Money Management Principles for Mechanical Traders

Shlok Datye

Royal Institute of Technology
Stockholm, Sweden
November 2012

Abstract

In his five books during 1990–2009, starting with *Portfolio Management Formulas*, Ralph Vince made accessible to mechanical traders with limited background in mathematics various important concepts in the field of money management. During this process, he coined and popularized the terms "optimal f" and "leverage space trading model."

This thesis provides a sound mathematical understanding of these concepts, and adds various extensions and insights of its own. It also provides practical examples of how mechanical traders can use these concepts to their advantage. Although beneficial to all mechanical traders, the examples involve trading futures contracts, and practical details such as the back-adjustment of futures prices are provided along the way.

Website

The author intends to maintain a companion website for this thesis at `http://shlok.is/thesis`, through which he can be contacted, and where additional material such as errata and further research can be found. However, he makes no guarantees to this effect.

Acknowledgments

I thank Filip Lindskog for the outstanding supervision that he provided me with during the creation of this thesis.

I dedicate this thesis to my parents and brother

Shlok Datye

Contents

Introduction

Mechanical traders are familiar with trading systems. These systems take in a certain amount of money and produce an amount of money that is indeterminate in advance. A mechanical trader repeatedly plays a system he deems to have an edge, hoping to make profits in the long run.

The primary purpose of this thesis is to formalize the concept of a trading system and to obtain various useful theoretical results about it in the field of money management. These results have in the trading community become known broadly as the "leverage space trading model" and contain the concept of the "optimal f"—both terms coined and popularized by Vince (1990–2009). The secondary purpose of this thesis is to demonstrate with practical examples how traders can use the theory to their benefit.

Chapter 1 formalizes the concept of a trading system and provides a mathematical recipe for obtaining that portion of our capital that we should place into each trade if we wish to maximize the long-term exponential growth of our capital. This portion, also referred to as the position size, can turn out to be greater than 1. In other words, the theory allows for taking on leverage when this is what is required in order to maximize the exponential growth. In fact, taking on leverage is nothing particularly special under the theory, but a natural and inherent part of it.

Chapter 2 provides an example of how the theory can be applied in practice. First, a historically profitable trading system is constructed. The theoretical apparatus from chapter 1 is then used to approximate the optimal position size. The trading system that we construct in this chapter involves trading futures contracts, and we outline along the way the important practical concept of the back-adjustment of futures prices.

Chapter 3 discusses the topic of diversification, i.e. how we should play more than one trading system at once. Under our theory, "diversification" is nothing but a natural extension of the theory of position sizing in chapter 1 to multiple dimensions. Chapter 4 provides an example of how such diversification can be carried out in practice.

Chapter 5 analyzes the concept of a drawdown, which is a measure of drops that occur in our capital as time goes by. Drawdowns are what traders perceive as being their "risk," and we explain how they can be constrained. Chapter 6 provides a practical example of this.

Finally, chapter 7 irons out two loose ends. First, we explain how to properly take into account contract sizes and margin requirements. (Up to this point, we will have been tacitly assuming that we can trade fractions of contracts.) Second, we provide detailed step-by-step instructions that mechanical traders can refer to during their trading operations; these steps explain how our theory fits into the picture, and what further research has to be carried out that is not covered in this thesis.

1

Position Sizing: Theory

1. Introduction

A *trading system* is a thing, or a machine, that takes in a dollars and ejects $a(1+X)$ dollars, where X is a random variable describing the percentage return of the system. This is depicted in figure 1.

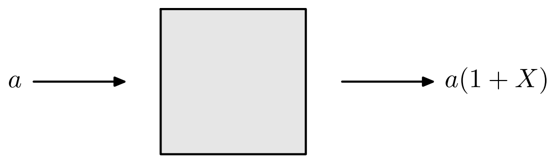

Figure 1. A Trading System.

The percentage returns from different plays in a trading system are assumed to be independent and identically distributed. The random variable X is assumed to be either discrete or continuous; to have a finite expectation; and to have a positive probability of being positive and a positive probability of being negative. It is also assumed that there exists an $L > 0$ such that X has a zero probability of being below $-L$ and a positive probability of being between $-L$ and $-L+\epsilon$ inclusive, for arbitrarily small $\epsilon > 0$. As an example, if X is continuous, its density could look as shown in figure 2. The number L will informally be referred to as

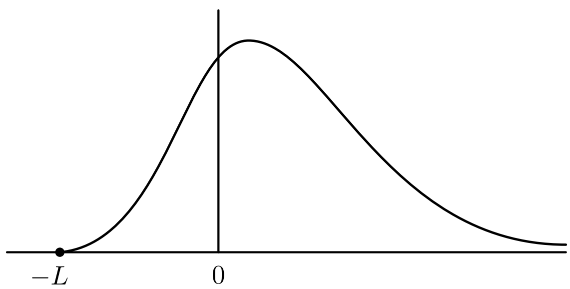

Figure 2. An Example of a Return Density of a Trading System with a Continuous Return Distribution.

the “largest possible loss,” even though it can have a zero probability of occurring; for discrete X, this probability is of course positive.

The assumption about X having a positive probability of being positive and a positive probability of being negative is a natural one. If this were not the case, there would hardly be a need for any theory: If we were guaranteed to lose, we would never trade; and if we were guaranteed to win, we could always play with our entire capital and more besides.

The reason for the assumption of X having a largest possible loss L will become clear in section 4. For now, note that this assumption is not unreasonable. For example, if a system involves buying a stock, we could have $L = 1$ (i.e. no chance of more than a 100% loss). If, in addition, a stop-loss order is employed, L could very well be less than 1. Short positions without stop-loss orders could be impossible to deal with in theory (since the largest possible loss could be infinite); but in practice, a reasonable compromise such as $L = 2$ could be applied.

2. *The Concept of Time Preference*

It should be mentioned that the concept of time preference is not included in our theory. The reason is that we do not necessarily know in advance how long time it is going to take a trading system to eject. This means, in particular, that the concept of a “risk-free interest rate” is not included in the theory. Money that is not at work in a trading system is assumed to remain constant. Criteria outside of this theory (or work done by someone else to incorporate time preference into the theory) will have to determine the placement of money in a risk-free bank account. When we mention such things as “time” or “long-term” in this thesis, we may be informally referring to the number of plays in a trading system.

3. *Betting Strategy*

A question now arises: How should we play a given trading system? Kelly (1956) suggested that we should place the same *proportionally fixed* portion of our available capital into each play, and derived for discrete return distributions the portion that maximizes the long-term exponential growth of our capital curve. Breiman (1961) showed for discrete return distributions that this fixed-portion strategy is indeed the “best” possible strategy to employ in various long-term senses. Finkelstein and Whitley (1981) generalized Breiman’s results about particular long-term benefits of this strategy to arbitrary return distributions with finite expectations. For these reasons, we will exclusively focus our attention to fixed-portion trading in this thesis.

4. *The Function $G(f)$*

Starting with one dollar, we now begin repeatedly playing a given trading system. We denote the percentage return from play n with X_n. We have already decided to play with a proportionally fixed portion of our capital. To that end, we let f/L be the portion of our available capital that we use in every play, where f is some number in $(0,1)$; this construction will be explained momentarily. We denote our starting capital with $C_0 = 1$ (one dollar), and our capital after n plays with C_n. Clearly, then, our capital after n plays is provided recursively by

$$C_n = \Big(1 - \frac{f}{L}\Big)C_{n-1} + \frac{f}{L}C_{n-1}(1 + X_n) = C_{n-1}\Big(1 + \frac{f}{L}X_n\Big),$$

which can be written in closed form as

$$C_n = \prod_{i=1}^{n}\Big(1 + \frac{f}{L}X_i\Big).$$

The ideas of the largest possible loss L, and of using the portion f/L to trade with in every play, appeared in Vince (1990). Their purpose should now be more clear. First, we are made sure that our capital always stays positive, which will be of benefit below when we are assured that we are not taking the logarithm of a nonpositive number. Second, we get the aesthetical benefit of working with a fraction f on the unit interval $(0,1)$. Finally, note that if $L < 1$, we have $f/L > 1$ for some values of f. In other words, the theory allows taking on leverage. As we just mentioned, however, it never allows too much aggressiveness—we never take so much leverage that our capital could possibly become nonpositive. (The reason for not allowing $f = 1$ is to eliminate the possibility of the capital ever becoming zero. The case of $f = 0$ is not allowed because it is uninteresting.)

We now define the random variables G_n with

$$C_n = e^{nG_n}.$$

In other words, G_n is the exponential growth of our capital curve after n plays. We wish to understand what happens to G_n in the long run. To that end, we rewrite the above equation as follows:

$$G_n = \frac{1}{n}\log C_n = \frac{1}{n}\log\prod_{i=1}^{n}\Big(1 + \frac{f}{L}X_i\Big) = \frac{1}{n}\sum_{i=1}^{n}\log\Big(1 + \frac{f}{L}X_i\Big).$$

Before continuing, we want to show that $\mathbb{E}|\log(1 + fX_1/L)| < \infty$. To that end, we look at the function $x \mapsto |\log(1 + fx/L)|$, which is shown in figure 3. The fact that $\log(1 + fx/L) < fx/L$ for $x > 0$, along with $\mathbb{E}[X_1]$ being finite (as was assumed in section 1), establishes the convergence of $\mathbb{E}|\log(1 + fX_1/L)|$ towards infinity. And since $-L/f < -L$, and L is the largest possible loss of X_1, there is no other concern about convergence. This establishes what we wanted.

Now, since $\mathbb{E}|\log(1 + fX_1/L)| < \infty$, the Kolmogorov strong law of large numbers (see Gut (2005), page 295) tells us that

$$G_n \xrightarrow{\text{a.s.}} \mathbb{E}\log\Big(1 + \frac{f}{L}X_1\Big) \quad \text{as} \quad n \to \infty.$$

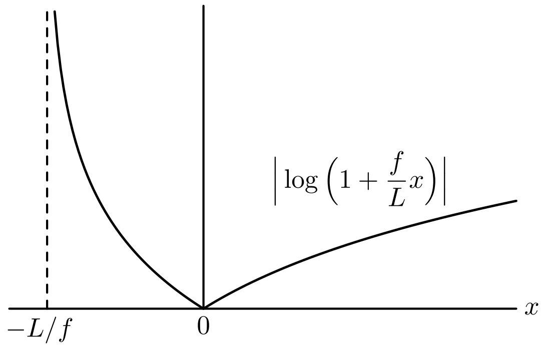

FIGURE 3. THE FUNCTION $x \mapsto |\log(1 + fx/L)|$.

This motivates the definition of a new function,

$$G(f) := \mathbb{E}\log\Big(1 + \frac{f}{L}X_1\Big), \tag{1}$$

which we call the *long-term exponential growth function* of our trading system.

Intuitively, we understand that we are supposed to find an f on $(0,1)$ that maximizes $G(f)$ if we wish to obtain the greatest long-term exponential growth for our capital. This intuition is theoretically confirmed in the next section.

5. The Importance of $G(f)$

The following theorems provide us with a deeper understanding of the importance of the function $G(f)$. Analogous theorems (with different proofs) are provided in Thorp (1969), page 285, for the simple case of coin tossing.

THEOREM 1. *(a) If* $G(f) > 0$, *then* $C_n \xrightarrow{\text{a.s.}} \infty$ *as* $n \to \infty$.
(b) If $G(f) < 0$, *then* $C_n \xrightarrow{\text{a.s.}} 0$ *as* $n \to \infty$.
(c) If $G(f) = 0$, *then* $\limsup_{n\to\infty} C_n = \infty$ *a.s. and* $\liminf_{n\to\infty} C_n = 0$ *a.s.*

In practice, this theorem tells us that if $G(f) > 0$, we will make infinite fortunes in the long run. However, if $G(f) < 0$, we will go broke in the long run. Finally, if $G(f) = 0$, our capital will fluctuate wildly between 0 and ∞ as time goes by.

PROOF. Define $\xi_i := \log(1 + fX_i/L)$. We know that X_i is nondegenerate, i.e. that the outcome of X_i is not known in advance (this was assumed in section 1). Therefore, ξ_i is also nondegenerate. Now define the nondegenerate random walk (S_n) with $S_n := \xi_1 + \cdots + \xi_n$. We know that $\log C_n = S_n$, which gives us $C_n = e^{S_n}$. We also know that $G(f) = \mathbb{E}[\xi_1]$. The entire theorem now follows from proposition 9.14 in Kallenberg (2002), page 167, which tells us that $S_n \xrightarrow{\text{a.s.}} \infty$ if $\mathbb{E}[\xi_1] > 0$, $S_n \xrightarrow{\text{a.s.}} -\infty$ if $\mathbb{E}[\xi_1] < 0$, and $\limsup_{n\to\infty}(\pm S_n) = \infty$ a.s. if $\mathbb{E}[\xi_1] = 0$. □

Let us now imagine that we play a given trading system in two ways simultaneously—one dollar going into plays where we use the fraction $f = f_1$, and another dollar going into plays where we use the fraction $f = f_2$. We denote the two capital trajectories with $C_n(f_1)$ and $C_n(f_2)$ respectively.

THEOREM 2. *If* $G(f_1) > G(f_2)$, *then* $C_n(f_1)/C_n(f_2) \xrightarrow{\text{a.s.}} \infty$ *as* $n \to \infty$.

In practice, this theorem tells us that if we have two fractions f_1 and f_2 such that $G(f_1) > G(f_2)$, we will miss out on infinite fortunes in the long run if we trade with the fraction f_2.

PROOF. The idea is similar as in the proof of theorem 1. Define

$$\xi_i := \log\Big(1 + \frac{f_1}{L}X_i\Big) - \log\Big(1 + \frac{f_2}{L}X_i\Big).$$

We know that X_i is nondegenerate. Hence, ξ_i is nondegenerate. Define the nondegenerate random walk (S_n) with $S_n := \xi_1 + \cdots + \xi_n$. Note that $\mathbb{E}[\xi_1] = G(f_1) - G(f_2) > 0$. From proposition 9.14 in Kallenberg (2002), page 167, it follows that $S_n \xrightarrow{\text{a.s.}} \infty$. But since

$$S_n = \sum_{i=1}^{n} \Big[\log\Big(1 + \frac{f_1}{L}X_i\Big) - \log\Big(1 + \frac{f_2}{L}X_i\Big)\Big] = \log\frac{C_n(f_1)}{C_n(f_2)},$$

it follows that $C_n(f_1)/C_n(f_2) \xrightarrow{\text{a.s.}} \infty$. □

To summarize this section, theorem 1 tells us that we should trade with a fraction f such that $G(f) > 0$. If we do this, we will make infinite fortunes in the long run. If we do not, we will either get unpredictable results or go broke. Theorem 2 tells us that if we trade using an f that does not maximize $G(f)$, we will miss out on infinite fortunes in the long run compared to what we could have gained by performing the maximization—even if we are already set to make infinite fortunes.

6. *The Shape of $G(f)$*

We have seen the importance of the function $G(f)$, which is given by

$$G(f) = \mathbb{E}\log\Big(1 + \frac{f}{L}X_1\Big).$$

In this section, we obtain some general results about the shape of $G(f)$. In the process, we also obtain an interesting result about our trading system's return expectation, $\mathbb{E}[X_1]$. We start by finding the first two derivatives of $G(f)$.

If X_1 is continuous with density $\rho(x)$, we find for f in $(0, 1)$ that

$$G(f) = \int_{-L}^{\infty} \log\Big(1 + \frac{f}{L}x\Big)\rho(x)\,dx,$$

$$G'(f) = \frac{1}{L}\int_{-L}^{\infty} \frac{x}{1 + fx/L}\rho(x)\,dx,$$

and

$$G''(f) = -\frac{1}{L^2}\int_{-L}^{\infty} \frac{x^2}{(1+fx/L)^2}\rho(x)\,dx.$$

On the other hand, if X_1 is discrete where the outcomes $x_1, \dots, x_r$ can occur with probabilities $p_1, \dots, p_r$, we find for f in $(0,1)$ that

$$G(f) = \sum_{i=1}^{r} p_i \log\Big(1 + \frac{f}{L}x_i\Big),$$

$$G'(f) = \frac{1}{L}\sum_{i=1}^{r} p_i \frac{x_i}{1 + fx_i/L},$$

and

$$G''(f) = -\frac{1}{L^2}\sum_{i=1}^{r} p_i \frac{x_i^2}{(1+fx_i/L)^2}.$$

In both the continuous and discrete case, we immediately notice that

$$G'(h) \xrightarrow[h\to 0^+]{} \frac{1}{L}\mathbb{E}[X_1] \qquad \text{and} \qquad G''(f) < 0 \quad \text{for all} \quad f \in (0,1).$$

This means that the slope of $G(f)$ starts out at $\mathbb{E}[X_1]/L$ near zero, and that thereafter, as $f \to 1$, the slope is strictly decreasing. From these observations, we can deduce the various possible shapes of $G(f)$.

We first consider the case of the return expectation, $\mathbb{E}[X_1]$, being zero or negative. In this case, $G(f)$ can look as shown in figure 4. Curves (a) and (b) correspond to a zero return expectation, and curves (c) and (d) correspond to a negative one. Curves (a) and (c) approach finite values as $f \to 1$, whereas curves (b) and (d) approach $-\infty$. From the results in section 5 above, we immediately

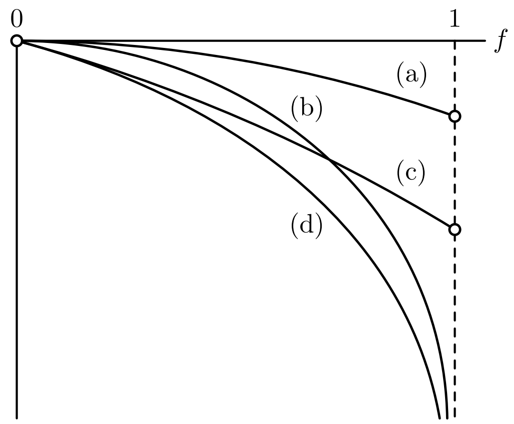

FIGURE 4. THE FUNCTION $G(f)$ FOR TRADING SYSTEMS WITH NONPOSITIVE RETURN EXPECTATIONS.

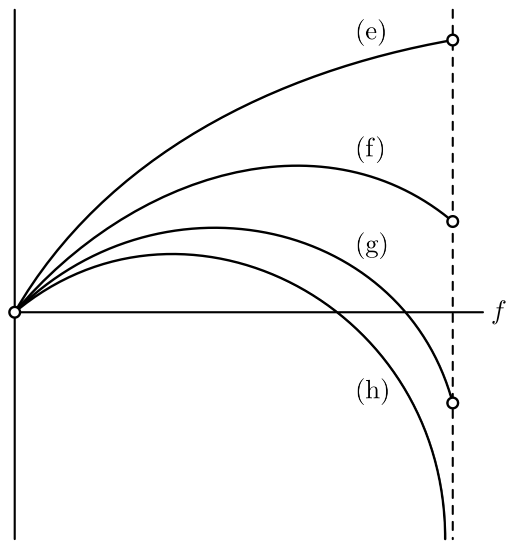

FIGURE 5. THE FUNCTION $G(f)$ FOR TRADING SYSTEMS WITH POSITIVE RETURN EXPECTATIONS.

notice that we will go broke in the long run no matter what f we choose. We have therefore theoretically confirmed what traders are already aware of—that trading with a negative return expectation is an exercise in futility. We have also shown the futility of trading with a zero return expectation.

If $\mathbb{E}[X_1]$ is positive, $G(f)$ can look as shown in figure 5, where all the curves begin with a positive slope, corresponding to the positive return expectation. Curves (e), (f), and (g) approach finite values as $f \to 1$, whereas curve (h) approaches $-\infty$. On curve (e), we will get most growth for f that is arbitrarily near 1. On the other three curves, the most growth is obtained with an f between 0 and 1. Curves (g) and (h) are particularly interesting; for each of them, there exists a fraction f^* such that $G(f^*) = 0$, below which we will be profitable, and above which we will face a disaster.

We now make an important observation about *discrete* return distributions. The function $G(f)$ for a discrete return distribution, where the outcomes $x_1, \ldots, x_r$ can occur with probabilities $p_1, \ldots, p_r$, was found above to be

$$G(f) = \sum_{i=1}^{r} p_i \log\Big(1 + \frac{f}{L} x_i\Big).$$

We know in this case that there exists an x_i such that $x_i = -L$ and $p_i > 0$, i.e. that the largest possible loss L has a positive probability of occurring. Hence, we immediately notice that $G(f) \to -\infty$ as $f \to 1$. This implies that for discrete return distributions with positive return expectations, $G(f)$ can only take on shape (h) in figure 5. (If the return expectation is nonpositive, $G(f)$ can only take on shapes (b) or (d) in figure 4.) Most traders will in practice only be using discrete distributions to analyze their trading systems (which is what we do in the next chapter). Such traders will, therefore, only encounter shape (h) in figure 5 when analyzing their profitable trading systems.

The value of f that maximizes shape (h) in figure 5 is what Vince (1990–2009) has popularized as the "optimal f." This term is becoming ever more established in the trading community, so we will hereafter write it without the surrounding quotes.

7. *Over- or Underestimating Profits or Losses*

We consider again a trading system with a discrete return distribution, where the outcomes $x_1, \ldots, x_r$ can occur with probabilities $p_1, \ldots, p_r$, and

$$G(f) = \sum_{i=1}^{r} p_i \log\left(1 + \frac{f}{L} x_i\right).$$

Let us now imagine that we overestimate the profits of this system by adding a new outcome x_{r+1}, with $x_{r+1} > x_i$ for all $i \in \{1, \ldots, r\}$, that can occur with probability p_{r+1}, such that each x_i, $i \in \{1, \ldots, r\}$, can now occur with probability $p_i(1 - p_{r+1})$. The exponential growth function corresponding to this scenario of overestimated profits is given by

$$\begin{aligned} G^*(f) &= \sum_{i=1}^{r} p_i(1 - p_{r+1}) \log\left(1 + \frac{f}{L} x_i\right) + p_{r+1} \log\left(1 + \frac{f}{L} x_{r+1}\right) \\ &= G(f) + p_{r+1}\left[\log\left(1 + \frac{f}{L} x_{r+1}\right) - \sum_{i=1}^{r} p_i \log\left(1 + \frac{f}{L} x_i\right)\right]. \end{aligned}$$

Since $x \mapsto \log x$ is an increasing function, we immediately notice that $G^*(f) > G(f)$ for all f. This means that overestimating profits (in the way we have done it) will "push $G(f)$ upwards."

We next show that if $G'(f_1) = 0$ and $(G^*)'(f_2) = 0$, then $f_2 > f_1$; this means that overestimating profits (in the way we have done it) will "shift the optimal f to the right." Note that

$$(G^*)'(f) = G'(f) + \frac{p_{r+1}}{L}\left(\frac{x_{r+1}}{1 + f x_{r+1}/L} - \sum_{i=1}^{r} p_i \frac{x_i}{1 + f x_i/L}\right).$$

Now, assume that $G'(f_1) = 0$. It can easily be shown that $x \mapsto x/(1 + bx)$, for some constant $b > 0$, is an increasing function (for $x > -1/b$), from which it follows that $(G^*)'(f) > G'(f)$ for all f, which gives us $(G^*)'(f_1) > 0$. With a second differentiation, it is easily shown that $(G^*)'(f)$ is a decreasing function, which tells us that if $(G^*)'(f_2) = 0$, we must have $f_2 > f_1$.

We have therefore shown that overestimating the profits of a trading system (in the way we have done it) results in us overestimating the exponential growth of our system at every f, and believing that the optimal f is larger than it actually

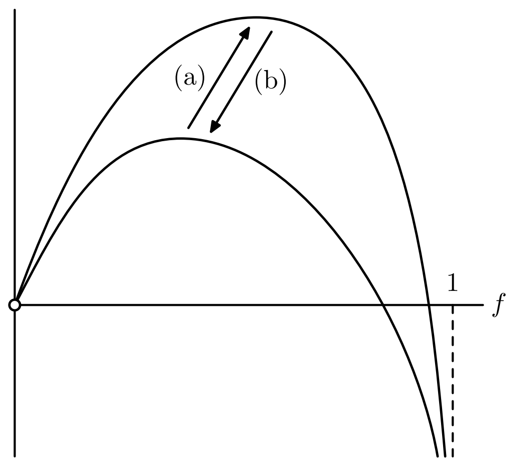

FIGURE 6. THE EFFECTS ON THE FUNCTION $G(f)$ OF OVER- OR UNDERESTIMATING PROFITS.

is. This is depicted with arrow (a) in figure 6. On the flip side, underestimating the profits (by removing the largest possible gain) will result in us believing the opposite, as depicted with arrow (b) in the same figure.

Let us now consider what happens if we, instead of overestimating profits, overestimate losses. To that end, let us add to the original return distribution—where $x_1, \ldots, x_r$ can occur with probabilities $p_1, \ldots, p_r$—a new outcome x_0, with $x_0 < x_i$ for all $i \in \{1, \ldots, r\}$, that can occur with probability p_0, such that each x_i, $i \in \{1, \ldots, r\}$, can now occur with probability $p_i(1 - p_0)$. We denote the largest possible loss corresponding to this scenario of overestimated losses with $L^* := -x_0$. The exponential growth function for this scenario is

$$\begin{aligned} G^*(f) &= p_0 \log\left(1 + \frac{f}{L^*} x_0\right) + \sum_{i=1}^{r} p_i (1 - p_0) \log\left(1 + \frac{f}{L^*} x_i\right) \\ &= \sum_{i=1}^{r} p_i \log\left(1 + \frac{f}{L^*} x_i\right) + p_0 \left[\log\left(1 + \frac{f}{L^*} x_0\right) - \sum_{i=1}^{r} p_i \log\left(1 + \frac{f}{L^*} x_i\right)\right]. \end{aligned}$$

Before continuing, we note that it is, in this case, rather meaningless to compare $G(f)$ directly with $G^*(f)$. The reason is that these functions are based on different largest possible losses, and that f measures, in each case, the position size in relation to the largest possible loss. Comparing $G(f)$ with $G^*(f)$ would thus be akin to comparing apples with oranges.

Define the functions $H\colon (0, 1/L) \to \mathbb{R}$ and $H^*\colon (0, 1/L^*) \to \mathbb{R}$ with $H(g) := G(gL)$ and $H^*(g) := G^*(gL^*)$. These functions describe the exponential growths of the trading systems corresponding to G and G^* respectively, as functions of the actual position sizes. It is therefore more appropriate to compare H and H^*.

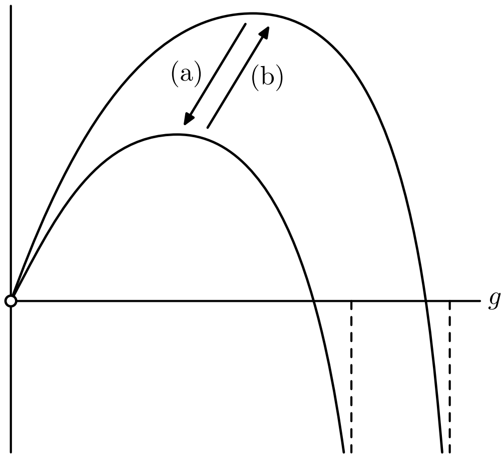

FIGURE 7. THE EFFECTS ON THE FUNCTION $H(g)$ OF OVER- OR UNDERESTIMATING LOSSES.

We find that

$$H(g) = \sum_{i=1}^{r} p_i \log(1 + gx_i)$$

and

$$H^*(g) = \sum_{i=1}^{r} p_i \log(1 + gx_i) + p_0 \left[\log(1 + gx_0) - \sum_{i=1}^{r} p_i \log(1 + gx_i) \right].$$

Using similar methods as we used when we were overestimating profits, we can easily show that $H^*(g) < H(g)$ for all $g \in (0, 1/L^*)$; and that if $(H^*)'(g_1) = 0$ for some $g_1 \in (0, 1/L^*)$ and $H'(g_2) = 0$ for some $g_2 \in (0, 1/L)$, then we must have $g_1 < g_2$.

This establishes that overestimating the losses of a trading system (in the way we have done it) results in us underestimating the exponential growth of our system for every g that is now allowed, and believing that our optimal position size is smaller than it actually is. This is depicted with arrow (a) in figure 7. On the flip side, underestimating the losses (by removing the largest possible loss) will result in us believing the opposite, as depicted with arrow (b) in the same figure.

We conjecture that the effects shown in figures 6 and 7 continue to hold under more general types of over- and underestimations than we have mathematically shown here. Hopefully, someone else will analyze this conjecture.

The real-life implications of this section are that if we are ever unsure about the exact shape of $G(f)$, we are *safer to underestimate profits than to overestimate them*, and *safer to overestimate losses than to underestimate them.* Underestimating losses is particularly dangerous; it may end up allowing us to take on too aggressive position sizes—ones that will eventually make our capital negative.

8. *Extension to a Nonstationary Environment*

In practice, we can rarely play the same trading system infinitely many times, as we have been doing above. This section provides an extension of our theory to a nonstationary environment.

Assume that we have access to infinitely many trading systems indexed with $m \in \{1, 2, \ldots\}$. Let N_m be the number of times we play system m, and let $X_i^{(m)}$ be the random variable describing the percentage return of play i in system m. All of these random variables are assumed to be independent of each other. However, percentage returns from the different systems do *not* necessarily have the same distribution. Continuing this natural extension of what we have already covered in the case of a single trading system, let L_m be the largest possible loss of system m, f_m be the fraction that we use to trade with in this system, and $G_m(f_m)$ be its long-term exponential growth function. Note that

$$G_m(f_m) = \mathbb{E} \log \Big(1 + \frac{f_m}{L_m} X_1^{(m)}\Big).$$

Let us now start trading with the amount $C_0 = 1$ (one dollar). It is easily seen that our capital after playing M trading systems is given by

$$C_M = \prod_{m=1}^{M} \prod_{i=1}^{N_m} \Big(1 + \frac{f_m}{L_m} X_i^{(m)}\Big).$$

Define the random variables G_M, which represent the exponential growth of our capital after trading M systems, with

$$C_M = \exp\Big(G_M \sum_{m=1}^{M} N_m\Big). \tag{2}$$

We find that

$$\begin{aligned} G_M &= \frac{1}{\sum_{m=1}^{M} N_m} \log \Big[\prod_{m=1}^{M} \prod_{i=1}^{N_m} \Big(1 + \frac{f_m}{L_m} X_i^{(m)}\Big) \Big] \\ &= \frac{1}{\sum_{m=1}^{M} N_m} \sum_{m=1}^{M} \sum_{i=1}^{N_m} \log \Big(1 + \frac{f_m}{L_m} X_i^{(m)}\Big). \end{aligned}$$

The following lemma, which can be found in Gut (2005), page 288, provides us with information about the average of infinitely many independent random variables that are not necessarily identically distributed.

Lemma 1. *(The Kolmogorov sufficient condition.) Let $Y_1, Y_2, \ldots$ be independent random variables with mean 0 and finite variances σ_n^2, $n \geq 1$, and set $S_n = \sum_{k=1}^{n} Y_k$, $n \geq 1$. Then,*

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} < \infty \qquad \Longrightarrow \qquad \frac{S_n}{n} \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty.$$

We can use this lemma to our benefit if we assume that

$$\sum_{m=1}^{\infty} \sum_{i=1}^{N_m} \frac{\operatorname{Var}\bigl(\log(1+f_m X_1^{(m)}/L_m)\bigr)}{\bigl(\sum_{j=0}^{m-1} N_j + i\bigr)^2} < \infty, \tag{3}$$

where we have set $N_0 := 0$. This assumption is true for all practical purposes: First, the quantity $\operatorname{Var}(\log(1+f_m X_1^{(m)}/L_m))$ is finite for each m. (This is an easy consequence of the fact that $(\log(1+x))^2 < x$ for all $x > 0$.) Second, there is no reason to believe that this quantity will, in practice, ever approach infinity as $m \to \infty$.

Using this assumption and lemma 1, we find that

$$\frac{1}{\sum_{m=1}^{M} N_m} \sum_{m=1}^{M} \sum_{i=1}^{N_m} \Bigl[\log\Bigl(1+\frac{f_m}{L_m} X_i^{(m)}\Bigr) - G_m(f_m)\Bigr] \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad M \to \infty. \tag{4}$$

Before continuing, the following lemma is in order.

Lemma 2. *Let $Y_1, Y_2, \ldots$ be random variables such that*

$$\frac{1}{n} \sum_{i=1}^{n} \bigl(Y_i - \mathbb{E}[Y_i]\bigr) \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty,$$

(a) If $\liminf_{n\to\infty} \mathbb{E}[Y_n] > 0$, then $\sum_{i=1}^{n} Y_i \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$.
(b) If $\limsup_{n\to\infty} \mathbb{E}[Y_n] < 0$, then $\sum_{i=1}^{n} Y_i \xrightarrow{\text{a.s.}} -\infty$ as $n \to \infty$.

Proof. We only prove (a); the proof of (b) is similar. Take some $c > 0$ such that $c < \liminf_{n\to\infty} \mathbb{E}[Y_n]$. We know that finitely many $\mathbb{E}[Y_n]$ are below or equal to c and that infinitely many $\mathbb{E}[Y_n]$ are above c. Therefore, there exists an N such that

$$\sum_{i=1}^{n} \mathbb{E}[Y_i] \geq nc \quad \text{for all} \quad n \geq N,$$

which gives us

$$\frac{1}{n} \sum_{i=1}^{n} \bigl(Y_i - \mathbb{E}[Y_i]\bigr) \leq \frac{1}{n} \sum_{i=1}^{n} Y_i - c \quad \text{for all} \quad n \geq N.$$

We know that the left-hand side of this inequality almost surely approaches zero. This implies that $\sum_{i=1}^{n} Y_i \xrightarrow{\text{a.s.}} \infty$, for otherwise the inequality would be contradicted. □

The following theorems confirm the importance of the exponential growth functions for the individual trading systems.

Theorem 3. *Given inequality* (3), *the following holds:*
(a) If $\liminf_{m\to\infty} G_m(f_m) > 0$, then $C_M \xrightarrow{\text{a.s.}} \infty$ as $M \to \infty$.
(b) If $\limsup_{m\to\infty} G_m(f_m) < 0$, then $C_M \xrightarrow{\text{a.s.}} 0$ as $M \to \infty$.

In practice, this theorem tells us that if we maintain "sufficiently positive" values for $G_m(f_m)$, we will make infinite fortunes in the long run, and that if we maintain "sufficiently negative" values for $G_m(f_m)$, we will go broke in the long run.

PROOF. Apply lemma 2 on formulas (2) and (4). □

Let us now imagine that we play the trading systems in two ways simultaneously—one dollar going into plays where we use the fraction f_m for system m, and another dollar into plays where we use the fraction f_m^* for system m. We denote the two capital trajectories with C_M and C_M^* respectively.

THEOREM 4. *Given inequality* (3), *the following holds: If* $\limsup_{m\to\infty} G_m(f_m) < \liminf_{m\to\infty} G_m(f_m^*)$, *then* $C_M^*/C_M \xrightarrow{\text{a.s.}} \infty$ *as* $M \to \infty$.

In practice, this theorem tells us that if we maintain $G_m(f_m^*)$ "sufficiently higher" than $G_m(f_m)$ and trade using the fractions f_m, we will miss out on infinite fortunes in the long run compared to what we could have made by using the fractions f_m^*.

PROOF. Formula (4) holds for arbitrary fractions f_m, which gives us

$$\frac{1}{\sum_{m=1}^{M} N_m} \sum_{m=1}^{M} \sum_{i=1}^{N_m} \Big[\log\Big(1 + \frac{f_m^*}{L_m} X_i^{(m)}\Big) - \log\Big(1 + \frac{f_m}{L_m} X_i^{(m)}\Big) \\ - \big(G_m(f_m^*) - G_m(f_m)\big)\Big] \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad M \to \infty.$$

Notice also that

$$\begin{aligned} 0 &< \liminf_{m\to\infty} G_m(f_m^*) - \limsup_{m\to\infty} G_m(f_m) \\ &= \liminf_{m\to\infty} G_m(f_m^*) + \liminf_{m\to\infty} \big(-G_m(f_m)\big) \\ &\le \liminf_{m\to\infty} \big(G_m(f_m^*) - G_m(f_m)\big). \end{aligned}$$

Now, since

$$\log \frac{C_M^*}{C_M} = \sum_{m=1}^{M} \sum_{i=1}^{N_m} \Big[\log\Big(1 + \frac{f_m^*}{L_m} X_i^{(m)}\Big) - \log\Big(1 + \frac{f_m}{L_m} X_i^{(m)}\Big)\Big],$$

it follows from lemma 2 that $C_M^*/C_M \xrightarrow{\text{a.s.}} \infty$ as $M \to \infty$. □

9. Summary

The concept of a trading system was defined in section 1. Section 2 mentioned that the concept of time preference is not included in our theory. Section 3 cited research that has been done on the "superiority" of trading with a fixed portion of one's capital. In section 4, we decided to trade with a fixed portion f/L of our capital, where L is the largest possible loss of our system and f is a number in $(0, 1)$. We then found out that every trading system has a corresponding function $G(f)$, as defined by equation (1), page 6. Section 5 established the importance of $G(f)$; we found out that unless $G(f) > 0$, we are in a dangerous place to be, and that we should maximize $G(f)$ if we want to get the greatest long-term exponential growth for our capital. Section 6 showed all the possible shapes that $G(f)$ can take, showed why one of these shapes is the only one that most traders will actually encounter in practice, and explained the concept of the optimal f. Section 7 analyzed the effects on $G(f)$ of over- or underestimating profits or losses. Finally, section 8 showed that in a nonstationary environment where we periodically move from one trading system to another, $G(f)$ for each individual system continues to be of importance.

2
Position Sizing: Practice

10. Introduction

The purpose of this chapter is to provide a starting point for applying in practice the theory developed in the previous chapter.

We start by constructing a real-world trading system that operates on a real financial market. Note that there is nothing overly special about our trading system; traders will ultimately supply their own systems. The only reason we construct a system here is to make this thesis interesting and self-contained for readers new to mechanical trading.

Our trading system operates on futures contracts, and we use the opportunity to explain an important practical concept known as the "back-adjustment" of futures prices. This has little to do with our theory per se, and we cover it primarily for self-containment purposes. Another reason to cover this subject is that we have not seen it adequately explained anywhere else.

We then proceed to approximate the return distribution of our system using historical data, find the corresponding approximated $G(f)$, and then trade historically with a few values of f and observe the difference in performance. We stress that further applied research has to be carried out before one starts to trade for real using these concepts. Such research is beyond the scope of this thesis, but a good starting point is provided in section 45, page 55.

11. Definitions

Before we construct our trading system, we need to define a few concepts. Let $(x_t)_{t=1}^{\infty}$ be a time series. (For notational simplicity, we allow t to be unbounded from above. In practice, when examining finite historical data, t is bounded from above.) Let $d \geq 1$ be some integer.

The d-day *simple moving average* of the time series at day $s \geq d$ is defined with

$$\mathrm{MA}\big((x_t)_{t=1}^{\infty}, d\big)_s := \frac{x_{s-d+1} + \cdots + x_s}{d}.$$

In other words, the simple moving average on a particular day is just the average of the d data points prior to and including that day. The d-day *exponential moving average* of the time series at day $s \geq d$ is defined recursively with

$$\mathrm{EMA}\big((x_t)_{t=1}^{\infty}, d\big)_s := \begin{cases} \mathrm{MA}\big((x_t)_{t=1}^{\infty}, d\big)_s & \text{if } s = d, \\ x_s K + \mathrm{EMA}\big((x_t)_{t=1}^{\infty}, d\big)_{s-1}(1-K) & \text{if } s > d, \end{cases}$$

where

$$K := \frac{2}{d+1}.$$

This definition of the exponential moving average can be found in Elder (1993), page 122. The simple moving average and the exponential moving average are similar in that they both measure some kind of an average of past data points. However, the latter is more "advanced" in the sense that it gives the highest weighting to the most recent data and slowly fades out the weighting of older data.

(Note our use of "days" as time periods above and onwards. The reader should note that there is nothing special about days, and that any time period—such as years, weeks, hours or minutes—could in theory be used instead. The only reason we refer to days is that the trading systems we construct in this thesis make use of days. Longer term traders and high-frequency traders alike can adapt the discussion to suit their own needs.)

In the financial markets, four prices are recorded for every day t: The opening price o_t, the highest price h_t, the lowest price l_t, and the closing price c_t.[1] We can represent a collection of such daily data with the sequence $(o_t, h_t, l_t, c_t)_{t=1}^{\infty}$. The *true range* at day $s \geq 2$ is defined with

$$\mathrm{TR}\big((o_t, h_t, l_t, c_t)_{t=1}^{\infty}\big)_s := \max(h_s - l_s, |c_{s-1} - h_s|, |c_{s-1} - l_s|).$$

In other words, the true range on a given trading day after the market has closed is the "distance" the market has undergone since it closed on the previous trading day. The d-day *average true range* at day $r \geq d+1$ is defined with

$$\mathrm{ATR}\big((o_t, h_t, l_t, c_t)_{t=1}^{\infty}, d\big)_r := \mathrm{EMA}\big\{\big[\mathrm{TR}\big((o_t, h_t, l_t, c_t)_{t=1}^{\infty}\big)_{s+1}\big]_{s=1}^{\infty}, d\big\}_{r-1}.$$

In other words, the average true range on a given day after the market has closed is the exponential moving average of true ranges from the "recent" past. The average true range is, therefore, nothing but a measure of market volatility in the "recent" past. The average true range is defined, albeit slightly differently, in Covel (2007), page 80, and in Faith (2007), page 252; the former uses a simple moving average instead of an exponential moving average, and the latter uses a different formula for calculating the exponential moving average (it uses $K = 1/d$ instead of $K = 2/(d+1)$).

[1] We will not dwell on what kind of "prices" these are. Some historical data sets show averages of bid-ask spreads. More commonly though, data sets show *actual prices*, i.e. exchange-ratios at which actual economic transactions took place. (See Elder (1993), chapter 2, on how prices are formed in the marketplace. A deeper analysis on how this process results from purposeful human behavior is a subject of economics; see e.g. Mises (1949) and Rothbard (1962).)

12. The Trading System Constructed

We will make use of trading rules known in the literature as the "Donchian trend following system," named after the futures trader Richard Donchian. In Faith (2007), page 139, these rules are described as follows: We enter a long [short] position if the market reaches the highest [lowest] point it has been for the past 20 days, under the conditions that if the 25-day exponential moving average is above [below] the 350-day exponential moving average, we are only allowed to take a long [short] position. We exit a long [short] position when the market reaches the lowest [highest] point it has been for the past 10 days. We furthermore employ a stop-loss order of two 20-day average true ranges from our entry; in other words, we instruct our broker to automatically exit our position before we experience a more loss than two 20-day average true ranges.

We can imagine many different ways to interpret these rules, if we are to precisely write them down as instructions for a computer. We have chosen the interpretation illustrated by the following pseudocode:

```
Trading days: t = 1,...,n
k ← an index in [1,n] that guarantees enough past data
a′: a variable to hold the average true range at trade entry
loop through trading days t = k,...,n
   o ← opening price for this day
   if we have a signal from the previous day to exit a trade
      do so now at the price o
   end if
   if we have a signal from the previous day to enter a long or short trade
      do so now at the price o
   end if
   h ← highest price for this day
   l ← lowest price for this day
   if a long trade is on
      x ← (entry price of trade) − 2a′
      if l < x, exit trade at price x
   end if
   if a short trade is on
      x ← (entry price of trade) + 2a′
      if h > x, exit trade at price x
   end if
   h1 ← highest highest price for the past 10 days (including this day)
   l1 ← lowest lowest price for the past 10 days (including this day)
   if a long trade is on and l = l1
      post a signal to exit the trade the next day
   end if
   if a short trade is on and h = h1
      post a signal to exit the trade the next day
```

    end if
    $h_2 \leftarrow$ highest highest price for the past 20 days (including this day)
    $l_2 \leftarrow$ lowest lowest price for the past 20 days (including this day)
    $a \leftarrow$ this day's 20-day average true range
    $e_1 \leftarrow$ this day's 25-day exponential moving average of closing prices
    $e_2 \leftarrow$ this day's 350-day exponential moving average of closing prices
    if no trade is on or an exit signal has been posted
      if $e_1 > e_2$ and $h = h_2$
        post a signal to enter a long trade the next day
        $a' \leftarrow a$
      end if
      if $e_1 < e_2$ and $l = l_2$
        post a signal to enter a short trade the next day
        $a' \leftarrow a$
      end if
    end if
end of loop

(We have, for the sake of simplicity, ignored slippage, commissions, and other transaction costs. These can easily be added by minor modifications to the above code, at places where we enter and exit trades.)

Applying these rules on a given market will provide us with a trading system. Applying them on historical prices of the market will provide us with a collection of trades. The percentage returns from these historical trades will, then, provide us with a rough approximation of the return distribution of the trading system. This, in turn, will allow us to approximate $G(f)$ and thus the optimal f for the trading system.

We have chosen to apply the rules on corn futures from July 1, 1959 to August 13, 2010. (The contracts were traded on the Chicago Board of Trade, later CME Group, with the ticker symbol C. The data was obtained from Commodity Research Bureau.) Unfortunately, there is no such thing as a single "corn futures price" that traders can trade in perpetuity. They have to regularly roll from one futures contract to the next in order to avoid having to take actual delivery of the underlying commodity. The next section describes how we take this practical issue into account. It turns out that it affects how we calculate the percentage returns of our trades.

13. The Back-Adjustment of Futures Prices

Figure 8 shows the prices of two futures contracts on corn, one expiring in September 1986 and the other in December 1986. (The price is quoted in cents per bushel of corn, and one contract is for 5,000 bushels; see Rogers (2004), page 228. Each bar, "┼", in the figure represents the opening, highest, lowest, and closing prices for a trading day; the vertical line connects the highest and lowest prices, the "left-lobe" represents the opening price, and the "right-lobe" represents the clos-

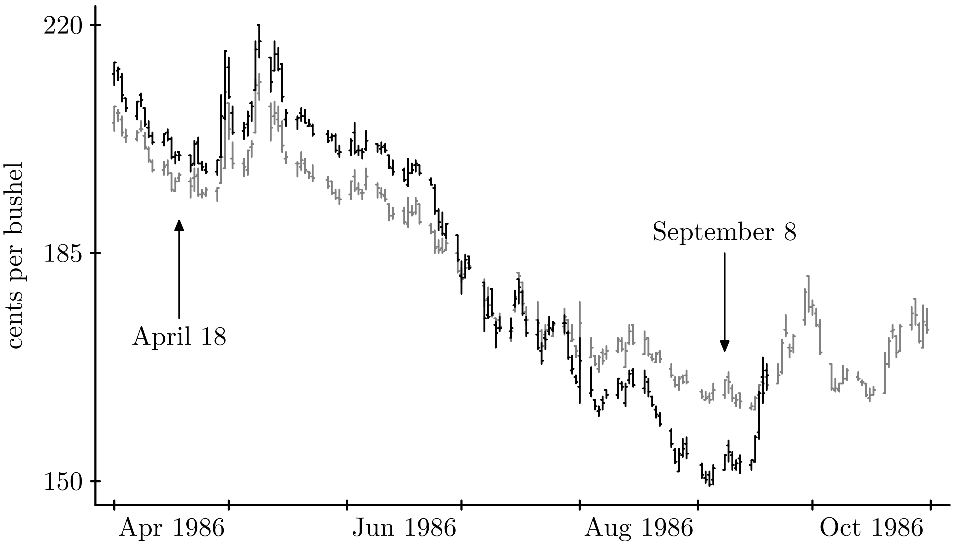

Figure 8. Two Futures Contracts on Corn in 1986.

ing price.) Suppose, by way of example, that we initiate a long position in one September contract on April 18 at 200 cents. On September 8, about a week before the contract expires, we exit our position at 154 cents. Immediately after exiting, we initiate a new long position in the December contract at 165.5 cents.

What was our total loss in these transactions? If we are rusty on how futures contracts work, we might think that we first lost money by entering the contract at 200 cents and exiting it at 154 cents, losing $200 - 154 = 46$ cents per bushel, and that we lost additional $165.5 - 154 = 11.5$ cents per bushel when rolling over to the next contract at a higher price. This, however, is not how futures contracts work. If we recall that a futures contract *does not cost anything to enter into*, and that we *only participate in its price fluctuations while we are bound by it*, we see clearly that we lost "only" 46 cents per bushel. We did not lose 11.5 cents per bushel when rolling between the contracts, because we did not participate in the price difference between them.

This is the reason for why we want to stitch together the historical prices the of the futures contracts in such a way that we eliminate the price gaps when rolling between contracts, the goal being to obtain an accurate depiction of the profits and losses of our trading operations. First, we have to decide when we will roll between contracts. We have decided to roll between contracts about one month prior to expiration. More precisely: We get out of a contract at the closing price on the last trading day of the second-last month during which the contract trades, and at the opening price on very next trading day enter the next contract, i.e. the contract having the second-earliest expiry date on that day.

We do the stitching by *back-adjusting* the prices in the following way: We start with the latest futures contract in our data set and go back to the day where we exited the previous contract. We shift the entire price series of the previous

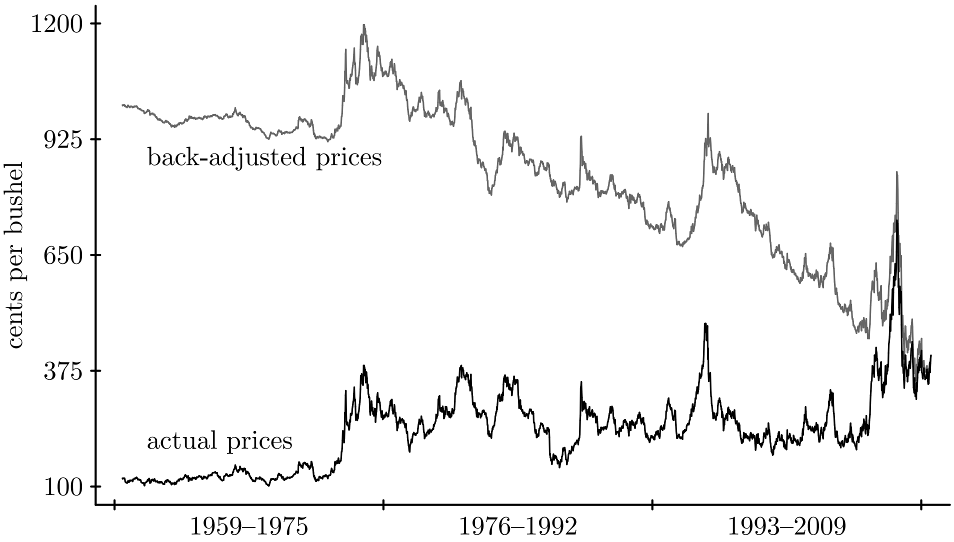

FIGURE 9. ACTUAL AND BACK-ADJUSTED FUTURES PRICES OF CORN FROM 1959 TO 2010.

contract up or down such that, on that day, the closing price of the previous contract equals the closing price of the later contract. We continue this process until we reach the beginning of our data set.

Figure 9 shows the actual and back-adjusted futures price series for corn (the figure actually shows weekly closing prices connected with straight lines). Notice the dramatic difference that the price gaps between futures contracts—price gaps that we as traders are never exposed to—create over a long period of time.

It should be clear by now how we actually calculate the percentage return of a given trade. The formula is quite simple:

$$(-1)^s \times \frac{\text{back-adjusted exit price} - \text{back-adjusted entry price}}{\text{actual entry price}},$$

where $s = 0$ if the trade is long or $s = 1$ if the trade is short. The numerator represents the profit or loss that we as traders actually experience. If we were to use *actual* prices in the numerator instead of *back-adjusted* prices, we would obtain a completely distorted view of the performance of our trading system—we would be pretending as if we participate in the price gaps upon rolling between futures contracts, when in fact we do not.

None of this discussion affects the validity of the pseudocode in the previous section. In fact, *that entire pseudocode is supposed to be run on the back-adjusted price series.* Of course, we should also record the actual entry price of each trade so we can obtain its percentage return (as we are about to do in the next section).

To conclude this section, we stress that back-adjusted prices are not actual prices of any real financial instrument. They are, in and of themselves, completely meaningless. As we have seen, they are merely a useful practical construct that allows us to accurately calculate the performance of a trading system that involves entering and exiting futures contracts. In fact, it is quite possible for back-adjusted prices to be negative. But this does not matter, since what we are concerned with is the *difference* between back-adjusted prices at two different points in time—a difference that represents the profit or loss that we as traders actually experience.

(It should be obvious that there are other ways of eliminating the price gaps between future contracts than to perform back-adjustment. One such way is *forward-adjustment*, where we start with the oldest data point and work our way towards the latest data point. All of these methods are as good as any other.)

14. Obtaining the Return Distribution and the Function $G(f)$

Running our pseudocode as explained in the previous section, we obtain a series of $r = 318$ trades with percentage returns $x_1, \ldots, x_r$. The smallest return is -0.1023 and the largest 0.3354. We can visualize the return distribution by creating a histogram of all the returns, as shown in figure 10. (The intervals in this figure are closed to the left and open to the right. For example, the return 0 lands in the interval $[0, 5)$.)

We now make the rough assumption that the return distribution of our trading system is exactly the discrete distribution provided by the historical data that we have obtained. In other words, we assume that the $x_1, \ldots, x_r$ are the only possible outcomes, and that each x_i has the same probability $p_i = 1/r$ of occurring. The distribution is therefore assumed to have the largest possible loss of $L = 0.1023$, and the expected value of the distribution assumed to be given by $\sum_{i=1}^{r} x_i p_i$, which we compute to be 0.01204.

We now have everything we need to obtain a rough approximation of the function $G(f)$. Recall from the previous chapter that $G(f)$ is in general provided

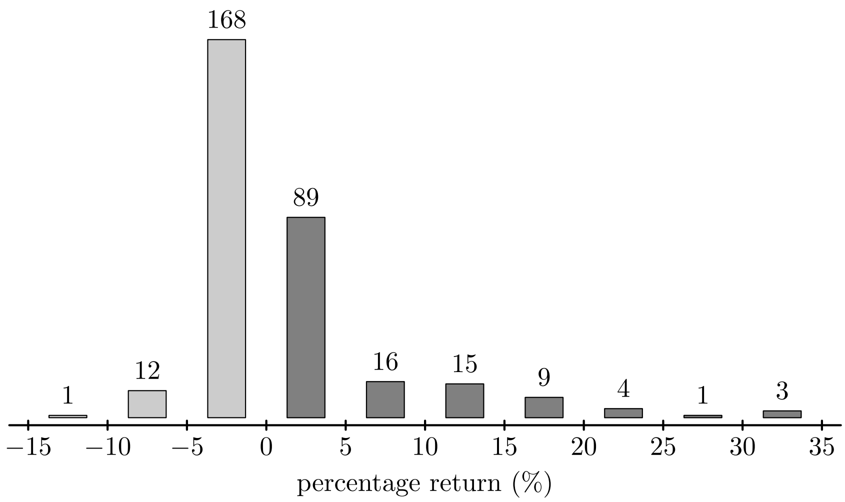

FIGURE 10. A HISTOGRAM OF OUR PERCENTAGE RETURNS.

by

$$G(f) = \mathbb{E} \log \Big(1 + \frac{f}{L} X\Big),$$

where X is a random variable describing the percentage return of our trading system. In our present case, this becomes

$$G(f) = \sum_{i=1}^{r} p_i \log \Big(1 + \frac{f}{L} x_i\Big) = \frac{1}{r} \sum_{i=1}^{r} \log \Big(1 + \frac{f}{L} x_i\Big).$$

The function $G(f)$ that we obtain is shown in figure 11. We find the optimal f to be 0.44. The fraction f^* at which $G(f^*) = 0$ is found to be $f^* = 0.90$. (These values were computed by applying the Newton-Raphson root-finding algorithm on $G'(f)$ and $G(f)$ respectively.)

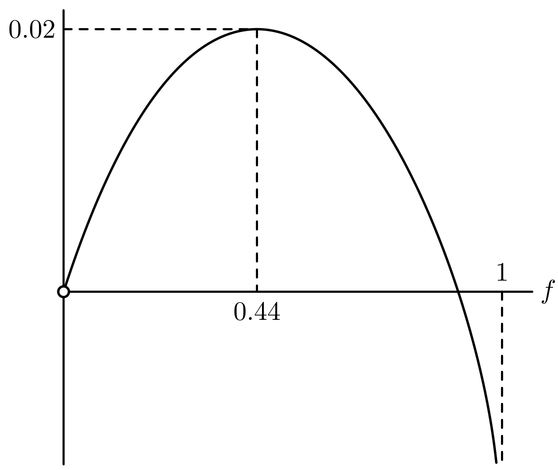

FIGURE 11. THE FUNCTION $G(f)$ FOR OUR TRADING SYSTEM.

15. Trading with Different Values of f

Let us now start with one dollar, and trade our system using three different values of f that, judging from figure 11, we believe to know something about: 0.2, 0.44 and 0.95. Figure 12 shows the results (the scale between 0 and 1 on the vertical axis is linear, whereas the scale above 1 is logarithmic).

Figure 12 is in and of itself, regardless of any mathematical theory, rather remarkable. It tells us that if we would have traded our system using $f = 0.44$, we would have 1412-folded our initial capital, and that using $f = 0.2$, we would have 197-folded it. Note that the actual *portions* of our capital that we use for each trade in these cases are, respectively, $0.44/L = 4.3$ and $0.2/L = 2.0$. In other words, both cases involve taking on leverage.

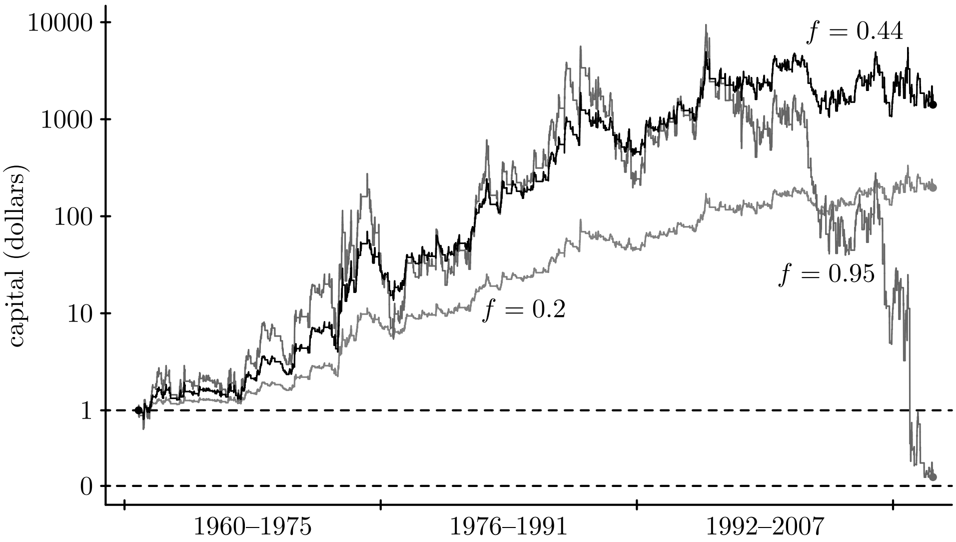

FIGURE 12. CAPITAL CURVES FOR DIFFERENT VALUES OF f.

16. Further Applied Research

The assumption we made in section 14 that the return distribution of our trading system is provided exactly by the discrete distribution obtained from our historical trades is obviously very suspect; we are assuming that we are trading with a single fixed trading system for about half a century! What this implies in particular is that, for all this time, the return distribution is stationary and the returns independent of each other. In so far as we are planning on using this supposed single system for trading in the future (we should not!), we are also assuming that the particular historical period we considered represents what the market will always look like.

The reader should take this chapter for what it is, and understand its limitations. Its only purpose is to provide a glimpse into the world of mechanical trading and to provide a basic starting point as to how our theory fits into the picture. Addressing the above issues is beyond the scope of this thesis, but needs to be done before we start trading for real. A good starting point for further applied research is provided in section 45, page 55.

17. Summary

In section 11, we defined various concepts that are commonplace in the world of trading. In section 12, we constructed an actual trading system that operates on a real financial market. Section 13 discussed the back-adjustment of futures prices, a practical technicality for futures traders. In sections 14 and 15, we approximated the return distribution of our trading system, found the corresponding $G(f)$, and then proceeded to trade using a few values of f. This is where we saw clearly how dramatic an effect the fraction f can have on a trader's capital curve in the long run. Finally, in section 16, we explained the limitations of our discussion and mentioned that we are far from finished when it comes to applied research.

3

Diversification: Theory

18. Introduction

This chapter explains how traders can allocate their capital across multiple trading systems. This will be a natural extension of the theory in chapter 1 to multiple dimensions. Instead of only one trading system, we now have access to a portfolio of K trading systems indexed with $k \in \{1, 2, \ldots, K\}$. We denote with $X_n^{(k)}$ the percentage return from play n in system k.

For now, we make the assumption that all the K systems have the *same frequency*, i.e. that for each n, play n of system k_1 takes the same time to eject as play n of system k_2, for different k_1 and k_2. In other words, we are able to diversify our initial capital across the different systems, wait for all of them to eject at the same time, diversify the proceeds into the systems again, and so on in perpetuity. Of course, this is usually impossible to do in reality, since different trading systems usually *do* have different frequencies. We will generalize our theory to cover this more realistic case in section 24, page 36.

All the assumptions that were made in chapter 1 about the return distribution of a single trading system continue to hold for each of the K systems. Note, however, that $X_n^{(k_1)}$ and $X_n^{(k_2)}$, for each n and different k_1 and k_2, are not necessarily independent and do not necessarily have the same distribution. We assume that the return distributions of the systems are either all continuous or all discrete, which provides us with either a continuous or a discrete joint distribution.

Continuing our extension, let L_k be the largest possible loss of system k, and let f_k be the fixed fraction that we use to trade with in this system, such that f_k/L_k is the actual portion of our capital, possibly greater than 1, that we use for every play in this system.

We assume that $f_k \geq 0$ for all k, with $f_k > 0$ for some k. (All of them being zero is not interesting.) We also assume that $(f_1, \ldots, f_K)$ is contained in the set

$$\mathcal{B} := \left\{ (f_1, \ldots, f_K) \;\middle|\; \mathbb{P}\left(1 + \sum_{k=1}^{K} \frac{f_k}{L_k} X_1^{(k)} \leq 0\right) = 0 \right\}.$$

This condition guarantees that our capital stays positive, as we will see in the next section. Note that if $(f_1, \ldots, f_K)$ is contained in $\mathcal{B}$, then so is $t(f_1, \ldots, f_K)$ for all $t \in (0, 1)$. More generally, note also that $\mathcal{B}$ is a convex set.

We also make the assumption that there exists an $\epsilon > 0$ such that

$$\mathbb{P}(X_1^{(1)} < -\epsilon, \ldots, X_1^{(K)} < -\epsilon) > 0.$$

This assumption guarantees that $\mathcal{B}$ is bounded, and will make parts of the following discussion somewhat more elegant than they otherwise would have been. This assumption is also likely to hold for most portfolios that traders will encounter in practice. (Those who do not like this assumption can either let ϵ be arbitrarily small, or simply get rid of it and adapt the following discussion accordingly.)

19. The Function $G(f_1, \ldots, f_K)$

Denoting our starting capital with $C_0 = 1$ (one dollar), our capital after n plays in each trading system (a total of nK plays) is provided recursively by

$$C_n = C_{n-1}\left(1 - \sum_{k=1}^{K} \frac{f_k}{L_k}\right) + C_{n-1} \sum_{k=1}^{K} \frac{f_k}{L_k}(1 + X_n^{(k)}) = C_{n-1}\left(1 + \sum_{k=1}^{K} \frac{f_k}{L_k} X_n^{(k)}\right),$$

which can be written in closed form as

$$C_n = \prod_{i=1}^{n} \left(1 + \sum_{k=1}^{K} \frac{f_k}{L_k} X_i^{(k)}\right).$$

(This is where the reader will notice the importance of $(f_1, \ldots, f_K)$ being contained in the set $\mathcal{B}$ for guaranteeing that our capital stays positive.)

We define the random variables G_n, which represent the exponential growth of our capital curve after n plays, with

$$C_n = e^{nG_n},$$

which we rewrite as

$$G_n = \frac{1}{n} \log C_n = \frac{1}{n} \log \prod_{i=1}^{n} \left(1 + \sum_{k=1}^{K} \frac{f_k}{L_k} X_i^{(k)}\right) = \frac{1}{n} \sum_{i=1}^{n} \log \left(1 + \sum_{k=1}^{K} \frac{f_k}{L_k} X_i^{(k)}\right).$$

Now, since

$$\log \left(1 + \sum_{k=1}^{K} \frac{f_k}{L_k} x_k\right) < \sum_{k=1}^{K} \frac{f_k}{L_k} x_k \quad \text{for} \quad x_1, \ldots, x_K > 0,$$

we can use similar arguments as we used in the case of a single trading system, on page 5, to show that

$$\mathbb{E}\left|\log \left(1 + \sum_{k=1}^{K} \frac{f_k}{L_k} X_1^{(k)}\right)\right| < \infty.$$

The Kolmogorov strong law of large numbers (see Gut (2005), page 295) then tells us that

$$G_n \xrightarrow{\text{a.s.}} \mathbb{E}\log\left(1+\sum_{k=1}^{K}\frac{f_k}{L_k}X_1^{(k)}\right) \quad \text{as} \quad n\to\infty,$$

which motivates the definition of the function

$$G(f_1,\ldots,f_K) := \mathbb{E}\log\left(1+\sum_{k=1}^{K}\frac{f_k}{L_k}X_1^{(k)}\right),$$

which we call the *long-term exponential growth function* of our trading systems.

20. The Importance of $G(f_1,\ldots,f_K)$

The following two theorems are natural extensions of theorems 1 and 2 in section 5, page 6, to our present multidimensional setting. The proofs are completely analogous, and are left as exercises for the reader.

THEOREM 5. *(a) If $G(f_1,\ldots,f_K)>0$, then $C_n \xrightarrow{\text{a.s.}} \infty$ as $n\to\infty$.*
(b) If $G(f_1,\ldots,f_K)<0$, then $C_n \xrightarrow{\text{a.s.}} 0$ as $n\to\infty$.
(c) If $G(f_1,\ldots,f_K)=0$, then

$$\limsup_{n\to\infty} C_n = \infty \text{ a.s.} \quad \text{and} \quad \liminf_{n\to\infty} C_n = 0 \text{ a.s.}$$

Let us now imagine that we play our trading systems in two ways simultaneously—one dollar going into plays where we use the fraction f_k for system k, and another dollar going into plays where we use the fraction f_k^* for system k. We denote the two capital trajectories with C_n and C_n^* respectively.

THEOREM 6. *If $G(f_1^*,\ldots,f_K^*)>G(f_1,\ldots,f_K)$, then $C_n^*/C_n \xrightarrow{\text{a.s.}} \infty$ as $n\to\infty$.*

To summarize this section, theorem 5 tells us that we should trade with fractions $f_1,\ldots,f_K$ such that $G(f_1,\ldots,f_K)>0$. If we do this, we will make infinite fortunes in the long run. If we do not, we will either get unpredictable results or go broke. Theorem 6 tells us that if we trade using fractions $f_1,\ldots,f_K$ that do not maximize $G(f_1,\ldots,f_K)$, we will miss out on infinite fortunes in the long run compared to what we could have gained by performing the maximization.

21. The Shape of $G(f_1,\ldots,f_K)$

It would be rather difficult to visualize the entire $(K+1)$-dimensional surface $G(f_1,\ldots,f_K)$ for $K=3$, and virtually impossible for $K>3$. Before we provide an important general result about this entire surface, we begin by letting $\mathbf{u}=(u_1,\ldots,u_K)$ be an arbitrary unit vector with $u_k\geq 0$ for all k, and then looking at how $G(f_1,\ldots,f_K)$ behaves along the straight line $t\mapsto t\mathbf{u}$ for $t\in(0,b_{\mathbf{u}})$, where $b_{\mathbf{u}} := \sup_{t\mathbf{u}\in\mathcal{B}} t$. By doing this, we are back to analyzing a function of one variable, and it turns out that this approach provides us with a good initial "feel" for the behavior of $G(f_1, \ldots, f_K)$.

Thus, our object of interest is the function

$$G_{\mathbf{u}}(t) := G(t\mathbf{u}) = \mathbb{E} \log \left(1 + t \sum_{k=1}^{K} \frac{u_k}{L_k} X_1^{(k)}\right), \quad t \in (0, b_{\mathbf{u}}).$$

If the $X_1^{(k)}$ are continuous with the joint density $\rho(x_1, \ldots, x_K)$, we see that

$$G_{\mathbf{u}}(t) = \int_{-L_K}^{\infty} \cdots \int_{-L_1}^{\infty} \log \left(1 + t \sum_{k=1}^{K} \frac{u_k}{L_k} x_k\right) \rho(x_1, \ldots, x_K)\, dx_1 \cdots dx_K.$$

On the other hand, if each $X_1^{(k)}$ is discrete with possible outcomes x_{k,i_k}, where $i_k \in \{1, 2, \ldots, r_k\}$, and the joint distribution is given by

$$\mathbb{P}\big(X_1^{(1)} = x_{1,i_1}, \ldots, X_1^{(K)} = x_{K,i_K}\big) =: p_{i_1,\ldots,i_K},$$

we see that

$$G_{\mathbf{u}}(t) = \sum_{i_1=1}^{r_1} \cdots \sum_{i_K=1}^{r_K} p_{i_1,\ldots,i_K} \log \left(1 + t \sum_{k=1}^{K} \frac{u_k}{L_k} x_{k,i_k}\right).$$

In both cases, we can easily differentiate to obtain

$$G'_{\mathbf{u}}(s) \xrightarrow[s \to 0^+]{} \sum_{k=1}^{K} \frac{u_k}{L_k} \mathbb{E}[X_1^{(k)}] \qquad \text{and} \qquad G''_{\mathbf{u}}(t) < 0 \quad \text{for all} \quad t \in \big(0, b_{\mathbf{u}}\big).$$

This shows that the surface $G(f_1, \ldots, f_K)$ behaves, along any straight line going out from the origin, in a somewhat similar manner as $G(f)$ did for a single trading

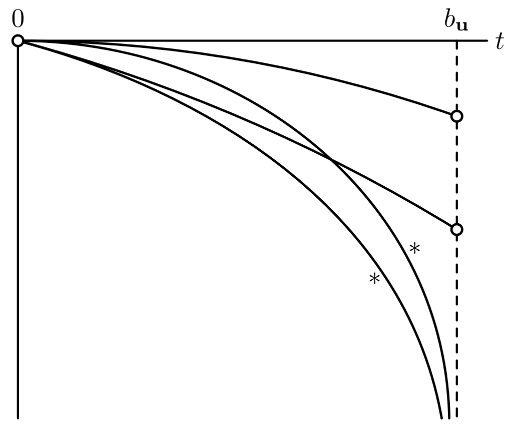

FIGURE 13. THE FUNCTION $G_{\mathbf{u}}(t)$ FOR $\sum_{k=1}^{K} u_k \mathbb{E}[X_1^{(k)}]/L_k \le 0$.

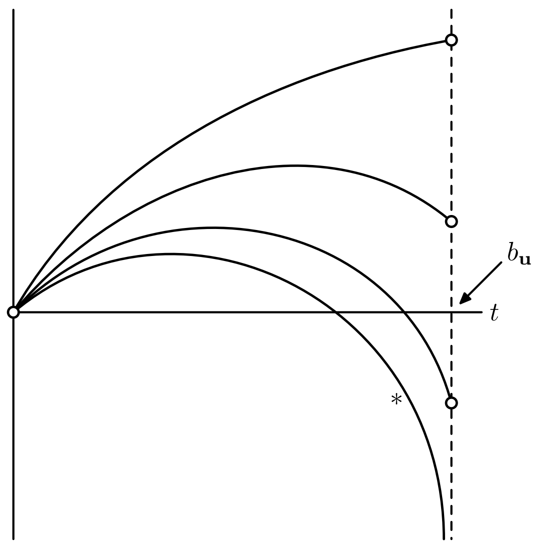

FIGURE 14. THE FUNCTION $G_{\mathbf{u}}(t)$ FOR $\sum_{k=1}^{K} u_k \mathbb{E}[X_1^{(k)}]/L_k > 0$.

system in chapter 1: The slope of $G_{\mathbf{u}}(t)$ starts out at $\sum_{k=1}^{K} u_k \mathbb{E}[X_1^{(k)}]/L_k$ near zero and is thereafter strictly decreasing. The various possibilities for the shape of $G_{\mathbf{u}}(t)$ are shown in figure 13 for the case $\sum_{k=1}^{K} u_k \mathbb{E}[X_1^{(k)}]/L_k \leq 0$, and in figure 14 for the case $\sum_{k=1}^{K} u_k \mathbb{E}[X_1^{(k)}]/L_k > 0$. In the discrete case, the only possible shapes are the ones marked with an asterisk.

We now turn our attention back to the entire surface $G(f_1, \ldots, f_K)$. Let $(f_1, \ldots, f_K)$ and $(f_1^*, \ldots, f_K^*)$ be two arbitrary points in its domain, and consider the straight line that goes through these points:

$$\boldsymbol{\ell}(t) := (1-t)(f_1, \ldots, f_K) + t(f_1^*, \ldots, f_K^*), \quad t \in [0, 1].$$

We find that

$$G(\boldsymbol{\ell}(t)) = \mathbb{E} \log \left(1 + \sum_{k=1}^{K} \frac{f_k + t(f_k^* - f_k)}{L_k} X_1^{(k)} \right).$$

In both the continuous and the discrete case, it is easy to show that

$$\frac{d^2 G(\boldsymbol{\ell}(t))}{dt^2} < 0, \quad t \in (0, 1).$$

Since our straight line was arbitrarily chosen, we have established that the surface $G(f_1, \ldots, f_K)$ is *strictly concave*. From this, we have the following result:

THEOREM 7. *The surface $G(f_1, \ldots, f_K)$ has at most one local maximum. If such a local maximum exists, it is the global maximum of the entire surface.*

Although interesting from a theoretical perspective, this theorem actually has an important practical application: It tells us that if we have found a local maximum using a computer, we do not have to waste any further computational resources to search for other maxima, because we know that we have already found *the* (global) maximum.

The above discussion should give us a feel for how "regular," in some sense, the surface $G(f_1, \ldots, f_K)$ is in general—even for large K.

22. Over- or Underestimating Profits or Losses

This section is a natural extension of section 7, page 10, to multiple dimensions. Not surprisingly, we will end up obtaining analogous results along each line going out from the origin. This will, in turn, provide us with information about the surface in general.

Recall that if our trading systems have discrete return distributions, and the joint distribution is given by

$$\mathbb{P}\big(X_1^{(1)} = x_{1,i_1}, \ldots, X_1^{(K)} = x_{K,i_K}\big) = p_{i_1,\ldots,i_K},$$

then we have

$$G_{\mathbf{u}}(t) = \sum_{i_1=1}^{r_1} \cdots \sum_{i_K=1}^{r_K} p_{i_1,\ldots,i_K} \log\left(1 + t\sum_{k=1}^{K} \frac{u_k}{L_k} x_{k,i_k}\right).$$

Let us now, without any loss of generality, imagine that we overestimate the profits of the first system ($k = 1$). (We can choose any other system by reordering the index k.) We do this overestimation by adding a new outcome x_{1,r_1+1} to the first system, with $x_{1,r_1+1} > x_{1,i_1}$ for all i_1, such that

$$\begin{aligned}
\mathbb{P}\big(X_1^{(1)} = x_{1,r_1+1}, X_1^{(2)} = x_{2,i_2}, \ldots, X_1^{(K)} = x_{K,i_K}\big) &=: p_{r_1+1,i_2\ldots,i_K},\\
\mathbb{P}\big(X_1^{(1)} = x_{1,i_1}, \ldots, X_1^{(K)} = x_{K,i_K}\big) = p_{i_1,\ldots,i_K}&\left(1 - \frac{p_{r_1+1,i_2\ldots,i_K}}{\sum_{i_1=1}^{r_1} p_{i_1,\ldots,i_K}}\right),
\end{aligned}$$

where $p_{r_1+1,i_2\ldots,i_K} < \sum_{i_1=1}^{r_1} p_{i_1,\ldots,i_K}$ for all $i_2, \ldots, i_K$. The exponential growth function corresponding to this scenario is given by

$$\begin{aligned}
G^*_{\mathbf{u}}(t) = &\sum_{i_1=1}^{r_1} \cdots \sum_{i_K=1}^{r_K} p_{i_1,\ldots,i_K} \left(1 - \frac{p_{r_1+1,i_2\ldots,i_K}}{\sum_{i_1=1}^{r_1} p_{i_1,\ldots,i_K}}\right) \log\left(1 + t\sum_{k=1}^{K} \frac{u_k}{L_k} x_{k,i_k}\right)\\
&+ \sum_{i_2=1}^{r_2} \cdots \sum_{i_K=1}^{r_K} p_{r_1+1,i_2\ldots,i_K} \log\left(1 + t\frac{u_1}{L_1} x_{1,r_1+1} + t\sum_{k=2}^{K} \frac{u_k}{L_k} x_{k,i_k}\right),
\end{aligned}$$

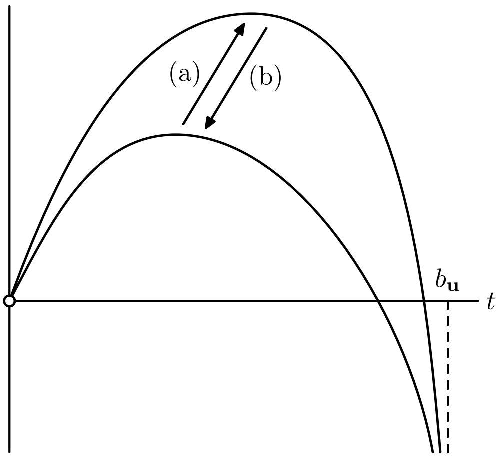

Figure 15. The Effects on the Function $G_{\mathbf{u}}(t)$ of Over- or Underestimating Profits.

which we rewrite as

$$G^*_{\mathbf{u}}(t) = G_{\mathbf{u}}(t) + \sum_{i_2=1}^{r_2} \cdots \sum_{i_K=1}^{r_K} p_{r_1+1,i_2\ldots,i_K} \left[\log\left(1 + t\frac{u_1}{L_1}x_{1,r_1+1} + t\sum_{k=2}^{K}\frac{u_k}{L_k}x_{k,i_k}\right) - \frac{1}{\sum_{i_1=1}^{r_1} p_{i_1,\ldots,i_K}} \sum_{i_1=1}^{r_1} p_{i_1,\ldots,i_K} \log\left(1 + t\sum_{k=1}^{K}\frac{u_k}{L_k}x_{k,i_k}\right)\right].$$

Using similar ideas as in section 7, we find—*given* $u_1 \neq 0$—that $G^*_{\mathbf{u}}(t) > G_{\mathbf{u}}(t)$ for all t, and that if $G'_{\mathbf{u}}(t_1) = 0$ and $(G^*_{\mathbf{u}})'(t_2) = 0$, then $t_2 > t_1$. Note, however, that if $u_1 = 0$, there is no difference between $G^*_{\mathbf{u}}(t)$ and $G_{\mathbf{u}}(t)$.

Since $\mathbf{u}$ was chosen arbitrarily, we have obtained the following results: If we overestimate the profits of one of our systems (in the way we have done it), we will, as long as we place a nonzero portion of our capital into that system, overestimate the exponential growth of our portfolio. We will also believe that the optimal point is located farther away from the origin than it actually is.

On the flip side, if we underestimate the profits of one of our systems (by removing its largest possible gain), we will, as long as we place a nonzero portion of our capital into that system, underestimate the exponential growth of our portfolio. We will also believe that the optimal point is located closer to the origin than it actually is.

For a given line $t \mapsto t\mathbf{u}$ with $u_1 \neq 0$, these scenarios are depicted, respectively, with arrows (a) and (b) in figure 15. (Of course, the "optimal point" in this figure may not be the actual optimal point for our portfolio, since there are, for $K > 1$, infinitely many other $\mathbf{u}$ to choose from.)

We now proceed to see what happens if we overestimate the losses of the first system ($k = 1$) by adding to it a new outcome $x_{1,0}$ with $x_{1,0} < x_{1,i_1}$ for all i_1,

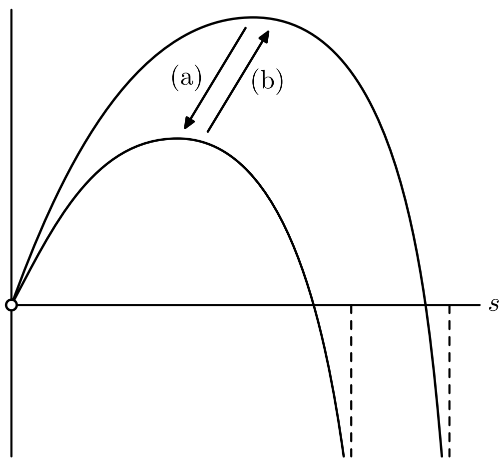

FIGURE 16. THE EFFECTS ON THE FUNCTION $H_{\mathbf{u}}(s)$ OF OVER- OR UNDERESTIMATING LOSSES.

such that

$$\mathbb{P}\big(X_1^{(1)} = x_{1,0}, X_1^{(2)} = x_{2,i_2}, \ldots, X_1^{(K)} = x_{K,i_K}\big) =: p_{0,i_2\ldots,i_K},$$
$$\mathbb{P}\big(X_1^{(1)} = x_{1,i_1}, \ldots, X_1^{(K)} = x_{K,i_K}\big) = p_{i_1,\ldots,i_K}\left(1 - \frac{p_{0,i_2\ldots,i_K}}{\sum_{i_1=1}^{r_1} p_{i_1,\ldots,i_K}}\right).$$

The idea is similar as in section 7: Comparing $G(f_1, \ldots, f_K)$ and the corresponding growth function for the scenario of overestimated losses is akin to comparing apples with oranges, since adding a larger loss to the first system completely changes the meaning of f_1. To get around this, we define

$$H(g_1, \ldots, g_K) := G(g_1 L_1, \ldots, g_K L_K).$$

We then look at H along the line $s \mapsto s\mathbf{u}$, where $\mathbf{u} = (u_1, \ldots, u_K)$ is an arbitrary unit vector with $u_k \geq 0$ for all k, by defining $H_{\mathbf{u}}(s) := H(s\mathbf{u})$. We can now compare $H_{\mathbf{u}}(s)$ with the corresponding growth function for the scenario of overestimated losses—$H_{\mathbf{u}}^*(s)$. Since it involves no new ideas or techniques, we leave it as an exercise for the reader to show, in the case $u_1 \neq 0$, that $H_{\mathbf{u}}^*(s) < H_{\mathbf{u}}(s)$, and that if $(H_{\mathbf{u}}^*)'(s_1) = 0$ and $H_{\mathbf{u}}'(s_2) = 0$, then $s_1 < s_2$.

Arrow (a) in figure 16 shows the effect on $H_{\mathbf{u}}(s)$ of overestimating losses (in the way we have done it). Arrow (b) in the same figure shows the effect of underestimating losses (by removing the largest possible loss from the first system). By now, the reader should understand exactly what this means, without any verbose elaboration.

We have thus seen evidence that the same general principles that we mentioned in section 7 continue to hold in multiple dimensions: If we are unsure about what $G(f_1, \ldots, f_K)$ is, we are (for our individual systems) safer to underestimate profits than to overestimate them, and safer to overestimate losses than to underestimate them. Underestimating losses continues to be particularly dangerous, because it may result in our capital becoming negative.

23. Extension to a Nonstationary Environment

This section is a natural extension of section 8, page 13, to multiple dimensions. Everything is completely analogous, so we will cover things very quickly, leaving all the details (including proofs to theorems) as exercises for the reader.

We assume that our K trading systems now change as time goes by; we play a first "batch" of K systems N_1 times, move on to a second batch of K systems and play it N_2 times, and so on in perpetuity. The percentage return from the i-th play in the k-th system in the m-th batch of K systems is denoted with $X_i^{(k,m)}$. Thus, the m-th batch of K systems corresponds to the percentage returns

$$X_i^{(1,m)}, \ldots, X_i^{(K,m)},$$

where i runs through $\{1, 2, \ldots, N_m\}$. Naturally, for fixed k and m, the random variables $X_i^{(k,m)}$ are independent and identically distributed; for fixed k, they are independent but not necessarily identically distributed; and for fixed m and i, they are not necessarily independent and not necessarily identically distributed. Continuing this natural extension, we let $L_{k,m}$ be the largest possible loss of the k-th system in the m-th batch of K systems, and $f_{k,m}$ be the fraction that we use to trade with in this system. Let $G_m(f_{1,m}, \ldots, f_{K,m})$ be the exponential growth function of the m-th batch of K systems. Note that

$$G_m(f_{1,m}, \ldots, f_{K,m}) = \mathbb{E} \log \left(1 + \sum_{k=1}^{K} \frac{f_{k,m}}{L_{k,m}} X_1^{(k,m)} \right).$$

Beginning our trading operations with the capital $C_0 = 1$ (one dollar), our capital after playing M batches of systems is given by

$$C_M = \prod_{m=1}^{M} \prod_{i=1}^{N_m} \left(1 + \sum_{k=1}^{K} \frac{f_{k,m}}{L_{k,m}} X_i^{(k,m)} \right).$$

Defining the random variables G_M with $C_M = \exp(G_M \sum_{m=1}^{M} N_m)$, we find that

$$G_M = \frac{1}{\sum_{m=1}^{M} N_m} \sum_{m=1}^{M} \sum_{i=1}^{N_m} \log \left(1 + \sum_{k=1}^{K} \frac{f_{k,m}}{L_{k,m}} X_i^{(k,m)} \right).$$

Given the realistic assumption that

$$\sum_{m=1}^{\infty} \sum_{i=1}^{N_m} \frac{\operatorname{Var}\big(\log(1 + \sum_{k=1}^{K} f_{k,m} X_i^{(k,m)} / L_{k,m}) \big)}{\big(\sum_{j=0}^{m-1} N_j + i \big)^2} < \infty, \tag{5}$$

where $N_0 := 0$, the Kolmogorov sufficient condition (lemma 1, page 13) gives us

$$\frac{1}{\sum_{m=1}^{M} N_m} \sum_{m=1}^{M} \sum_{i=1}^{N_m} \left[\log\left(1 + \sum_{k=1}^{K} \frac{f_{k,m}}{L_{k,m}} X_i^{(k,m)}\right) - G_m(f_{1,m}, \ldots, f_{K,m}) \right] \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad M \to \infty.$$

The following theorems confirm the importance of the exponential growth functions for the individual batches of trading systems.

THEOREM 8. *Given inequality* (5)*, the following holds:*

(a) If $\liminf_{m\to\infty} G_m(f_{1,m}, \ldots, f_{K,m}) > 0$*, then* $C_M \xrightarrow{\text{a.s.}} \infty$ *as* $M \to \infty$*.*

(b) If $\limsup_{m\to\infty} G_m(f_{1,m}, \ldots, f_{K,m}) < 0$*, then* $C_M \xrightarrow{\text{a.s.}} 0$ *as* $M \to \infty$*.*

In practice, this theorem tells us that if we maintain "sufficiently positive" values for $G_m(f_{1,m}, \ldots, f_{K,m})$, we will make infinite fortunes in the long run, and that if we maintain "sufficiently negative" values for $G_m(f_{1,m}, \ldots, f_{K,m})$, we will go broke in the long run.

Let us now imagine that we play the trading systems in two ways simultaneously—one dollar going into plays where we use the fractions $f_{k,m}$, and another dollar into plays where we use the fractions $f^*_{k,m}$. We denote the two capital trajectories with C_M and C^*_M respectively.

THEOREM 9. *Given inequality* (5)*, the following holds: If*

$$\limsup_{m\to\infty} G_m(f_{1,m}, \ldots, f_{K,m}) < \liminf_{m\to\infty} G_m(f^*_{1,m}, \ldots, f^*_{K,m}),$$

then $C^*_M / C_M \xrightarrow{\text{a.s.}} \infty$ *as* $M \to \infty$*.*

In practice, this theorem tells us that if we maintain $G_m(f^*_{1,m}, \ldots, f^*_{K,m})$ "sufficiently higher" than $G_m(f_{1,m}, \ldots, f_{K,m})$ and trade using the fractions $f_{k,m}$, we will miss out on infinite fortunes in the long run compared to what we could have made by using the fractions $f^*_{k,m}$.

In this section, we have provided an extension of our theory to a nonstationary environment. In the next section, we revert back to the simpler case of a single fixed $G(f_1, \ldots, f_K)$ surface.

24. The Frequencies of Trading Systems

Until now, we have been assuming that all of our K trading systems have the same frequency (see section 18, page 27). In reality though, the systems are very likely to have *different* frequencies, as depicted in figure 17. (The figure depicts the first trades of two systems; where the trade of the upper system begins at time t_1 and ejects at t_3; and the trade of the lower system begins at t_2 and ejects at t_4; such

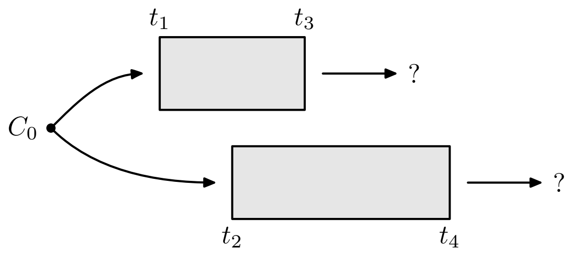

FIGURE 17. TRADING SYSTEMS WITH DIFFERENT FREQUENCIES.

that $t_1 < t_2 < t_3 < t_4$.) At first, it may seem that this situation is incompatible with our theory. For instance, what should we do with the capital that comes out the upper system? Should we wait until the lower system ejects? Although it is possible that we could somehow reconcile this with our theory, such attempts should strike us as rather awkward. After all, $G(f_1, \ldots, f_K)$ is the expected value of a certain function of K random variables, and it would not seem very natural to measure their dependencies (including, say, their correlations) with respect to occurrences that did not even occur at the same time.

We can resolve this confusion by making our trading operations more dynamic, such that we reallocate our capital every single time any of our systems begins or ejects. If one system is not active while another one is, we imagine that the former is active anyway—with a zero return. This is depicted in figure 18.

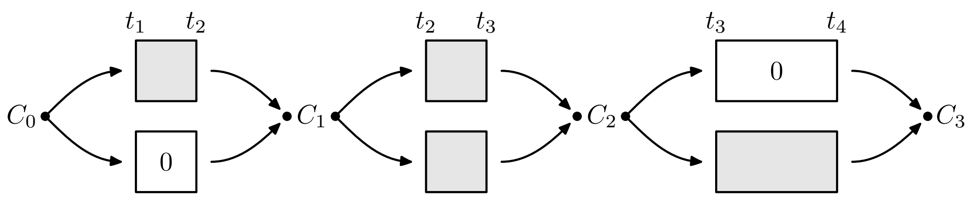

FIGURE 18. THE CONFUSION OF DIFFERENT FREQUENCIES RESOLVED.

Clearly, splitting up the trades of a particular system—i.e. splitting one gray block into two or more gray blocks, as shown in our figures—is going to change the return distribution of the system. For our theory to continue working in this situation, it is important that this splitting *does not introduce any dependency* between the blocks. (In practice, if traders find that splitting up their trades introduces a dependency, they should try to create new systems that exploit this dependency to their advantage.)

We end this section by analyzing what effect it has on a particular system to add zero returns into it, i.e. adding white "zero trades" to it as shown in figure 18. Imagine that we are back to analyzing a single trading system with a return distribution X, a largest possible loss L, and a growth function $G(f) = \mathbb{E}\log(1 + fX/L)$. Imagine that we create a new trading system with a return distribution X^* that has a certain probability q of being zero and a probability $(1 - q)$ of being determined by X. The growth function of this new system is given by

$$G^*(f) = \mathbb{E}\log\Big(1+\frac{f}{L}X^*\Big) = q\cdot 0 + (1-q)\,\mathbb{E}\log\Big(1+\frac{f}{L}X\Big) = (1-q)\,G(f).$$

Thus, we see that adding zero outcomes to a system merely scales the function $G(f)$ vertically, while all of its major properties, such as its optimal point and its root, stay exactly the same. In other words, the general behavior stays exactly the same; the only thing that changes is the time it takes to obtain that behavior.

25. Summary

In this chapter, we extended the entire theory we developed in chapter 1 to multiple dimensions. Sections 19 and 20 defined $G(f_1,\dots,f_K)$ and established its importance. Section 21 gave a good feel for its shape; particularly interesting was theorem 7, page 31, about the number of local maxima on $G(f_1,\dots,f_K)$. Section 22 discussed the over- or underestimation of profits or losses, and section 23 covered nonstationary environments. Finally, section 24 explained how our theory can be applied even when our K trading systems have different frequencies.

4

Diversification: Practice

26. Introduction

In this chapter, we construct a new trading system, in addition to the one we constructed in chapter 2, and show how the theory developed in the previous chapter can assist us in allocating our capital into a portfolio consisting of both systems.

27. A Two-System Portfolio

Since the construction of trading systems is not the subject of this thesis, our second system will for the sake of simplicity consist of the exact same rules as those of the first one—the rules in section 12, page 19—but applied on a different market. We have chosen cotton futures, during the same time period as the one we chose for the corn futures, i.e. July 1, 1959 to August 13, 2010. (The cotton futures were traded on the New York Board of Trade, later IntercontinentalExchange, with the ticker symbol CT; their data was, as for the corn futures, obtained from Commodity Research Bureau. Note that we remove from the data set all

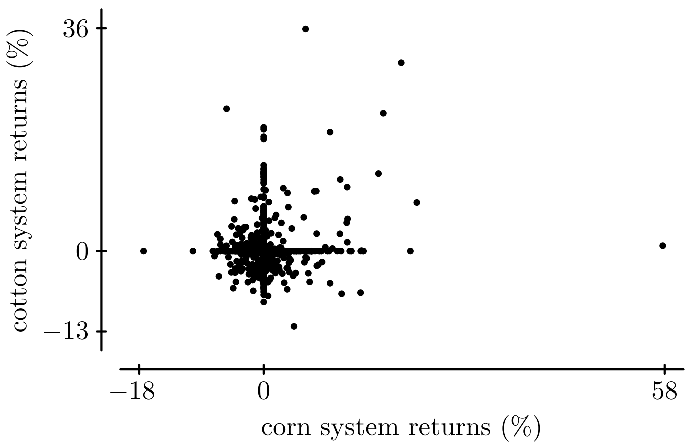

Figure 19. A Scatter Plot of the Returns of Our Two Systems.

days in which both markets are not open.) We will refer to the first system from chapter 2 as the "corn system" and this second system as the "cotton system."

In accordance with the discussion in section 24, page 36, about splitting up trades and adding "zero trades," we produce all the trades of our two-system portfolio. (Note that if one system exits *during* the day due to a stop-loss, we have to exit the other at the *closing* of that day. This is due to the nature of the historical data, which consists only of opening, highest, lowest, and closing prices for each day.)

We can visualize the joint return distribution by looking at a scatter plot of the returns, as shown in figure 19. There are a total of 957 points on this scatter plot. The smallest returns for the corn and cotton systems are, respectively, -0.1738 and -0.1217. The largest returns are, respectively, 0.5772 and 0.3589. (It should come as no surprise that these figures for the corn system differ from the ones in chapter 2; this is due to the splitting of trades and the adding of "zero trades.")

28. Obtaining $G(f_1, f_2)$

If $X^{(1)}$ and $X^{(2)}$ are random variables denoting, respectively, the percentage returns of the corn and cotton systems, we have from the previous chapter the general formula

$$G(f_1, f_2) = \mathbb{E} \log \left(1 + \frac{f_1}{L_1} X^{(1)} + \frac{f_2}{L_2} X^{(2)} \right)$$

for the exponential growth function of our two-system portfolio.

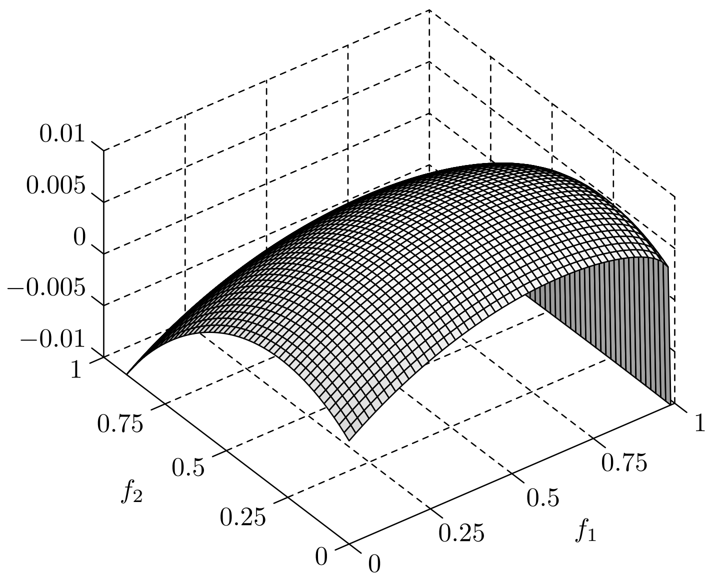

FIGURE 20. THE SURFACE $G(f_1, f_2)$ FOR OUR TWO-SYSTEM PORTFOLIO.

Let $r := 957$ be the number of points on our scatter plot (figure 19), and denote the points with $(x_{1,1}, x_{2,1}), \dots, (x_{1,r}, x_{2,r})$. The largest possible losses of the corn and cotton systems are, respectively, $L_1 = 0.1738$ and $L_2 = 0.1217$.

We now approximate $G(f_1, f_2)$ of our two-system portfolio in a similar manner as we did for $G(f)$ of the corn system in chapter 2: We assume that the joint return distribution of our two-system portfolio is exactly the discrete joint distribution provided by our scatter plot. In other words, we assume that each point $(x_{1,i}, x_{2,i})$ has the same probability $1/r$ of occurring. We thus obtain the approximation

$$G(f_1, f_2) = \frac{1}{r} \sum_{i=1}^{r} \log \left(1 + \frac{f_1}{L_1} x_{1,i} + \frac{f_2}{L_2} x_{2,i} \right),$$

which is shown in figure 20.

29. Trading at the Optimal Point

We find that our $G(f_1, f_2)$ is maximized at $(f_1, f_2) = (0.62, 0.25)$. (This was computed by applying the two-dimensional Newton-Raphson root-finding algorithm on the gradient of $G(f_1, f_2)$.)

Figure 21 shows our capital curve when trading our two-system portfolio at this optimal point. The figure also shows the optimal curve for the corn system (the same as the one in figure 12, page 25), and the optimal curve for the cotton system. (The curve for the cotton system was, of course, obtained using exactly the same methods that we used for the corn system in chapter 2.)

Note that the cotton system is, all by itself, not nearly as good as the corn system; the former only 14-folds our initial capital, whereas the latter 1412-folds

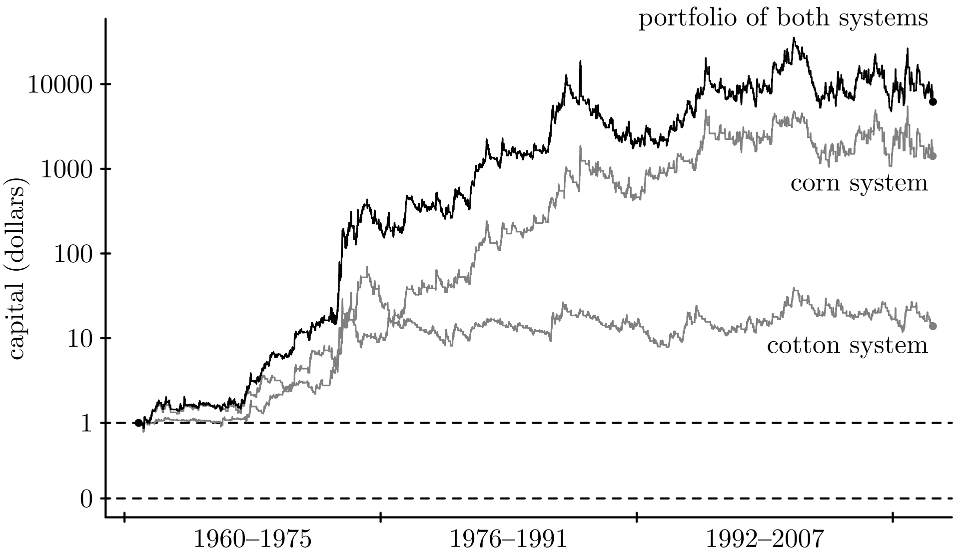

Figure 21. Optimal Capital Curves.

it. However, by combining the two, we are able to gain a significant performance boost—we end up 6183-folding our initial capital.

30. Further Applied Research

Similarly to what we mentioned in section 16, page 25, about chapter 2, the reader should take this present chapter for what it is and understand its limitations. We certainly have not created any bulletproof moneymaking machine. Not only do all the same caveats continue to hold, but new ones have been introduced. For instance, we never performed any checks—as section 24, page 36, calls for—on whether splitting up the trades of the corn and cotton systems introduces any dependencies. Section 45, page 55, should provide a good starting point for further applied research.

31. Summary

In section 27, we constructed a two-system portfolio. In section 28, we obtained the function $G(f_1, f_2)$ for the portfolio. In section 29 we traded the portfolio optimally and compared the performance with trading the systems individually. Finally, in section 30, we stressed the limitations of our discussion.

5

Drawdown Constraining: Theory

32. Introduction

Until now, we have been focusing solely on "growth aspect" of our story. Readers may have been deceived into believing that a trader's goal should always be to maximize $G(f_1, \ldots, f_K)$. (This chapter is written in the general setting of trading K systems, as shown in chapter 3. In the case of $K = 1$, readers can easily substitute the simpler notation from chapter 1, if they prefer.) Note, however, that we never actually mentioned that traders *should* perform the maximization; we only mentioned that they should do so *if* they want to obtain the optimal growth. This, however, is a very big "if."

Figure 22 introduces the concept of a *drawdown*, which is the maximum "peak to bottom" drop that occurs in our capital curve during a particular time period. As we will show in this chapter, the farther $(f_1, \ldots, f_K)$ is from the origin, along a given K-dimensional straight line going out from the origin, the greater the drawdowns in our capital curve will be. Based on our intuitions, this is what we would expect to hold, and we have in fact already seen an example of this—in figure 12, page 25.

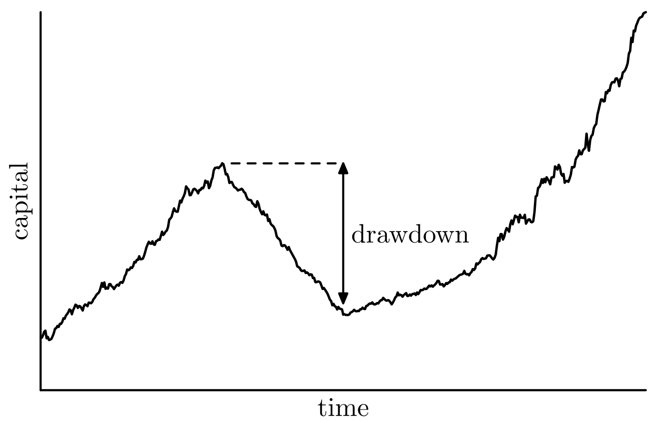

Figure 22. A Drawdown.

Some readers may be wondering why we should bother with analyzing drawdowns at all. After all, if we are trading at the optimal point, we will in the long run make infinitely more money than at any other point! The problem, however, is that too large drawdowns can be very painful psychologically.[2] Although individual traders with confidence in their strategy may be able to tolerate relatively large drawdowns, they will usually want *some* control over them. The situation is often more serious for traders managing money for less tolerant clients, who may be prone to angrily withdrawing their funds should they experience too large drawdowns.

Managing drawdowns is therefore of utmost importance to traders. Indeed, drawdowns are what traders perceive as being their "risk." This chapter provides the theoretical apparatus behind drawdown constraining, and the next chapter provides a practical example.

33. The Drawdown Defined

Recall, from chapter 3, that our capital after n plays in each of K trading systems is given by

$$C_n = \prod_{i=1}^{n} \left(1 + \sum_{k=1}^{K} \frac{f_k}{L_k} X_i^{(k)} \right),$$

where $f_k \geq 0$ for all k, $f_k > 0$ for some k, and $(f_1, \ldots, f_K) \in \mathcal{B}$. (See section 18, page 27.)

The drawdown of our capital after N plays, expressed as a percentage, is then defined with

$$D_N := \sup_{n \in \{1,\ldots,N\}} \sup_{m \in \{0,\ldots,n-1\}} \frac{C_m - C_n}{C_m}.$$

For example, $D_N = 0.2$ means that the drawdown during the first N plays is 20%. Note that a positive drawdown signifies a drop in our capital, and that the drawdown is always less than 1.

34. The Impossibility of Long-Term Drawdown Constraining

The following theorem shows that it is impossible to constrain drawdowns in any meaningful way over an infinite time horizon, because any positive drawdown will eventually take place.

Theorem 10. *For all $d \in (0, 1)$, we have $\mathbb{P}(D_N > d) \to 1$ as $N \to \infty$.*

[2] Controlling one's own emotions is considered by many to be a far more important element of successful trading than one's trading strategy. For some insights on this topic, see the interviews in Schwager (1989, 1992) with some of the world's most successful traders. See also Douglas (2000).

Proof. Take $y \in (0,1)$ such that $q := \mathbb{P}\big(C_n < yC_{n-1}\big) > 0$, and let $n^* > 0$ be a large enough integer such that $y^{n^*} < 1 - d$. We then see that

$$\begin{aligned}
\mathbb{P}(D_{Nn^*} > d) &= \mathbb{P}\left(\sup_{n\in\{1,\dots,Nn^*\}} \sup_{m\in\{0,\dots,n-1\}} \frac{C_m - C_n}{C_m} > d \right) \\
&= \mathbb{P}\left(\inf_{n\in\{1,\dots,Nn^*\}} \inf_{m\in\{0,\dots,n-1\}} \frac{C_n}{C_m} < 1-d \right) \\
&\geq \mathbb{P}\bigg(\bigcup_{n=1}^{N} \Big\{ \frac{C_{nn^*}}{C_{(n-1)n^*}} < y^{n^*} \Big\} \bigg) \\
&= 1 - \mathbb{P}\bigg(\bigcap_{n=1}^{N} \Big\{ \frac{C_{nn^*}}{C_{(n-1)n^*}} < y^{n^*} \Big\}^{\mathrm{c}} \bigg) \\
&= 1 - \prod_{n=1}^{N} \mathbb{P}\left(\Big\{ \frac{C_{nn^*}}{C_{(n-1)n^*}} < y^{n^*} \Big\}^{\mathrm{c}} \right) \\
&= 1 - \prod_{n=1}^{N} \left[1 - \mathbb{P}\bigg(\Big\{ \frac{C_{nn^*}}{C_{(n-1)n^*}} < y^{n^*} \Big\} \bigg) \right] \\
&\geq 1 - \prod_{n=1}^{N} \left[1 - \mathbb{P}\bigg(\bigcap_{m=0}^{n^*-1} \Big\{ \frac{C_{nn^*-m}}{C_{nn^*-(m+1)}} < y \Big\} \bigg) \right] \\
&= 1 - \prod_{n=1}^{N} \left[1 - \prod_{m=0}^{n^*-1} \mathbb{P}\bigg(\Big\{ \frac{C_{nn^*-m}}{C_{nn^*-(m+1)}} < y \Big\} \bigg) \right] \\
&= 1 - (1 - q^{n^*})^N \to 1 \quad \text{as} \quad N \to \infty,
\end{aligned}$$

which shows that $\mathbb{P}(D_N > d) \to 1$ as $N \to \infty$. □

35. Short-Term Probabilistic Statements

We saw in the previous section that we cannot obtain any long-term "almost sure" results for constraining drawdowns, as we were able to do for capital growth in chapters 1 and 3.

To control our drawdowns, we have to content ourselves with making some kind of short-term probabilistic statements. For instance, we can set up a constraint like this: "I want there to be less than a 5% probability of the drawdown becoming greater than 20% in the next 50 trades." More generally, for a probability b, a drawdown d, and a number of trades N, this constraint would be

$$\mathbb{P}(D_N > d) < b.$$

Suppose, then, that $(f_1, \dots, f_K)$ is the optimal point but does not satisfy our drawdown constraint. What other point should we choose instead? For high K, there is obviously no shortage of points to try. The next section suggests some natural canditates.

36. Moving Closer to the Origin

This section confirms the intuition of ours that if we are dissatisfied with the potential drawdown for a particular $(f_1, \ldots, f_K)$, we can ameliorate the situation by choosing a point on the straight line from the origin to $(f_1, \ldots, f_K)$ that is closer to the origin.

To show this, we first fix the outcomes of the returns $X_i^{(k)}$ to the numbers $x_i^{(k)}$. We do this because our end result will in effect be deterministic; no matter what the returns end up being, the drawdown would always have been lower if we would have chosen an allocation point that is, along a straight line, closer to the origin.

We next restate the fractions f_k as being located on a particular straight line going out from the origin (similarly to what we did when analyzing the function $G_{\mathbf{u}}(t)$ in chapter 3). To that end, let $\mathbf{u} = (u_1, \ldots, u_K)$ be a unit vector with $u_k \geq 0$ for all k.

Our fixed capital after n plays can now be written as

$$c_n(\mathbf{u}; t) := \prod_{i=1}^{n} \left(1 + t \sum_{k=1}^{K} \frac{u_k}{L_k} x_i^{(k)}\right), \qquad t \in (0, b_{\mathbf{u}}),$$

where $b_{\mathbf{u}} := \sup_{t\mathbf{u} \in \mathcal{B}} t$, and our fixed drawdown after N plays can be written as

$$d_N(\mathbf{u}; t) := \sup_{n \in \{1,\ldots,N\}} \sup_{m \in \{0,\ldots,n-1\}} \frac{c_m(\mathbf{u}; t) - c_n(\mathbf{u}; t)}{c_m(\mathbf{u}; t)}.$$

For convenience, the fixed capital and drawdown are being written with the parameter $\mathbf{u}$ and as functions of the variable t. After all, t determines the distance of our allocation point from the origin, which is precisely the quantity that we are interested in.

The main result now follows after a lemma.

LEMMA 3. *Let $a > 0$ be a number. Then, for all $y \geq -1/a$ and all t_1, t_2 such that $0 < t_1 < t_2 < a$, we have $(1 + t_2 y)^{t_1/t_2} \leq 1 + t_1 y$.*

PROOF. This is nothing but a variation of the generalized version of Bernoulli's inequality shown in Steele (2004), page 31. □

THEOREM 11. *$d_N(\mathbf{u}; t) > 0$ for any $t \in (0, b_{\mathbf{u}})$ implies*

(a) that $d_N(\mathbf{u}; t) > 0$ for all $t \in (0, b_{\mathbf{u}})$, and

(b) that $t \mapsto d_N(\mathbf{u}; t)$ is a strictly increasing function.

PROOF. Note first that the drawdown $d_N(\mathbf{u}; t)$, for all t, can be written as

$$\begin{aligned} d_N(\mathbf{u}; t) &= 1 - \inf_{n \in \{1,\ldots,N\}} \inf_{m \in \{0,\ldots,n-1\}} \frac{c_n(\mathbf{u}; t)}{c_m(\mathbf{u}; t)} \\ &= 1 - \inf_{n \in \{1,\ldots,N\}} \inf_{m \in \{0,\ldots,n-1\}} \prod_{i=m+1}^{n} \left(1 + t \sum_{k=1}^{K} \frac{u_k}{L_k} x_i^{(k)}\right). \end{aligned}$$

(a) Assume that $d_N(\mathbf{u}; t_1) > 0$ for some t_1. This implies that there exists a j such that $t_1 \sum_{k=1}^K u_k x_j^{(k)}/L_k < 0$. For some arbitrary t_2, we then clearly have $t_2 \sum_{k=1}^K u_k x_j^{(k)}/L_k < 0$, which gives

$$\inf_{n\in\{1,\ldots,N\}} \inf_{m\in\{0,\ldots,n-1\}} \prod_{i=m+1}^{n} \left(1 + t_2 \sum_{k=1}^{K} \frac{u_k}{L_k} x_i^{(k)}\right) \le 1 + t_2 \sum_{k=1}^{K} \frac{u_k}{L_k} x_j^{(k)} < 1,$$

which implies that $d_N(\mathbf{u}; t_2) > 0$.

(b) Pick some t_1, t_2 such that $0 < t_1 < t_2 < b_{\mathbf{u}}$, and use lemma 3 (with $a := b_{\mathbf{u}}$) to obtain for each i that

$$\left(1 + t_2 \sum_{k=1}^{K} \frac{u_k}{L_k} x_i^{(k)}\right)^{t_1/t_2} \le 1 + t_1 \sum_{k=1}^{K} \frac{u_k}{L_k} x_i^{(k)}.$$

Multiplying on both sides for arbitrary n, m with $m < n$, we obtain

$$\left[\prod_{i=m+1}^{n} \left(1 + t_2 \sum_{k=1}^{K} \frac{u_k}{L_k} x_i^{(k)}\right)\right]^{t_1/t_2} \le \prod_{i=m+1}^{n} \left(1 + t_1 \sum_{k=1}^{K} \frac{u_k}{L_k} x_i^{(k)}\right).$$

Since $t_1/t_2 > 0$, we can take infimum twice on both sides to obtain

$$(1 - d_N(\mathbf{u}; t_2))^{t_1/t_2} \le 1 - d_N(\mathbf{u}; t_1).$$

Now, assume that $d_N(\mathbf{u}; t_2) > 0$. Then, $1 - d_N(\mathbf{u}; t_2) \in (0, 1)$, and since $t_1/t_2 < 1$, we find that

$$d_N(\mathbf{u}; t_1) < d_N(\mathbf{u}; t_2). \qquad \square$$

37. The Allowed and Forbidden Regions

From the above sections, we deduce that any particular drawdown constraint implies the existence of a "boundary" in $\mathcal{B}$, "below" which our constraint will be satisified and "above" which it will not. The boundary will thus partition $\mathcal{B}$ into an "allowed region" and a "forbidden region." This is depicted in figure 23 for $K = 2$.

The exact shape of the boundary of the allowed region will, of course, depend on the joint return distribution of the K trading systems. In practice, for high K, we will not necessarily have resources to compute the entire boundary; we can in this case, as we have seen, search for the optimal point, and then (unless the optimal point satisifies our constraint) head towards the origin along a straight line until we hit the boundary.

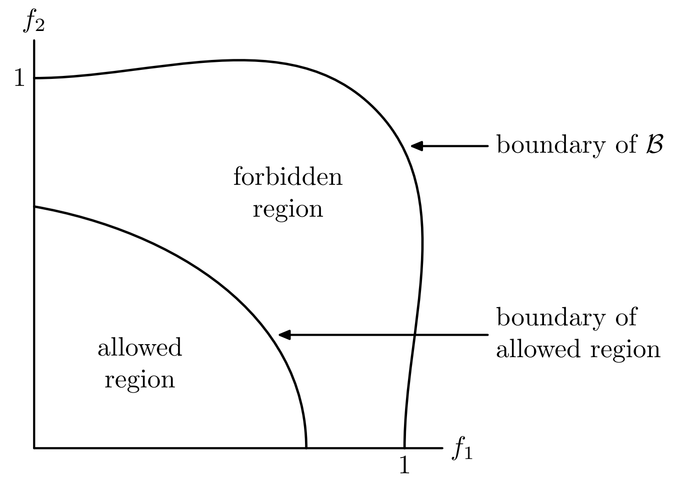

FIGURE 23. THE ALLOWED AND FORBIDDEN REGIONS.

38. Summary

It would have been nice if we could have obtained some long-term "almost sure" results for constraining drawdowns, as we did for capital growth in chapters 1 and 3. Unfortunately, section 34 showed that this is impossible. When constraining drawdowns, we have to content ourselves with short-term probabilistic statements, as mentioned in section 35. Section 36 suggested that if we are not satisfied with the drawdowns that a particular $(f_1, \ldots, f_K)$ may result in, we can try moving closer to the origin along a straight line. Finally, section 37 showed that any drawdown constraint gives rise to corresponding allowed and forbidden regions in which $(f_1, \ldots, f_K)$ may and may not be, respectively.

6

Drawdown Constraining: Practice

39. Introduction

The largest drawdown in the optimal capital curve of the two-system portfolio that we constructed in chapter 4—the topmost curve in figure 21, page 41—is 91%. Some might find this rather aggressive. In this chapter, we attempt to constrain the drawdowns when trading our portfolio, using the ideas from the previous chapter. But before we do that, we describe how we can calculate the probability $\mathbb{P}(D_N > d)$ using Monte Carlo simulations.

40. Monte Carlo Simulations

We are interested in calculating the probability $\mathbb{P}(D_N > d)$ using Monte Carlo simulations. Define the random variable $Y := \mathbb{I}_{\{D_N > d\}}$, where "$\mathbb{I}$" denotes the indicator function. Let $Y_1, \ldots, Y_n$ be independent and identically distributed random variables with the same distribution as Y. (The letter n denotes the number of simulations in the chapter. This should not cause any confusion, even though this letter has been used for other purposes before.) Note that

$$\mathbb{P}(D_N > d) = \mathbb{E}[Y].$$

Denote with z_δ the $1 - \delta$ quantile of the standard normal distribution, and define

$$\bar{Y}_n := \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

According to the discussion in Glasserman (2004), appendix A, $\mathbb{P}(D_N > d)$ can now be estimated with

$$\bar{Y}_n \pm z_{\delta/2} \frac{s_n}{\sqrt{n}},$$

where

$$s_n := \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2}.$$

The above interval is an asymptotically valid $1 - \delta$ confidence interval for the quantity $\mathbb{P}(D_N > d)$. In other words, the probability that the interval covers $\mathbb{P}(D_N > d)$ approaches $1 - \delta$ as $n \to \infty$. For moderately high n, this interval should give us at least some idea of the accuracy of our simulation.

Since $Y = \mathbb{I}_{\{D_N > d\}}$, we simulate each Y_i by simply generating a random capital curve up to C_N. If the largest drawdown in this capital curve exceeds d, we have $Y_i = 1$; otherwise we have $Y_i = 0$. Hence, $\sum_{i=1}^{n} Y_i$ is the number of capital curves that have a drawdown exceeding d. Since n is the total number of simulated capital curves, it should now make sense that $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is an estimate for $\mathbb{P}(D_N > d)$.

41. Drawdown Constraining

Suppose, when trading our two-system portfolio from chapter 4, that we want to control the probability of the drawdown in our capital curve being greater than 40%. We do this by calculating the probability $\mathbb{P}(D_N > d)$, with $d = 0.4$ and $N = r = 957$, for a few different allocation points on the straight line from the origin to the optimal point $(0.62, 0.25)$. We can let $t \in (0, 1)$ correspond to such an allocation point $t(0.62, 0.25)$. We simulate a capital curve up to C_N by randomly (and uniformly) choosing, at each step, among the $r = 957$ different points on the scatter plot in figure 19, page 39.

The following table shows a few values we find for $\mathbb{P}(D_N > d)$ at a few allocation points. We use $n = 100{,}000$ simulations; and a 95% confidence interval, i.e. $\delta = 0.05$, which makes $z_{\delta/2} = 1.96$.

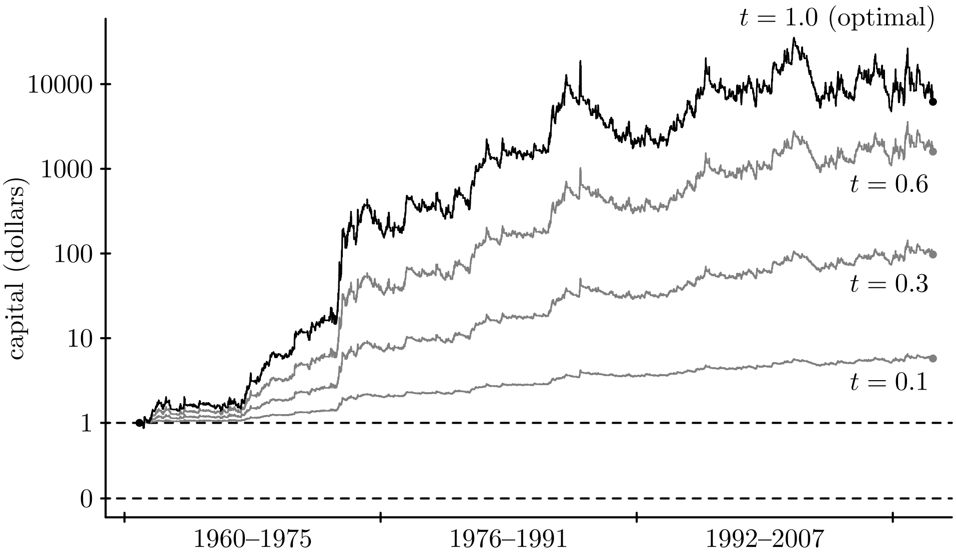

Figure 24. Constrained Drawdowns.

t	$\bar{Y}_n$	$z_{\delta/2}s_n/\sqrt{n}$
1.0	1.0	0.0
0.6	1.0	0.0001
0.3	0.696	0.003
0.1	0.001	0.0002

The resulting capital curves are shown in figure 24. We see that by moving the allocation point closer to the origin, we are indeed able to reduce the drawdowns; the drawdowns are, as t descends, 91%, 71%, 43%, and 17%. However, we sacrifice a lot of growth in the process; the end values of the capital curves are 6183, 1597, 97, and 6. This, of course, is a manifestation of the age-old adage that if one wants a particular reward, one has to be willing to take the risks associated with that reward. Which t we choose depends on our appetite for risk, in this case our drawdown tolerance. (Of course, this is not entirely accurate. For a given $t \in (0, 1)$, there may well be points *outside the straight line* where the growth stays the same but the risk reduces, or where the risk stays the same but the growth increases. Finding such points would require further computational research.)

42. Summary

In this short chapter, we first covered how to use Monte Carlo simulations to calculate the probability that the drawdowns in a capital curve exceed a certain level. We then demonstrated how we can constrain the drawdowns when trading the two-system portfolio that we constructed in chapter 4.

7

Loose Ends

43. Introduction

This chapter covers two loose ends. First, we look at how futures traders can take into account contract sizes and margin requirements. Second, we provide step-by-step instructions for traders who are preparing to apply our theory in practice.

44. Contract Sizes and Margin Requirements

Until now, we have been assuming that we can trade fractions of futures contracts, and that we go broke when our capital drops to zero. In practice though, we go broke when we cannot even trade one contract, or when we get a margin call from our broker. In this section, we suggest a method that makes sure we can trade in perpetuity under these realistic conditions.

Let C be our currently available capital (i.e. C_n, for some n that does not concern us here). Suppose we have constructed a portfolio of K trading systems that we have decided to trade with, and let f_k/L_k be the portion that we have decided to commit of our capital into each trade of system k. Suppose also that we are just about to enter trades in our systems.

Let S_k be the current price of one futures contract of market k, i.e. the market that system k operates on. Let furthermore M_k denote the initial margin requirement per contract in this market; we assume that this is a fixed quantity. (Readers may at this point want to revise the mechanics of futures markets; see e.g. Hull (2006), chapter 2. In particular, they should be familiar with initial and maintenance margins.) For example, if the quoted price of market k is currently 800 cents/bushel, each contract is for 5,000 bushels, and the initial margin requirement per contract is \$2,000, we have $S_k = \$40{,}000$ and $M_k = \$2{,}000$.

We want to make sure that we can trade in perpetuity. We can accomplish this by imagining that our available capital is only $C - \sum_{k=1}^{K} x_k M_k$ instead of C, where x_k is the proper number of contracts (yet to be determined) that we commit to in market k. If we then apply our theory, we guarantee that we will always have enough capital to cover the initial margin. In particular, since the maintenance margin is always lower than the initial margin, we will never get a margin call.

Our goal is to find the x_k. For each k, we have the relation

$$\frac{f_k}{L_k}\left(C - \sum_{\kappa=1}^{K} x_\kappa M_\kappa\right) = x_k S_k,$$

which we rewrite as

$$x_k \frac{S_k L_k}{f_k} + \sum_{\kappa=1}^{K} x_\kappa M_\kappa = C.$$

This can be written with matrix notation:

$$\begin{pmatrix} M_1 + \dfrac{S_1 L_1}{f_1} & M_2 & \cdots & M_K \\ M_1 & M_2 + \dfrac{S_2 L_2}{f_2} & \cdots & M_K \\ \vdots & \vdots & \ddots & \vdots \\ M_1 & M_2 & \cdots & M_K + \dfrac{S_K L_K}{f_K} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix} = \begin{pmatrix} C \\ \vdots \\ C \end{pmatrix}.$$

Using the Sherman-Morrison formula (see Hager (1989)), we now easily obtain for each k the solution

$$x_k = \frac{C \dfrac{f_k}{S_k L_k}}{1 + \displaystyle\sum_{\kappa=1}^{K} \frac{M_\kappa f_\kappa}{S_\kappa L_\kappa}}.$$

We conclude this section by providing an example that demonstrates how we can apply this result in practice. (Note that all the figures, except for the contract sizes, are hypothetical. However, they are realistic in the sense that they are approximately based on the figures that applied at the time of this writing.)

Example. We have accumulated a trading stake of $C = \$20{,}000$, and we are interested in trading the portfolio of corn and cotton systems that we constructed in chapter 4. We wish to trade at one of the levels of reduced drawdowns that we obtained in chapter 6, namely $(f_1, f_2) = 0.3(0.62, 0.25)$, which corresponds to $f_1/L_1 = 1.07$ and $f_2/L_2 = 0.62$. The currently quoted contract prices (of the particular contracts we are just about to enter) for corn and cotton are, respectively, 800 cents/bushel and 80 cents/pound. The contracts sizes are, respectively, 5,000 bushels and 50,000 pounds. Therefore, $S_1 = S_2 = \$40{,}000$. We furthermore know that the margin requirements are $M_1 = M_2 = \$2{,}500$.

Using the above formula, we compute $x_1 = 0.48$ and $x_2 = 0.28$. Since we cannot trade fractions of contracts, we have to use $x_1 = x_2 = 0$. We see that we cannot even enter into one contract in either corn or cotton. We thus come to the conclusion that we have no business trading this portfolio, given our limited capital and our particular level of risk.

Example (Continued).

However, if we can accumulate a larger stake, say $C = \$250{,}000$, we find that $x_1 = 6.05$ and $x_2 = 3.48$. If our capital drops by 50%, to $C = \$125{,}000$, we find $x_1 = 3.03$ and $x_2 = 1.74$. If, in addition, the margin requirements are hiked up by 40%, to $M_1 = M_2 = \$3{,}500$, we find $x_1 = 2.91$ and $x_2 = 1.68$. And if, in addition, the prices of the contracts rise by 40%, to $S_1 = S_2 = \$56{,}000$, we have $x_1 = 2.16$ and $x_2 = 1.24$. It thus looks like we can somewhat safely start trading the portfolio with a starting capital of $250,000.

Smaller traders can rejoice in the fact that there are "mini-versions" available of many futures contracts. For instance, CME Group provides a mini-sized corn contract of 1,000 bushels. Although IntercontinentalExchange does not seem to provide a mini-sized cotton contract, CME Group provides mini-sized contracts for many other products, including wheat, soybeans, crude oil, natural gas, gold, silver, and copper.

45. Step-by-Step Instructions for Traders

The following steps outline the activities of mechanical traders, how our theory fits into the picture, and what additional research has to be done along the way that was not covered in this thesis.

1. Construct a portfolio of K trading systems, and use historical data to approximate their joint return distribution. Chapters 2 and 4 went through this process for $K = 1$ and $K = 2$ respectively.

2. If desired, optimize the portfolio to the historical data. For example, in our case, optimization could have found out that the 20 days that we chose for entry breakouts (in section 12, page 19) is not the best figure to use. Optimization is covered e.g. in Faith (2007), pages 163–177, and in Vince (1990), chapter 2.

3. Analyze how robust the portfolio is, i.e. how well the historical performance is likely to hold up in the future. One way to do this is to simulate alternative histories and see how the portfolio would have performed under those conditions. See e.g. Faith (2007), chapter 12, and Chande (2001), chapter 8.

4. Statistically test the return distributions of our individual systems for clear evidence of dependencies. If such evidence is found, we should try to find the source of the dependencies and create new systems that exploit them to our benefit. Dependency tests are covered e.g. in Vince (1990), pages 26–40, and Balsara (1992), pages 175–177.

5. We now have a fairly good view of our portfolio's joint return distribution, and hence the surface $G(f_1, \ldots, f_K)$, for a certain time into the future. If we are still somewhat unsure, we can gain more safety by overestimating

losses or underestimating profits, as we discussed in section 22, page 32 (and section 7, page 10, for $K = 1$).

6. Decide how large drawdowns will be acceptable to us, and find an appropriate $(f_1, \dots, f_K)$ to trade with. This was covered in chapter 6.

7. When we find, in light of new evidence, that the joint return distribution has changed, we compute a new $G(f_1, \dots, f_K)$ and adjust our $(f_1, \dots, f_K)$ accordingly. Our drawdown tolerance may also change with time, which also results in an adjustment of our $(f_1, \dots, f_K)$. Recall that this process of dynamically changing $G(f_1, \dots, f_K)$ and $(f_1, \dots, f_K)$ as time goes by was given a theoretical footing in section 23, page 35 (and section 8, page 13, for $K = 1$).

46. Summary

In this chapter, we explained how to properly take into account contract sizes and margin requirements; we found out that if we have a too small starting capital or too strict drawdown constraints, we may not be able to trade a given portfolio at all. The chapter also outlined the various steps that mechanical traders will go through during their trading operations.

Summary

Chapter 1 introduced the highly important function $G(f)$, which describes the long-term exponential growth of our capital curve, where f measures how much of our capital we place into each play of our trading system. Chapter 3 extended this function to the surface $G(f_1, \ldots, f_K)$, where we are now simultaneously playing K trading systems, and each f_k measures how much of our capital we place into system k. We showed that it is crucial to choose an $(f_1, \ldots, f_K)$ such that $G(f_1, \ldots, f_K) > 0$. If we do, we will make infinite fortunes in the long run; if we do not, we will go broke or get unpredictable results. Chapter 5 showed how constraining drawdowns further narrows down our choices for $(f_1, \ldots, f_K)$.

Chapters 2, 4, 6, and 7 explained how the theory can be applied in practice. Without these chapters, this thesis would have been a mere theoretical exercise of limited use to practitioners. Now, practitioners can at least program our examples into a computer, check their results against ours, and then move on to apply the ideas on their own, superior, trading systems.

We have seen that it is absolutely essential for traders to be aware of where they are located on the surface $G(f_1, \ldots, f_K)$; infinite fortunes are on the line, in the long run. Our theory provides a *framework* that traders can use to navigate themselves along this surface. It is this framework that Vince (2009) has popularized as the "leverage space trading model." This term should make sense. After all, we are navigating ourselves along a $(K + 1)$-dimensional surface, or "space," and our position on this surface describes exactly how much leverage we are (or are not) taking. As we mentioned in the introduction, leverage is nothing particularly special under our theory, but a natural and inherent part of it.

References

1. Balsara, Nauzer J. (1992). *Money Management Strategies for Futures Traders.* John Wiley & Sons.

2. Breiman, L. (1961). *Optimal Gambling Systems for Favorable Games.* Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1, pages 65–78.

3. Chande, Tushar S. (2001). *Beyond Technical Analysis.* Second Edition. John Wiley & Sons.

4. Covel, Michael W. (2007). *The Complete TurtleTrader.* Harper. Republished, 2009. Page numbers refer to the 2009 edition.

5. Douglas, Mark (2000). *Trading in the Zone.* New York Institute of Finance.

6. Elder, Alexander (1993). *Trading for a Living.* John Wiley & Sons.

7. Faith, Curtis M. (2007). *Way of the Turtle.* McGraw-Hill.

8. Finkelstein, Mark, and Robert Whitley (1981). *Optimal Strategies for Repeated Games.* Advances in Applied Probability, Volume 13, Number 2, pages 415–428.

9. Glasserman, Paul (2004). *Monte Carlo Methods in Financial Engineering.* Springer.

10. Gut, Allan (2005). *Probability: A Graduate Course.* Springer.

11. Hager, William W. (1989). *Updating the Inverse of a Matrix.* SIAM Review, Volume 31, Number 2, pages 221–239.

12. Hull, John C. (2006). *Options, Futures, and Other Derivatives.* Sixth Edition. Prentice Hall.

13. Kallenberg, Olav (2002). *Foundations of Modern Probability.* Second Edition. Springer.

14. Kelly, J. L., Jr. (1956). *A New Interpretation of Information Rate.* Bell System Technical Journal, Volume 35, Issue 4, pages 917–926.

15. Mises, Ludwig von (1949). *Human Action.* Yale University Press. Republished by the Ludwig von Mises Institute, 1998.

16. Rogers, Jim (2004). *Hot Commodities.* Beeland Interests. Republished by Random House, 2007. Page numbers refer to the 2007 edition.

17. Rothbard, Murray N. (1962). *Man, Economy, and State.* William Volker Fund and D. Van Nostrand. Republished by the Ludwig von Mises Institute as *Man, Economy, and State with Power and Market*, 2009.

18. Schwager, Jack D. (1989). *Market Wizards.* New York Institute of Finance. Republished by Marketplace Books, 2006.

19. ——— (1992). *The New Market Wizards.* HarperCollins. Republished by Marketplace Books, 2008.

20. Steele, J. Michael (2004). *The Cauchy-Schwarz Master Class.* Cambridge University Press.

21. Thorp, E. O. (1969). *Optimal Gambling Systems for Favorable Games.* Review of the International Statistical Institute, Volume 37, Number 3, pages 273–293.

22. Vince, Ralph (1990). *Portfolio Management Formulas.* John Wiley & Sons.

23. ——— (1992). *The Mathematics of Money Management.* John Wiley & Sons.

24. ——— (1995). *The New Money Management.* John Wiley & Sons.

25. ——— (2007). *The Handbook of Portfolio Mathematics.* John Wiley & Sons.

26. ——— (2009). *The Leverage Space Trading Model.* John Wiley & Sons.