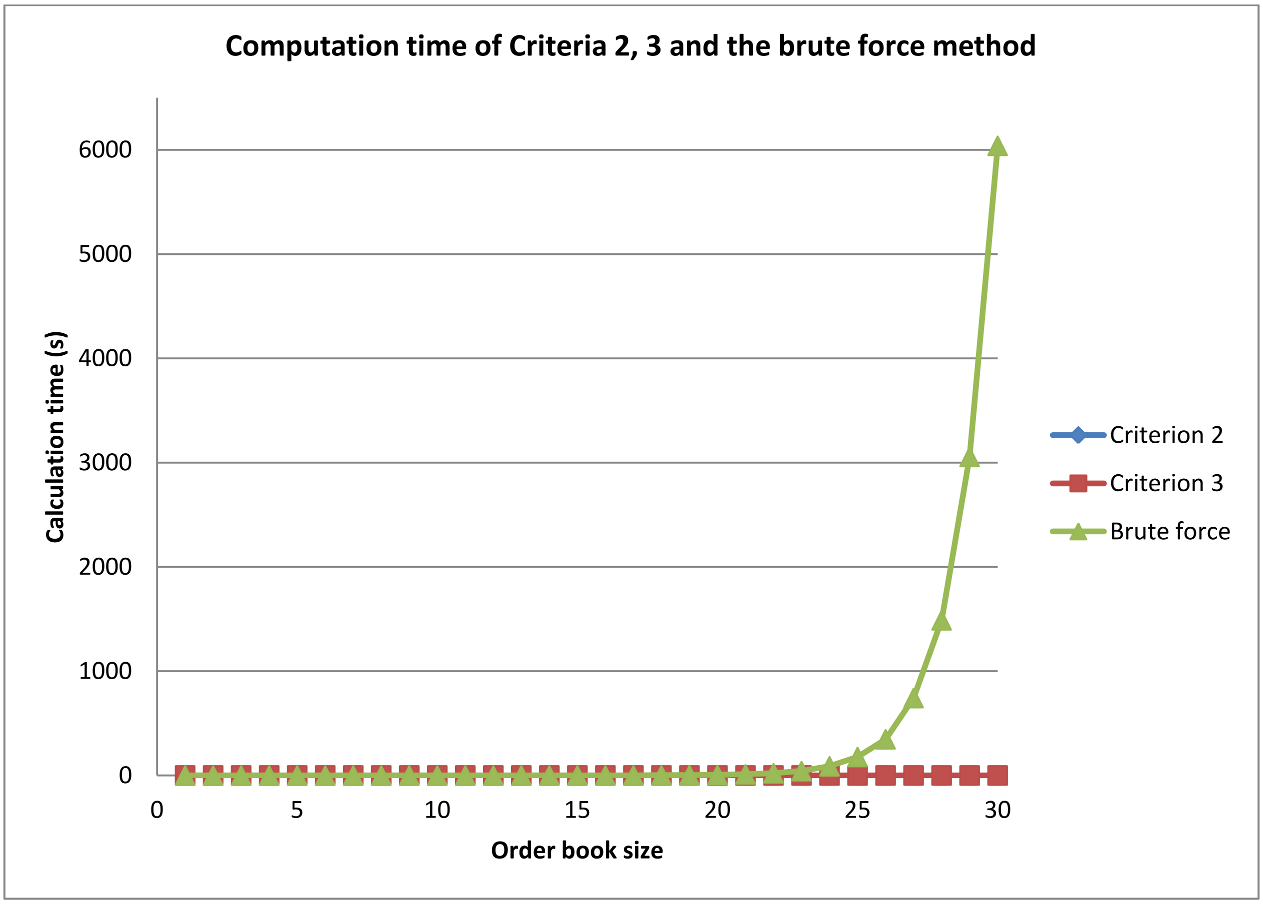
It seems as though the more lean Criterion 2 is the better choice when selecting a worst-case portfolio from a randomized order book. The reasons for the poor showing of Criterion 3 can partially be attributed to the rather rough approximations made in the Charge Impact evaluation step. As evidenced by the simulation results, the Charge Impact is not accurate enough to reliably estimate the contribution an order has through the formation of spreads. In addition, the Charge Impact introduces an element of dependence on the order in which orders are considered by the algorithm. A third factor is the rather fleeting proportion of the final initial margin that is made up of the Intermonth Spread Charge and the Delivery Month Charge. The better choice in the majority of cases is to simply ignore the contributions to these steps when orders are evaluated for inclusion into the worst-case portfolio. This is supported by the results posted by Criterion 2. This should also manifest itself in a lower computation time for Criterion 2 than Criterion 3, as will be investigated next.

4.3 Time performance analysis

Time complexity of the brute force method

In the initial problem formulation, the brute force approach was described as having an exponential time complexity, i.e. its calculation time is proportional to 2^N, where N is the size of the order book. Plot 4.2 plots the calculation times for the individual results of simulation 1, listed in figure 4.1.

Plot 4.2 The diagram shows the computation times of the individual steps of simulation 1. The computation time for the brute force method grows exponentially, and requires roughly 1:40 hours to find the worst-case portfolio for a order book size of 30.

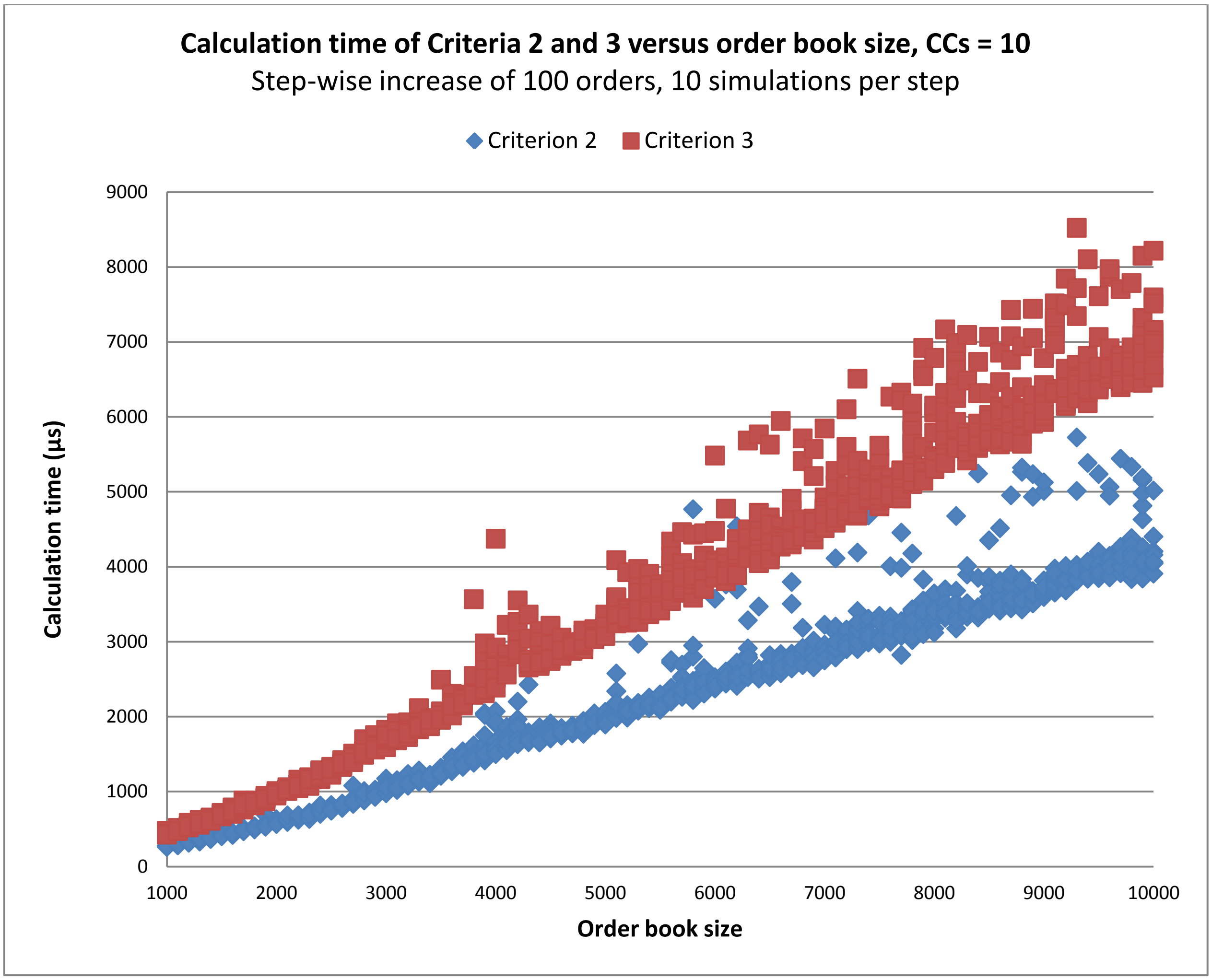


It is clear from plot 4.2 that the brute force method grows exponentially in computation time as the order book grows. What the plot does not show is the behavior of Criteria 2 and 3. In the next plots , the brute force method is excluded, and the size of the order book is increased by several orders of magnitude.

Time complexity of Criteria 2 and 3: moderate order book size

Without the burden of the brute force calculation, a time performance analysis of Criteria 2 and 3 can be performed with order books ranging from 1,000 up to 10,000, reflecting more realistic market conditions. Plot 4.3 details the result of such a simulation, with the orders spread across ten combined commodities.

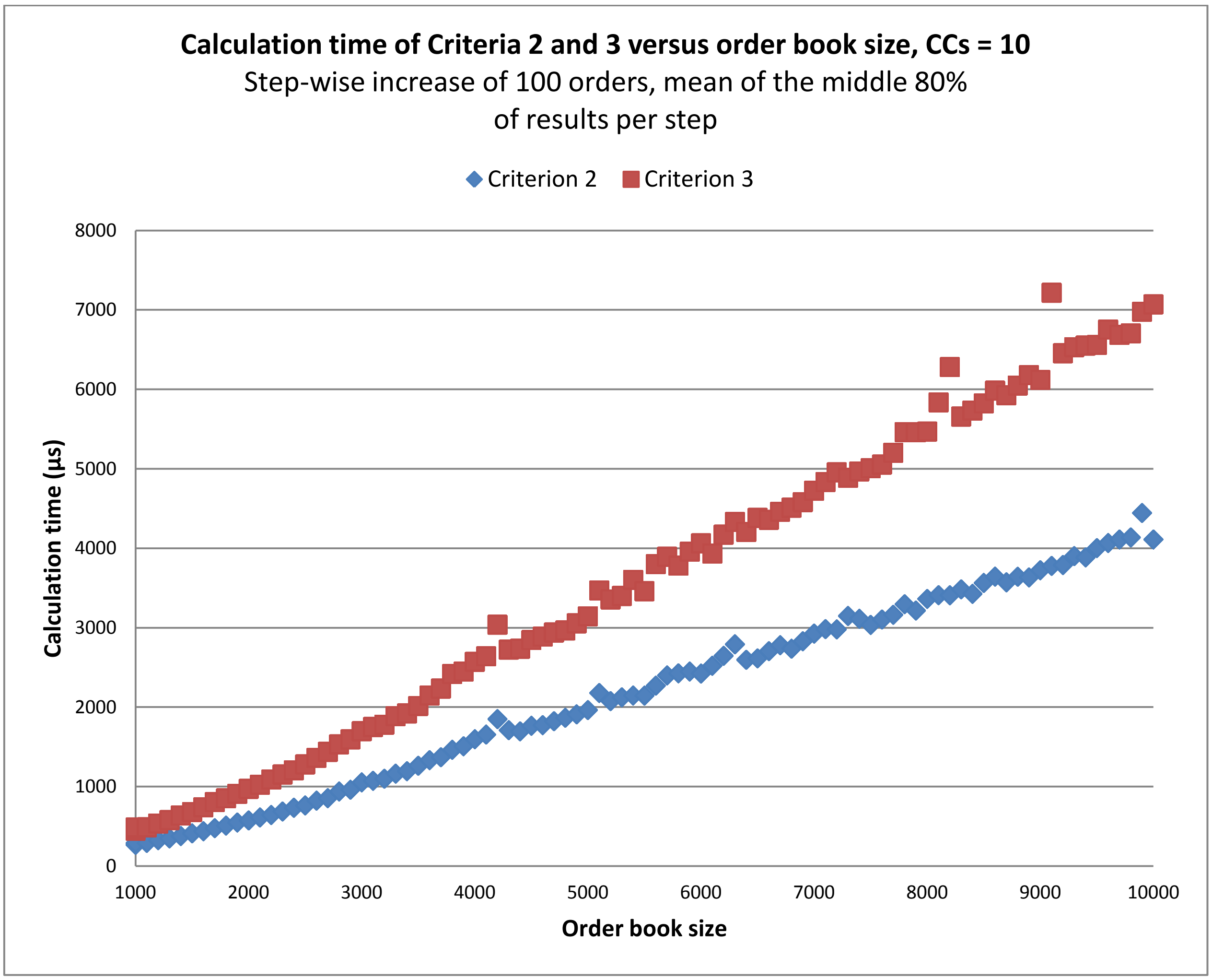
Plot 4.3 The diagram shows the computation times of Criteria 2 and 3 for increasing sizes of the order book. The order book size is increased in increments of 100 from 1,000 to 10,000, with 10 simulations performed at each step.



There is considerable variation at each point of the x-axis, indicating that the 10 simulations for each order book size yield different results. This, however, can be attributed to the specific JAVA implementation, and factors such as memory management, rather than a flaw inherent to the criteria. The variation notwithstanding, there are two clear trends being exhibited in plot 4.3. First, Criteria 2 and 3 both have a linear time complexity, with computation times apparently proportional to the size of the order book. Second, Criteria 2 is less computationally cumbersome than Criterion 3. This is to be expected: Criterion 3 is an extended version of Criterion 2, and the added Charge Impact component clearly has a negative effect on the general computation time.

Plot 4.4 shows a trimmed version of plot 4.3, where the mean of the middle 80% of results of the 10 simulations at each point in the x-axis is used as the final result. The variation is reduced as outliers are discarded, and the trends are even more clear. This mode of display will be used in the simulations ahead.

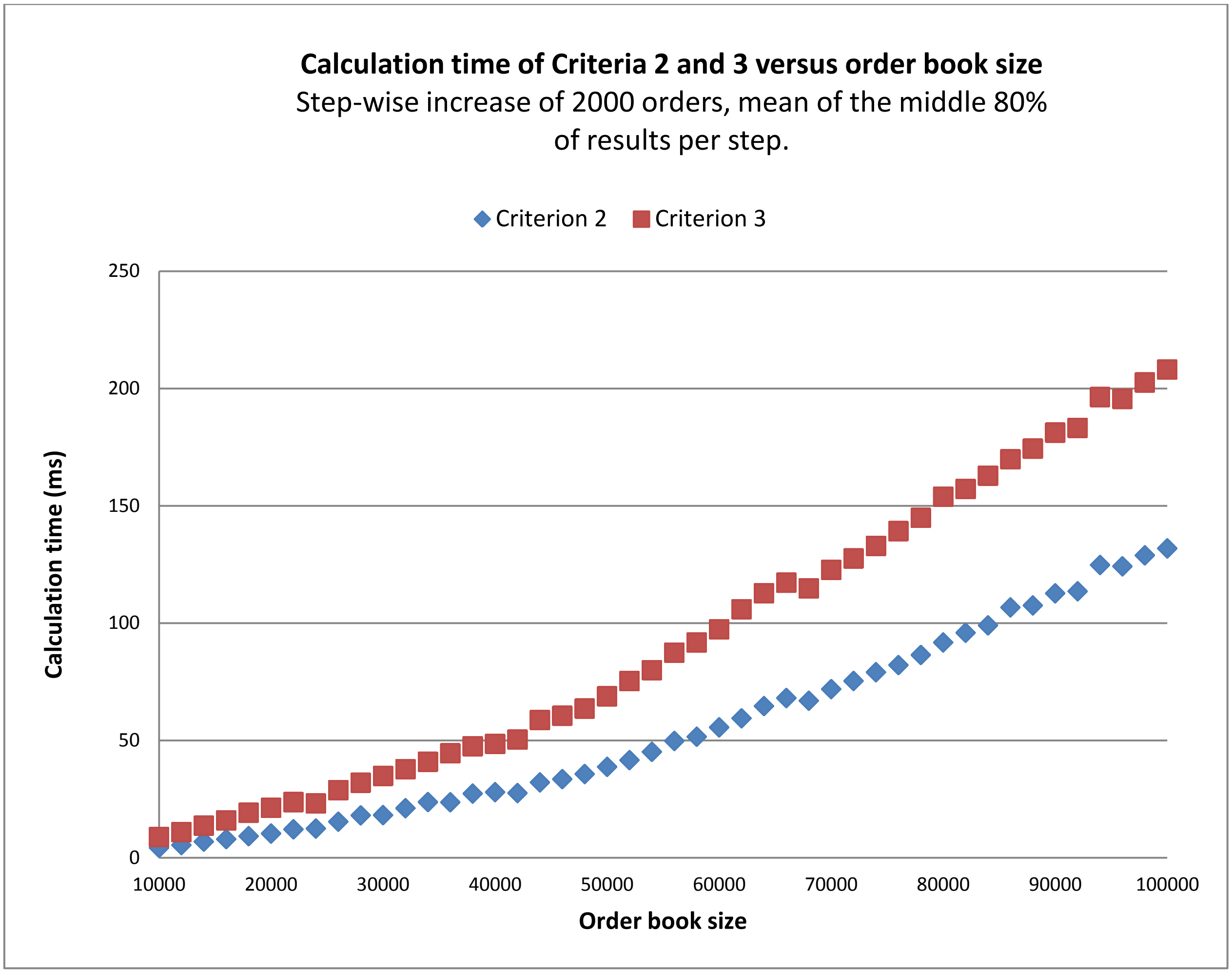
Plot 4.4 The diagram shows the computation times of Criteria 2 and 3 for increasing sizes of the order book. The order book size is increased in increments of 100 from 1,000 to 10,000. Every data point indicates the mean of the middle 80% of results of the 10 simulations performed at each step.



Time complexity of Criteria 2 and 3: large order book size

With moderately sized order books, both criteria display linear time complexity. However, does this behavior persist as the order book size grows even further. Plot 4.5 shows the results of a simulation similar to that of plots 4.3 and 4.4, but with the order book ranging from 10,000 up to 100,000.

Plot 4.5 The diagram shows the computation times of Criteria 2 and 3 for increasing sizes of the order book. The order book size is increased in increments of 2,000 from 10,000 to 100,000. Every data point indicates the mean of the middle 80% of results of the 10 simulations performed at each step.

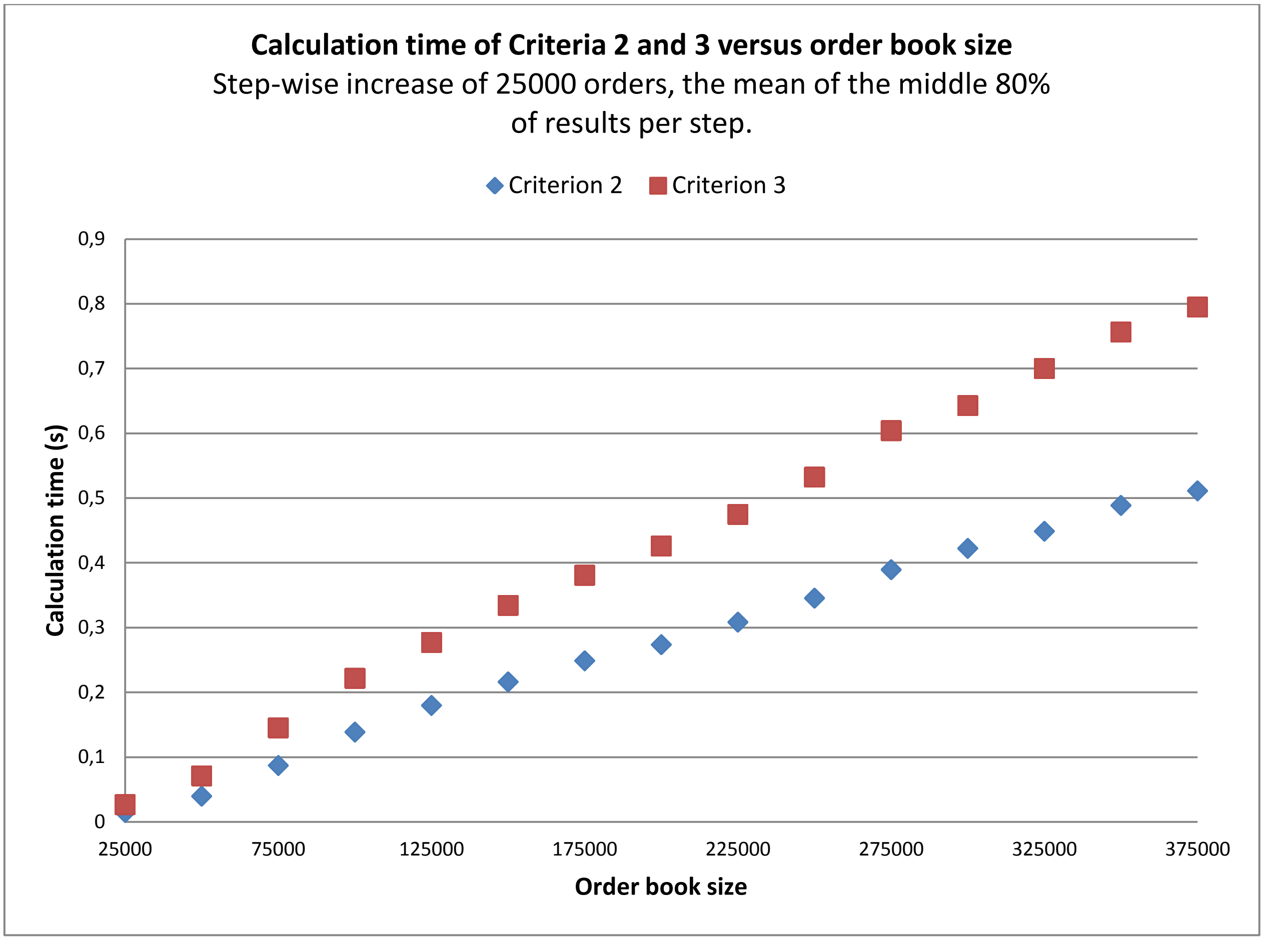


Here, the trends differ ever so slightly from a linear shape, suggesting that both Criteria 2 and 3 grow faster as the order book size increases. A simulation comprising even larger order books will determine if this is the case.

Time complexity of Criteria 2 and 3: very large order book size

The final simulation uses as input order books ranging from 25,000 up to 375,000. The results are shown in plot 4.6.

Plot 4.6 The diagram shows the computation times of Criteria 2 and 3 for increasing sizes of the order book. The order book size is increased in increments of 25,000 from 25,000 to 375,00,000. Every data point indicates the mean of the middle 80% of results of the 10 simulations performed at each step.



As plot 4.6 clearly shows, neither Criterion 2 nor 3 exhibit any exponential trends for very large order books. It seems to revert to a more linear pattern, akin to that for more moderately sized order books.

Similarly to the accuracy analysis, the results here suggest that Criterion 2 is the superior choice for worst-case portfolio selection from a time performance standpoint. Criterion 3 leverages extra calculations in order to better inform its decision to include or exclude an order, but this is not translated into better results, but worse. Criterion 2 presents a method for selection that is independent on the order in which orders are evaluated, correctly ignores spread charge contributions of orders to lessen computational burden, and produces the best results out of all proposed criteria. Finally, its computation time grows linearly with the size of the order book. It should therefore be a suitable algorithm from a pre-trade risk analysis standpoint.

Conclusion

General results

The initial problem formulation outlines a pre-trade risk validation algorithm that is able to find the worst-case portfolio with linear time complexity. The algorithm suggested in the following section, which makes use of a specified selection criterion to find the worst-case portfolio, fulfills this requirement. Three variations of such a selection criterion have been proposed and evaluated thoroughly. They each leverage combinations of different steps of the full SPAN risk calculation to evaluate the marginal impact of each order to the final initial margin requirement. Criterion 1 solely relies on maximizing the Scanning Risk step, whereas Criterion 2 adds the Net Option Value of each order in its evaluation. Criterion 3 takes it several steps further by also incorporating an approximation of the Intermonth Spread Charge and the Delivery Month Charge, called the Charge Impact.

Initial evaluations of accuracy of each of the three criteria results in Criterion 1 being dismissed. Further simulations are made to differentiate the performance of Criteria 2 and 3. Simulations 1 through 9 point to the same conclusion: Criterion 2 performs better overall than Criterion 3 in selecting orders out of a given order book into a worst-case portfolio. There is a considerable portion of runs in every simulation where Criterion 3 produces better results, and this grants a modicum of merit to the underlying approximations made in the Charge Impact calculation step. However, forgoing completely the spread charges in the selection process is the better option in a majority of cases, as Criterion 2 consistently outperforms its more comprehensive version.

Finally, the results from the time performance analysis indicate that Criteria 2 and 3 both have linear time complexity, and that Criterion 2 requires less computation time than Criterion 3. These results, together with the accuracy simulations, beg the conclusion that Criterion 2 is the superior method of selecting a worst-case portfolio out of a given order book in linear time.

Suggestions for further investigation

The results tell a biased tale: Criterion 3 does not produce results on par with its simpler predecessor. This indicates that the approximation of the spread charge steps is too crude to guide the decision of whether to include or exclude an order. The approximations of the Intermonth Spread Charge and the Delivery Month Charge made in the Charge Impact step rely on a single spread charge in a single tier. There are ways to refine this approximation, tailored for instance to the tier of each order to be included, that still allows for a proper evaluation of the marginal effect of each order.

An as of yet unexplored way of increasing time performance is to handle the selection process through parallel processing. Criterion 2 is especially suitable for this. Since the evaluation to include or discard an order rests on the characteristics of that order alone, the order book can be divided over several threads to more quickly determine which orders are to be included and which are to be discarded. With the order selection determined, the full SPAN calculation can then be performed on a single thread.

References

(n.d.). Retrieved 05 24, 2012, from Yahoo Finance: http://finance.yahoo.com/q/fc?s=HGK12.CMX

Benninga, S. (1989). *Numerical Techniques in Finance.* Cambridge, Massachusetts: The MIT Press.

CME Group. (n.d.). *SPAN.* Retrieved 05 22, 2012, from CME Group: http://www.cmegroup.com/clearing/risk-management/span-overview.html

Gottlieb, O. (2007, 07 11). *Find an Explicit Solution for Delta in Black-Scholes.* Retrieved 06 20, 2012, from http://www.soarcorp.com/research/Explicit_Derivation_of_Black_Scholes_Delta.pdf

Grant, J. (2012, 09 02). High-frequency trading: Up against a bandsaw. *Financial Times* .

Otten, R. (n.d.). *ElectronicSystems.* Retrieved 06 20, 2012, from Eindhoven University of Technology: http://www.es.ele.tue.nl/education/5MC10/Solutions/knapsack.pdf

Appendix

An explicit derivation of the Black-Scholes Delta (Gottlieb, 2007)

Start out with the expression for the call option:

$$c(S,t) = \Phi(d_1)S - \Phi(d_2)Ke^{-r(T-t)} \quad (1)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (2)$$

Differentiate (1) with respect to S:

$$\frac{\partial c}{\partial S} = \Phi(d_1) + \phi(d_1)S\frac{\partial d_1}{\partial S} - \phi(d_2)Ke^{-r(T-t)}\frac{\partial d_2}{\partial S} \quad (2)$$

The Chain Rule gives:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{\sigma S\sqrt{T-t}} \quad (3)$$

Inserting (3) into (2) gives:

$$\frac{\partial c}{\partial S} = \Phi(d_1) + \frac{1}{\sigma S\sqrt{T-t}}\left(\phi(d_1)S - \phi(d_2)Ke^{-r(T-t)}\right) \quad (4)$$

$$\phi(d_2)Ke^{-r(T-t)} = Ke^{-r(T-t)}\phi(d_1 - \sigma\sqrt{T-t}) = Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{\frac{-\left(d_1-\sigma\sqrt{T-t}\right)^2}{2}}$$

$$= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{\frac{-d_1{}^2}{2}}e^{\frac{-(-2d_1\sigma\sqrt{T-t}+\sigma^2(T-t))}{2}}$$

$$= Ke^{-r(T-t)}\phi(d_1)e^{d_1\sigma\sqrt{T-t}}e^{-\frac{\sigma^2(T-t)}{2}} \quad (5)$$

Inserting (2) into (5) gives:

$$K\phi(d_1)e^{-r(T-t)+\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)-\frac{\sigma^2(T-t)}{2}} = K\phi(d_1)\frac{S}{K} = \phi(d_1)S$$

$$\phi(d_1)S = \phi(d_2)Ke^{-r(T-t)} \quad (6)$$

Inserting (6) into (4) gives the expression for the delta of the call option:

$$\frac{\partial c}{\partial S} = \Phi(d_1)$$

For the put option:

$$p(S,t) = \Phi(-d_2)Ke^{-r(T-t)} - \Phi(-d_1)S = \big(1 - \Phi(d_2)\big)Ke^{-r(T-t)} - (1 - \Phi(d_1))S$$

$$= \Phi(d_1)S - \Phi(d_2)Ke^{-r(T-t)} + Ke^{-r(T-t)} - S = c(S,t) + Ke^{-r(T-t)} - S$$

which is the put-call parity. Lastly, the expression for the delta of a put option:

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1 = \Phi(d_1) - 1$$

The approximated Normal Distribution (Benninga, 1989)

Since the cumulative Normal Distribution cannot be evaluated exactly, the following numerical approximation is made:

$$\Phi_{num}(x) = 1 - h(x)(a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5) + error$$

where

$$h(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$t(x) = \frac{1}{(1 + px)}$$

$$p = 0.2316419$$

$$a_0 = 0.319381530$$

$$a_1 = -0.356563782$$

$$a_2 = 1.781477937$$

$$a_3 = -1.821255978$$

$$a_4 = 1.330274429$$

The approximation error is

$$error < 7.5 * 10^{-8}$$