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Name Concentration Risk and Pillar 2 Compliance

The Granularity Adjustment

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Abstract

A credit portfolio where each obligor contributes infinitesimally to the risk is said to be infinitely granular. The risk related to the fact that no real credit portfolio is infinitely granular, is called name concentration risk.

Under Basel II, banks are required to hold a capital buffer for credit risk in order to sustain the probability of default on an acceptable level. Credit risk capital charges computed under pillar 1 of Basel II have been calibrated for a specific level of name concentration. If a bank deviates from this benchmark it is expected to address this under pillar 2, which may involve increased capital charges.

Here, we look at some of the difficulties that a bank may encounter when computing a name concentration risk add-on under pillar 2. In particular, we study the granularity adjustment for the Vasicek and CreditRisk$^+$ models. An advantage of this approach is that no vendor software products are necessary. We also address the questions of when the granularity adjustment is a coherent risk measure and how to allocate the add-on to exposures in order to optimize the credit portfolio. Finally, the discussed models are applied to real data.

Keywords: Credit Risk; Basel II; IRB formula; Concentration risk; Name concentration; Idiosyncratic risk; Granularity adjustment; Vasicek model; CreditRisk$^+$ model; Coherence; Euler capital allocation

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Notation and Abbreviations

CCF_i	credit conversion factor of obligor i
COMM_i	commitment of obligor i
D_i	default variable of obligor i
DLGD_i	downturn LGD_i
EAD_i	exposure at default of obligor i
$\widetilde{\mathrm{EAD}}_i$	random exposure at default of obligor i
ELGD_i	expected loss given default of obligor i
$F_X(x)$	cumulative distribution function of the random variable X
$f_X(x)$	probability density function of the random variable X
$\mathrm{GA}_q(L)$	granularity adjustment of L at the confidence level q
$\mathrm{HHI}(L)$	Herfindahl-Hirschman index of L
i.i.d.	independent and identically distributed
K	regulatory capital for credit risk as a share of total EAD
L	loss variable of a credit portfolio
L_i	loss variable of obligor i
LGD_i	loss given default of obligor i
M_i	effective maturity of obligor i
m	number of risk factors
n	number of obligors in the credit portfolio
OUT_i	current outstanding amount to obligor i
PD_i	probability of default of obligor i
PIT	point-in-time
RWA_i	risk-weighted asset of obligor i
TTC	through-the-cycle
$\mathrm{UL}_q(L)$	unexpected loss of L at confidence level q
s_i	EAD_i as a share of total EAD
$\mathrm{V}(X)$	variance of the random variable X
VLGD_i	variance of the loss given default of obligor i
w.p.	with probability
$\boldsymbol{Z} = (Z_1, \ldots, Z_m)^\top$	vector of risk factors
λ_i	liabilities of obligor i
$\Phi(\cdot)$	cumulative distribution function of the standard normal distribution
$\varphi(\cdot)$	probability density function of the standard normal distribution

1 Introduction

All banks face risks. In fact, in order to profit banks actively take on risks. By diversifying stochastic cash flows banks can offer its costumers deterministic cash flows. This is demanded since the costumers then can make more long-term plans for the future. From this point of view the idea of banks is the same as that of insurance companies: making the future more predictable. Risk management is concerned with how these stochastic cash flows are managed and aims to create shareholder value in a competitive market.

1.1 Bank regulation

From a societal point of view one may say that the function of banks is to effectively allocate capital between consumers and investors by diversifying risk. As an intermediary of capital, banks play an important role in the economy.

A credit migration of a company implies credit migrations of its lenders, which in its turn implies credit migrations of the lenders' lenders, and so forth. Default dependence may also be caused by other business links than borrower-lender relationships, e.g., buyer-seller interactions. The probability of default of a company conditional on the default of another company, is known as *default contagion*. That is why defaults of banks are devastating for the economy. The consequences of a bank failure are augmented by the considerable lending between banks. In the credit crunch that erupted 2007, banks were unable to assess the credit quality of other banks and hence the lending between banks dried up.

Banks can achieve economies of scale since larger banks can diversify their portfolios more effectively. This enhances the impact of default contagion due to bank failures. As banks become larger we can expect fewer but more severe bank failures. Presumably, the the major lesson of the Great Depression in the 30s is to not let bank failures take place. Therefore, in the recent financial crisis, measures were taken by governments to prevent bank failures.

A deterioration of the credit quality of a bank may induce bank runs, i.e., a considerable part of the bank's customers withdraw their money at the same time due to the fear that the bank won't fulfill its obligations. If a bank run takes place, it doesn't really matter whether the fear is well-founded or only based on rumors, the result will always be the same, bank failure. No bank can survive if all their customers withdraw their money at the same time. Thus, a bank run is a self-fulfilling prophecy, increases the likelihood of bank failures and destabilizes the economy. This may give governments reasons to implement a tax-funded deposit insurance since it would remove the motives for bank runs.

To sum up, we have observed two shortcomings of the banking system. The immensely negative consequences to the economy that from follow bank failures and the unstable nature of the banking system due to bank runs. However, these market failures can be eliminated through effective measures taken by governments. Bank runs can be prevented by deposit insurance and bank failures by government bailouts.

Since both deposit insurances and bailouts are tax-financed it may seem reasonable for governments to impose regulations that enforces banks to uphold prudent risk appetites. However, it should be clear that bank failures most likely will take place anyhow. The risks faced by banks cannot easily be assessed. If we enter a casino the probabilities of the games are known to us. But we cannot determine the probabilities of the risks that the banks face, all we can do is to make an educated guess. History, however, has shown that this is everything but an easy task.

In order to understand the risk management of a bank it isn't sufficient to only consider it from a societal point of view. One also has to look at it from the shareholders' perspective. After all, the risk management of a bank works in the interests of the shareholders. There is, however, no obvious way to determine the optimal risk appetite for a company. It is easily understood that the risk appetite that creates the most shareholder value, which is just another way of posing the same question, depends on a multitude of factors. However, it may be worthwhile to point out a few important factors. Decreasing the risk or increasing the expected profit of a company, ceteris paribus, creates shareholder value. A clearly defined risk appetite will make investors more informed about the risks they are taking and if the risk appetite isn't changed too often, the investors won't have to rebalance their portfolios often, which reduces transaction costs.

There is an inherent trait of risk taking among companies that can achieve economies of scale. A competitive advantage can be gained through expansion, but expansion can only be reached through risk taking. This reasoning is analogical to Texas hold'em where you either play aggressive, in which case you win big or lose fast, or you take small risks and bet little by little, in which case you will most likely lose, but slowly. If we look at this from a game theoretical point of view we can infer that all players will play aggressively.

Additional lending always increases the risk of banks. The expected profit, however, will only increase as lending increases to a certain extent after which it will decrease. The reason is that additional lending also increases the risk, which will affect the bank's cost of borrowing money. The level of lending for which the expected profit is maximized provides a natural risk appetite for banks. But if large banks are going to be bailed out anyhow, additional lending doesn't entail any risk and the banks should always be able to borrow at a low cost. Thus, moral hazard makes an effective regulation of the risks that large banks face even more important.

1.2 A Brief History of the Basel Accords

Bank regulation has in various forms existed for a long time but it is not until the first Basel Accord (Basel I) of 1988 that a more unified framework has been established. In order to establish a competition among banks on equal terms it is important for the regulatory framework to be widely spread. The Basel Accords, however, only constitute recommendations for regulations that states may or may not ratify. The main focus in Basel I was *credit risk*, i.e., the risk of losses due to default among the bank's obligors. But in 1996 an amendment to Basel I was published which included *market risk*, i.e., the risk of losses due to changes in the value of the bank's assets. The second Basel Accord (Basel II) was published 2004 and included several new kinds of risks, of which operational risk was given special emphasis. Operational risk is defined as the risk of losses due to failed internal processes, such as fraud and programming errors. Basel II was fully implemented in Sweden in 2007. The third Basel Accord (Basel III) was published in 2010 and contains very much the same division of risks as Basel II. One of the major changes in Basel III is the countercyclical capital buffers. These are supposed to be larger during economical expansion and smaller during recession, which would mitigate economic cycles and crises. The implementation of Basel III in Sweden will begin in 2013. Even though several more kinds of risks have been included during the development of the Basel Accords, credit risk remains the single most important kind of risk. One of the main results of the Basel Accords is that they specify a minimum capital buffer that the banks must hold in order to sustain an adequate credit quality.

In Basel I credit exposures were divided into a few categories, e.g., state, bank and mortgages. The outstanding amount of every exposure was then multiplied with a number that depended on the category assigned to the exposure. The total capital requirement was then defined as the sum of the resulting quantities. As a consequence, it was more profitable for banks to keep exposures of low credit quality since these would yield a higher interest but the same capital requirement. Another problem was that the capital requirement didn't depend on diversification and, thus, didn't give banks any incentive to diversify their credit portfolios. The calculation of credit risk in Basel II is much more sophisticated than in Basel I and the treatment of credit risk in Basel III is similar to that of Basel II.

1.3 The Pillars of Basel II

Basel II is built on three pillars. Under pillar 1 banks compute a *minimum capital requirement* for credit, market and operational risk. The total minimum capital requirement, also referred to as *regulatory capital*, is simply the sum of the regulatory capital for each specific kind of risk. Since the regulatory capital for credit, market and operational risk are calculated as if they were independent it seems reasonable for the regulatory capital to depend on how diversified the bank is among these risks. However, simplifications have been made to make the pillar 1 computations tractable.

Under pillar 2, also referred to as the *supervisory review process*, a more holistic approach is taken towards risk and it covers risk types not considered under pillar 1. Relatively to pillar 1 the Basel II framework doesn't provide much information about how to assess the risks under pillar 2. As a consequence, the supervisory authorities have to study the methods used by the banks to make sure that they are reasonable. The bank gives details about their chosen methods in a report of the Internal Capital Adequacy Assessment Processes (ICAAP). The third pillar is concerned with establishing *market discipline*, primarily by increasing the transparency of banks by making information about capital adequacy public.

Under pillar 1 the bank may choose Standardized Approach or one of the two more advanced Internal Ratings-Based (IRB) approaches to compute regulatory capital[1]. In the IRB approaches the banks may use own estimates of credit risk related data instead of the often more conservative data provided by rating agencies. The IRB approaches are subdivided into the Foundation Internal Ratings Based approach (FIRB) approach and the Advanced Internal-Ratings Based (AIRB) approach. The difference is that the bank uses more own estimates in the AIRB approach than in the FIRB approach. Both approaches, however, use the same formula, which is called the IRB formula. If a bank wishes to use one of the the IRB approaches this first has to be approved by the supervisory authorities, why these approaches are more common among large banks.

1.4 Concentration Risk and Basel II

The computation of regulatory capital has been designed to meet the requirement of *portfolio invariance*, i.e., the increase of regulatory capital for a new credit will be the same regardless of the composition of the portfolio it is added to. This simplification was made to make the computations sufficiently practical for regulatory purposes (BCBS 2006b, p.4). However, credit portfolios with exposure concentrated in a single country and industry[2] are usually considered to be more risky than portfolios that are well diversified among sectors. This kind of risk is called *sector concentration risk*. We also have that a portfolio with large exposures usually is considered more risky than a portfolio that consists of more but smaller exposures. This kind of risk is called *name concentration risk*.

[1] Since only credit risk will be considered in the sequel, from this point we will by *regulatory capital* mean *regulatory capital for credit risk*, unless otherwise stated.

[2] A specific geographical area and industry will henceforth be denoted *sector*.

These risk types, sector and name concentration risk, aren't necessarily independent. Any two portfolios that are equal in every sense except in their distribution of large exposures among sectors will usually not be considered to exhibit the same risk. The portfolio with large exposures more concentrated among sectors will usually be considered more risky.

Sector and name concentration risk are together referred to as *concentration risk* and is, due to the simplification made in the computations of the regulatory capital mentioned above, not accounted for under pillar 1. However, it is to be accounted for under pillar 2 (BCBS 2005, p.4). Banks are free to choose any model to assess the concentration risk but the result has to be considered reasonable by the supervisory authorities. No particular model is recommended by the Basel framework. The computation of regulatory capital was however calibrated to accurately estimate the credit risk for a number of large internationally active banks. Unfortunately, no more information about the benchmark portfolio is available. The commonly used benchmark for name concentration risk is the infinitely granular portfolio.

It is difficult to say what methods of computing concentration risk that are most common in the banking sector since this information isn't public. The document *Studies on credit risk concentration* (BCBS 2006b), however, gives an overview over some methods used by financial institutions at that time. The Swedish Financial Supervisory Authority (Finansinspektionen (FI)) gives in three memoranda (Edlund (2009*a*), Edlund (2009*b*) and Edlund (2009*c*)) on its website some information on how FI computes concentration risk. For banks using the IRB approach FI also suggests a particular method of computing the add-on for name concentration risk (see Section 5.2.4).

1.5 Purpose

The objective of this thesis is to examine some of the existing methods to compute an add-on[3] with emphasis on the granularity adjustment and consider them in relation to the challenges that arise in the implementation process, including capital allocation. Thus, this objective is neither to cover all methods available, nor to determine the accuracy of the methods. The thesis mainly considers banks with IRB permission but many results can also be applied by banks that use the Standardized Approach. The objective is merely to point out some of the characteristics of the models that may come in hand as a bank is considering what method to implement. Thus, the thesis will not point out any model as superior but provide information in what circumstances one method is to be preferred over another. All models have advantages and disadvantages. The point is to know what model to choose in what situation.

[3]If not otherwise stated, by *add-on* we mean the add-on computed under pillar 2 for name concentration risk.

1.6 Outline

In Section 2, two of the most common credit risk models, the Vasicek and CreditRisk$^+$ models, are presented. The main objective of this section is to present how the default event is modeled. In Section 3 we introduce the notion of risk measures, define a coherent risk measure and discuss the use and misuse of risk measures. In Section 4 credit risk in Basel II is presented in a more technical form than in Section 1. In Section 5 we discuss different ways to estimate the add-on, with special emphasis on the granularity adjustment for the Vasicek and CreditRisk$^+$ models. We also study the granularity adjustment in respect to coherence and how the retail portfolio can be included in the computations. In Section 6 we examine different ways to allocate the add-on in order to optimize the portfolio. Results from when the models have been applied to real data are presented in Section 7 and then discussed in Section 8.

2 Credit Risk Models

Credit risk models can be either *dynamic* or *static*. Dynamic credit risk models are used when the particular time of default is important, e.g., in the pricing of different kinds of credit derivatives. In static credit risk models the total loss over some time horizon due to defaults in the credit portfolio is considered. A typical time horizon in practice is one year. In this thesis only static credit risk models will be considered but many results carry over to dynamic credit risk models as well. Below we introduce some commonly used notation.

$\mathbf{EAD}_i$ The *random exposure at default* ($\widetilde{\mathrm{EAD}}_i$) at some future time point is defined as

$$\widetilde{\mathrm{EAD}}_i = \mathrm{OUT}_i + \mathrm{CCF}_i \cdot \mathrm{COMM}_i,$$

where the *current outstanding amount* ($\mathrm{OUT}_i \geq 0$) and the *commitment* ($\mathrm{COMM}_i \geq 0$), i.e., the maximum amount that can be drawn by the obligor, are known quantities. A revolving loan is an example where $\mathrm{COMM}_i > 0$ and a term loan is an example where $\mathrm{COMM}_i = 0$. The *credit conversion factor* ($0 \leq \mathrm{CCF}_i \leq 1$) is random. Note that these definitions imply that $\widetilde{\mathrm{EAD}}_i \geq 0$.

The *(expected) exposure at default* (EAD_i) is defined as

$$\mathrm{EAD}_i = \mathrm{E}\left[\widetilde{\mathrm{EAD}}_i\right] = \mathrm{OUT}_i + \mathrm{E}\left[\mathrm{CCF}_i\right] \cdot \mathrm{COMM}_i.$$

$\mathbf{LGD}_i$ The *loss given default*[4] ($\mathrm{LGD}_i \geq 0$) over some specified time horizon is the

[4]In some texts the *recovery rate* (RR_i) is used instead and is defined as $\mathrm{RR}_i = 1 - \mathrm{LGD}_i$.

fraction of EAD that isn't recovered in the event of default. LGD_i is random and may be greater than one due to legal and other costs, even though this is uncommon.

$\boldsymbol{\mathrm{PD}_i}$ The *probability of default* ($0 \leq \mathrm{PD}_i \leq 1$) over some specified time horizon can be estimated by various methods. There exist two kinds of PD_i: *point-in-time* (PIT) and *through-the-cycle*[5] (TTC). PIT PD_i is the probability of default over some specified time horizon given the state of the economy today (i.e., PD_i is estimated using all available information). TTC PD_i is the average probability of default over a business cycle, i.e., the probability of default over some specified time horizon where the state of the economy is unknown. It would be natural to use PIT PD_i if it would be known since this is the PD_i under the "true" probability measure in the sense that it includes all available information. However, since PIT PD_i isn't known, the reliability of PD_i also has to be taken into account when choosing measure.

$\boldsymbol{D_i}$ The *default variable* (D_i) over some specified time horizon is a random variable that models the default event. It is natural to let D_i be Bernoulli distributed and let D_i take the value one if obligor i defaults within the specified time horizon. In this case the distribution of D_i is

$$D_i = \begin{cases} 1, & \text{w.p. } \mathrm{PD}_i, \\ 0, & \text{w.p. } 1 - \mathrm{PD}_i, \end{cases} \tag{2.1}$$

where PD_i usually is assumed to be known.

$\boldsymbol{L}$ The *loss variable of obligor i* ($L_i \geq 0$) over some specified time horizon is defined as the outstanding amount that will not be retrieved due to default within considered time period. The *loss variable of a credit portfolio with n obligors* (L) is defined as $L = \sum_{i=1}^{n} L_i$ and an outcome of a loss variable is referred to as a *loss*.

$\boldsymbol{Z}$ The vector of random variables, $\big(\boldsymbol{Z} = (Z_1, \ldots, Z_m)\big)$, is said to be a *vector of risk factors* if the random variables $D_1 \mid \boldsymbol{Z} = \boldsymbol{z}, \ldots, D_n \mid \boldsymbol{Z} = \boldsymbol{z}$, $\mathrm{LGD}_1 \mid \boldsymbol{Z} = \boldsymbol{z}, \ldots, \mathrm{LGD}_n \mid \boldsymbol{Z} = \boldsymbol{z}$ are mutually independent. In practice, risk factors typically represent macroeconomic variables, industries or geographical regions. The rationale is that D_i depends on the state of the economy and that companies in the same sector often are more dependent.

The risk associated with the dependence between obligors is called *systematic risk* and the risk that is associated with individual obligors is called *idiosyncratic risk*. Here, we make an exact definition by defining the systematic risk of a portfolio as $\mathrm{E}[L \mid \boldsymbol{Z}]$ and the idiosyncratic risk as $L - \mathrm{E}[L \mid \boldsymbol{Z}]$. Also, we define the *a priori distribution* of L as the distribution of L and the *a posteriori distribution* of L as the distribution of $L \mid \boldsymbol{Z} = \boldsymbol{z}$.

[5] TTC PD_i is sometimes also called *average* PD_i.

$\mathbf{ELGD}_i$ The *expected loss given default* (ELGD_i) is defined as $\mathrm{ELGD}_i = \mathrm{E}[\mathrm{LGD}_i]$. In some texts LGD_i is used to denote ELGD_i. This notation will, however, not be used here.

$\mathbf{VLGD}_i$ The *variance of the loss given default* (VLGD_i) is defined as $\mathrm{VLGD}_i = \mathrm{V}(\mathrm{LGD}_i)$.[6] There is no regulatory demand for banks to estimate VLGD_i. However, a method proposed in BCBS 2001, § 447, is

$$\mathrm{VLGD}_i = 0.25 \cdot \mathrm{ELGD}_i(1 - \mathrm{ELGD}_i).$$

This estimation is sometimes also used for regulatory purposes and in industry models. A great benefit is that it doesn't require any additional workload to use.

2.1 The Loss Variable

There exist two principal approaches to assess the distribution of the loss variable: Monte Carlo simulations and analytical methods. The main advantage of analytical methods is that they aren't at all as time-consuming as Monte Carlo methods, for which the loss distribution may take days or even weeks to simulate. In analytical models, however, additional assumptions often have to be made in order to achieve a closed-form solution. In many applications the following assumption is used to reach a model that is simple enough.

Assumption 2.1. Unless otherwise stated, we will assume that $\widetilde{\mathrm{EAD}}_i = \mathrm{EAD}_i$, that there exists a vector of risk factors $\boldsymbol{Z}$ and that $\mathrm{LGD}_1, \dots, \mathrm{LGD}_n$ are mutually independent and independent of $D_1, \dots, D_n$ and $\boldsymbol{Z}$.

Especially the assumption that $\mathrm{LGD}_1, \dots, \mathrm{LGD}_n$ are mutually independent and independent of $D_1, \dots, D_n$ and $\boldsymbol{Z}$ is not entirely realistic. However, using Assumption 2.1 we get that

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot D_i, \tag{2.2}$$

$$L \,|\, \boldsymbol{Z} = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot (D_i \,|\, \boldsymbol{Z}). \tag{2.3}$$

Even if software solutions used by banks differ from each other in many aspects the main modeling difference is how the distributions of $\boldsymbol{Z}$ and $D_i \,|\, \boldsymbol{Z}$ are defined. The a priori distribution is then fully determined by the a posteriori distribution and the law of total probability. Depending on how $\boldsymbol{Z}$ and $D_i \,|\, \boldsymbol{Z}$ are defined, a credit risk model may either

[6] In some texts VLGD_i denotes the standard deviation of LGD_i. This notation will, however, not be used here.

be a *structural model*[7] or a *reduced-form model*[8]. In structural models the mechanism of default is based on the Merton model (Merton 1974), where corporate debt is priced by modeling it as a European put option on the asset value of the firm with EAD_i as the strike price and using the results of Black and Scholes (1973). The Merton model can also be used to estimate PDs of firms. In reduced-form models the actual mechanism causing the default isn't modeled directly. Instead $\boldsymbol{Z}$ and $D_i \,|\, \boldsymbol{Z}$ are modeled to make a good fit to historical data and to provide mathematical tractability. Two examples of widely used reduced-form models are the CreditRisk$^+$ model and the Credit Portfolio View model and two examples of widely used structural models are the MKMV model and the CreditMetrics model. One could also say that the difference between these models is that the information used to asses the distribution of the loss variable for reduced-form models is on a more macroscopic level than for structural models. In general, it is difficult to say what detail level that is preferable. On the one hand, the more microscopic level that is applied the more information is used, on the other hand it might happen that we can't see the wood for the trees and it's better to get the big picture than to be exactly wrong. In the two next sections, we will take a closer look at the CreditRisk$^+$ model and the Vasicek model, of which the latter underpins both the MKMV model and the IRB formula.

2.2 The Multi-Factor Vasicek Model

Vasicek (1987) turned the Merton model upside down and used it to model the dependence of default events instead of pricing corporate debt or estimating PDs, as was done by Merton (1974). In the Vasicek model, we let $\lambda_i \geq 0$ denote the value of the liabilities of obligor i, which is a known quantity. The asset value of obligor i at time t, $V_{i,t}$, is modeled as a multivariate geometric Brownian motion, i.e.,

$$\mathrm{d}V_{i,t} = \mu_i V_{i,t}\,\mathrm{d}t + V_{i,t}\sum_{k=1}^{m}\sigma_{i,k}\,\mathrm{d}W_{k,t} + \eta_i V_{i,t}\,\mathrm{d}B_{i,t}, \tag{2.4}$$

where $\mu_i, \sigma_{i,1}, .., \sigma_{i,m}, \eta_i$ are constants and $W_{1,t}, .., W_{m,t}, B_{i,t}$ are mutually independent Wiener processes. Of course, μ_i, $\sigma_{i,1}, .., \sigma_{i,n}$ are not known a priori, but have to be estimated. The Wiener processes $W_{1,t}, \ldots, W_{n,t}$ may be shared among obligors and often represent different macroeconomical variables. In practice, these are typically shared to a greater extent by companies in the same region or industry. The Wiener process $B_{i,t}$, however, is only associated with obligor i and is not shared with other obligors. Thus,

[7] Structural models are also known as *firm-value models*, *asset-value models*, *latent variable models* and *threshold models*.

[8] Reduced-form models are also known as *default-rate models* and *mixture models*.

$W_{1,t},..,W_{m,t}$ are associated with systematic risk and $B_{i,t}$ with idiosyncratic risk.

It isn't obvious to use mutually independent Wiener processes, e.g., if one Wiener process represents the economical development in Sweden and another the economical development in Norway, then it seems reasonable that they are correlated. However, if $W_{1,t},..,W_{m,t}$ in (2.4) were correlated it would be possible to rewrite the expression with independent Wiener processes on the same form by changing $\sigma_{i,1},\dots,\sigma_{i,m}$ for $i=1,\dots,n$ (see Björk 2009, Section 4.7). We can also write (2.4) on the form (see Björk 2009, Proposition 5.2)

$$V_{i,1} = V_{i,0}\exp\left(\mu_i + \sum_{k=1}^{m}(\sigma_{i,k}Z_k - \frac{1}{2}\sigma_{i,k}^2) + \eta_i\epsilon_i - \frac{1}{2}\eta_i^2\right),$$

where $Z_{i,1},\dots,Z_{i,n}$ and ϵ_i are i.i.d. and $\mathcal{N}(0,1)$ distributed. In the Vasicek model it is assumed that D_i is Bernoulli distributed where a default event, i.e., $D_i = 1$ occurs if $V_{i,1} < \lambda_i$. Thus, we get that

$$\begin{aligned}
\mathrm{PD}_i &= \mathrm{P}\left(V_{i,0}\exp\left(\mu_i + \sum_{k=1}^{m}(\sigma_{i,k}Z_k - \frac{1}{2}\sigma_{i,k}^2) + \eta_i\epsilon_i - \frac{1}{2}\eta_i^2\right) \; < \; \lambda_i\right) \\
&= \mathrm{P}\left(\sigma_{i,1}Z_1 + \dots + \sigma_{i,m}Z_m + \eta_i\epsilon_i \le \ln\frac{\lambda_i}{V_{i,0}} + \frac{1}{2}\sum_{k=1}^{m}\sigma_{i,k}^2 + \frac{1}{2}\eta_i^2 - \mu_i\right) \\
&= \mathrm{P}\left(\frac{\sigma_{i,1}Z_1 + \dots + \sigma_{i,m}Z_m + \eta_i\epsilon_i}{\sqrt{\sigma_{i,1}^2 + \dots + \sigma_{i,m}^2 + \eta_i^2}} \le \frac{\ln\frac{\lambda_i}{V_{i,0}} + \frac{1}{2}\sum_{k=1}^{m}\sigma_{i,k}^2 + \frac{1}{2}\eta_i^2 - \mu_i}{\sqrt{\sigma_{i,1}^2 + \dots + \sigma_{i,m}^2 + \eta_i^2}}\right) \\
&= \Phi\left(\frac{\ln\frac{\lambda_i}{V_{i,0}} + \frac{1}{2}\sum_{k=1}^{m}\sigma_{i,k}^2 - \mu_i}{\sqrt{\sigma_{i,1}^2 + \dots + \sigma_{i,m}^2 + \eta_i^2}}\right).
\end{aligned}$$

From this we infer that obligor i defaults if

$$\frac{\sigma_{i,1}Z_1 + \dots + \sigma_{i,m}Z_m + \eta_i\epsilon_i}{\sqrt{\sigma_{i,1}^2 + \dots + \sigma_{i,m}^2 + \eta_i^2}} \le \frac{\ln\frac{\lambda_i}{V_{i,0}} + \frac{1}{2}\sum_{k=1}^{m}\sigma_{i,k}^2 + \frac{1}{2}\eta_i^2 - \mu_i}{\sqrt{\sigma_{i,1}^2 + \dots + \sigma_{i,m}^2 + \eta_i^2}}.$$

This condition can be rewritten in a way that is more suitable for our purposes. If we use the notation

$$\rho_i = \frac{\sigma_{i,1}^2 + \dots + \sigma_{i,m}^2}{\sigma_{i,1}^2 + \dots + \sigma_{i,m}^2 + \eta_i^2} \quad \text{and} \quad \alpha_{i,k} = \frac{\sigma_{i,k}}{\sqrt{\sigma_{i,1}^2 + \dots + \sigma_{i,m}^2}}$$

we get that

$$\frac{\sigma_{i,1}Z_1+\cdots+\sigma_{i,m}Z_m+\eta_i\epsilon_i}{\sqrt{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2+\eta_i^2}}=\frac{\sigma_{i,1}Z_1+\cdots+\sigma_{i,m}Z_m}{\sqrt{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2+\eta_i^2}}+\frac{\eta_i\epsilon_i}{\sqrt{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2+\eta_i^2}}$$

$$=\sqrt{\rho_i}\cdot\sqrt{\frac{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2+\eta_i^2}{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2}}\cdot\frac{\sigma_{i,1}Z_1+\cdots+\sigma_{i,m}Z_m}{\sqrt{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2+\eta_i^2}}+\sqrt{1-\rho_i}\cdot\epsilon_i$$

$$=\sqrt{\rho_i}\cdot\left(\frac{\sigma_{i,1}}{\sqrt{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2}}\cdot Z_1+\cdots+\frac{\sigma_{m,1}}{\sqrt{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2}}\cdot Z_m\right)+\sqrt{1-\rho_i}\cdot\epsilon_i$$

$$=\sqrt{\rho_i}\,(\alpha_{i,1}Z_1+\cdots+\alpha_{i,m}Z_m)+\sqrt{1-\rho_i}\epsilon_i=\sqrt{\rho_i}\,\boldsymbol{\alpha}_i^\top\boldsymbol{Z}+\sqrt{1-\rho_i}\,\epsilon_i,$$

where we have used the notation $\boldsymbol{\alpha}_i=(\alpha_{i,1},\dots,\alpha_{i,m})^\top$ and $\boldsymbol{Z}=(Z_1,\dots,Z_m)^\top$. Since

$$\frac{\ln\frac{\lambda_i}{V_{i,0}}+\frac{1}{2}\sum_{k=1}^m\sigma_{i,k}^2+\frac{1}{2}\eta_i^2-\mu_i}{\sqrt{\sigma_{i,1}^2+\cdots+\sigma_{i,m}^2+\eta_i^2}}=\Phi^{-1}(\mathrm{PD}_i),$$

we have that

$$D_i=\begin{cases}1, & \text{if}\quad \sqrt{\rho_i}\,\boldsymbol{\alpha}_i^\top\boldsymbol{Z}+\sqrt{1-\rho_i}\,\epsilon_i\le\Phi^{-1}(\mathrm{PD}_i),\\ 0, & \text{if}\quad \sqrt{\rho_i}\,\boldsymbol{\alpha}_i^\top\boldsymbol{Z}+\sqrt{1-\rho_i}\,\epsilon_i>\Phi^{-1}(\mathrm{PD}_i).\end{cases}$$

Whence it immediately follows that

$$\rho_i\in[0,1],\qquad\sum_{k=1}^m\alpha_{i,k}^2=1,\qquad\sqrt{\rho_i}\boldsymbol{\alpha}_i^\top\boldsymbol{Z}+\sqrt{1-\rho_i}\epsilon_i\sim\mathcal{N}(0,1).$$

We also notice that $D_i|\boldsymbol{Z}=\boldsymbol{z},\dots,D_n|\boldsymbol{Z}=\boldsymbol{z}$ are mutually independent. Hence, $\boldsymbol{Z}$ is a vector of risk factors. The a posteriori distribution is

$$\begin{aligned}&\mathrm{P}\left(D_i=1\,|\,\boldsymbol{Z}=\boldsymbol{z}\right)=\mathrm{P}\left(\sqrt{\rho_i}\,\boldsymbol{\alpha}_i^\top\boldsymbol{Z}+\sqrt{1-\rho_i}\,\epsilon_i\le\Phi^{-1}(\mathrm{PD}_i)\,|\,\boldsymbol{Z}=\boldsymbol{z}\right)\\&=\mathrm{P}\left(\sqrt{\rho_i}\,\boldsymbol{\alpha}_i^\top\boldsymbol{z}+\sqrt{1-\rho_i}\,\epsilon_i\le\Phi^{-1}(\mathrm{PD}_i)\right)=\mathrm{P}\left(\epsilon_i\le\frac{\Phi^{-1}(\mathrm{PD}_i)-\sqrt{\rho_i}\,\boldsymbol{\alpha}_i^\top\boldsymbol{z}}{\sqrt{1-\rho_i}}\right)\\&=\Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_i)-\sqrt{\rho_i}\,\boldsymbol{\alpha}_i^\top\boldsymbol{z}}{\sqrt{1-\rho_i}}\right).\end{aligned}\tag{2.5}$$

Formula 2.5 is sometimes also known as the *Vasicek formula*. If we let $\boldsymbol{Z}=Z$ we get the *single-factor* Vasicek model.

2.3 The Multi–Factor CreditRisk$^+$ Model

In CreditRisk$^+$ the actual mechanism causing the default isn't modeled directly. Instead the model has been chosen to provide a good fit to data and mathematical tractability.

Risk Factors. The risk factors are defined as independent random variables with $Z_j \sim \Gamma(1/\sigma_j^2, \sigma_j^2)$ for $j = 1, \ldots, m$ which implies that $\mathrm{E}[Z_j] = 1$ and $\mathrm{V}(Z_j) = \sigma_j^2$.

A Posteriori Distribution. Let $w_{i,k} \geq 0$ for $k = 0, \ldots, m$, $\boldsymbol{w_i} = (w_{i,1}, \ldots, w_{i,m})$ and $\sum_{k=0}^{m} w_{i,k} = 1$. The a posteriori distribution is then defined as

$$\mathrm{P}(D_i = k \,|\, \boldsymbol{Z} = \boldsymbol{z}) = \frac{\big(\mathrm{PD}_i(w_{i,0} + \boldsymbol{w}_i^\top \boldsymbol{z})\big)^k}{k!} \cdot \exp\Big[- \big(\mathrm{PD}_i(w_{i,0} + \boldsymbol{w}_i^\top \boldsymbol{z})\big)\Big], \tag{2.6}$$

where $k \in \{0, 1, 2, ...\}$. This is equivalent to $D_i \,|\, \boldsymbol{Z} = \boldsymbol{z} \sim \mathrm{Po}\,\big(\mathrm{PD}_i(w_{i,0} + \boldsymbol{w}_i^\top \boldsymbol{z})\big)$.

The a posteriori distribution leads to the unnatural economical interpretation that an obligor may default several times. This simplification, however, is a good approximation of the Bernoulli distribution if PD_i is small (see Lütkebohmert 2008, Section 6.2).

If we suppress the risk factor index, then for the single-factor setting we get that $w_{i,0} = 1 - w_i$ and $D_i \,|\, Z = z \sim \mathrm{Po}\,\big(\mathrm{PD}_i(1 - w_i + w_i z)\big)$.

2.4 Expected Loss

The *expected loss* (EL) is defined as $\mathrm{EL} = \mathrm{E}[L]$. Thus, the expected loss is

$$\mathrm{EL} = \sum_{i=1}^{n} \mathrm{E}\big[\mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot D_i\big] = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{E}\big[D_i\big].$$

In the multi-factor Vasicek model we have that $D_i \sim \mathrm{Be}(\mathrm{PD}_i)$, which gives that $\mathrm{E}[D_i] = \mathrm{PD}_i$. By the law of iterated expectations and that, if $X \sim \mathrm{Po}(m)$ then $\mathrm{E}[X] = m$, we get that $\mathrm{E}[D_i] = \mathrm{E}[\mathrm{E}[D_i \,|\, \boldsymbol{Z}] = \mathrm{E}[\mathrm{PD}_i(w_{i,0} + \boldsymbol{w}_i^\top \boldsymbol{z})] = \mathrm{PD}_i$ in the CreditRisk$^+$ model. Thus, for both the multi-factor Vasicek model and the CreditRisk$^+$ model we have that the expected loss is

$$\mathrm{EL} = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{PD}_i.$$

Notice that the expected loss of a portfolio is portfolio invariant and can therefore be computed by a bottom-up approach, which significantly eases the computations for banks. Also, what matters more is that banks don't have to aggregate the exposures to obligors in order to compute the expected loss.

2.5 Market-to-Market and Default Mode

Two credits, whose loss variables are identically distributed, may in fact have different net present values. The explanation is that even if two credits have identically distributed loss variables, the credit with a longer time to maturity has a greater risk of defaulting at some point in time, not necessarily within the time horizon that is specified for the loss variables. This fact is reflected by the concave shape of the yield curve, even though it isn't the only explanation of the shape. The effect of the time to maturity on the net present value is accentuated for loans with low PD.

This leads to an alternative definition of the loss variable. Instead of defining the loss variable as on page 12, where an obligor either defaults or not, we may define the loss variable as the net present value of the losses at the end of the time interval considered. If the latter definition is used, the loss variable is said to be in *market-to-market mode* (MtM mode) and if the definition on page 12 is used, the loss variable is said to be in *default mode*. Note that the loss can be negative in MtM mode due to credit migration, whereas this is impossible in default mode.

There are several reasons why to prefer MtM mode to default mode when measuring risk. A common argument is that since credits with longer maturities are riskier, this should also be reflected in the distribution of the loss variable (see BCBS 2005, Section 4.6). Another argument is that MtM mode is more likely lead to smooth changes in the risk measured over time. But there are also reasons why to prefer default mode to MtM mode. If a bank wishes to sustain a certain credit quality for a given time period, it is the default mode for that time period it ought to consider. Another deficiency of the MtM mode can be illustrated by an example from the insurance industry. Using MtM mode would then entail that the risk of selling life insurances to people in their 20s would be considered the same as selling life insurances to people in their 80s. For a bank to maximize its profit, which includes sustaining a desirable credit quality, it should consider the loss variable in default mode for different time horizons.

3 Risk Measures

All information associated with the risk of a portfolio w.r.t. some future time point is contained in its loss distribution. From this point of view, it seems natural to compare the risk of two different portfolios by comparing their loss distributions. However, due to two reasons this is not feasible. First, we don't know the loss distribution, we can only estimate it. Second, when considering risk we are usually not interested in all information contained in the loss distribution, e.g., typically we don't consider negative losses at all.

If we instead use a risk measure, i.e., a function $\rho : \mathbb{X} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$, where $\mathbb{X}$ is a set of random variables over a fixed probability space, we get around these obstacles. Two examples of risk measures are the standard deviation and the expected loss.

When considering different portfolios for an investor, usually the portfolio with the highest expected utility is chosen. To compute the expected utility of a portfolio its distribution and the utility function of the investor have to be known. The problem is then solved by return/risk optimization. However, on many occasions we want to maximize the return given a target level of risk. From the reasoning about the risk appetite we infer that this is also the case for banks. We recapitulate the two main drivers for the risk appetite of banks:

- To meet regulatory demands.
- To achieve the credit rating that maximizes profits (borrowing/lending optimization).

In the case when there is a target level of risk it is natural to let the risk measure denote the amount of cash that has to be added to the portfolio in order to meet the risk target. This means that instead of return/risk optimization, only return optimization is to be performed. The risk is then accounted for implicitly, as a cost in form of the amount of cash that has to be added to the portfolio. However, we are still faced by the problem of how to define risk. Since the bank is interested in how its credit quality is perceived by supervisory authorities and credit rating agencies, one way is to study how they define risk. An intuitive way of defining the credit quality of a company would be to map the estimated figure $\mathrm{E}\left[\mathrm{PD}_i \cdot \mathrm{LGD}_i\right]$ to a credit rating. In reality, however, different credit rating institutes use different estimations and information. However, it seems reasonable that the credit quality of a company should depend both on PD_i and ELGD_i, but if there is a strong dependence between PD_i and ELGD_i, it might be sufficient to estimate only PD_i. From a regulatory point of view it is, as concluded above, of utmost importance for the economy to ensure a low risk of bank failure, i.e., to impose a limit on PD_i for banks. But the value of ELGD_i is also of significance for regulators. After all, in the event of a bank failure the taxpayers will pay for the bailout. Also, there are ways that banks can increase their profit in which PD_i remains unaltered but ELGD_i increases. Some common risk measures are presented next.

Value at Risk (VaR). The risk measure *Value at Risk* of X at a specified future time point and at confidence level $q \in (0, 1)$ equals the minimum amount of cash that needs to be added to the portfolio today in order to achieve a PD_i that is less or equal to q over the specified time horizon. Or, in other words, here we define VaR as

$$\mathrm{VaR}_q(X) = \inf\{x \in \mathbb{R} : \mathrm{P}(X > x) \le 1 - q\}.$$

If F_X is continuous and strictly increasing we have that $\mathrm{VaR}_q(X) = F_X^{-1}(q)$.

Unexpected Loss[9] **(UL).** The *unexpected loss* of X at a specified future time point and at confidence level $q \in (0, 1)$ is defined as

$$\mathrm{UL}_q(X) = \mathrm{VaR}_q(X) - \mathrm{E}[X].$$

An interpretation is how much more capital than the expected loss that need to be added to the portfolio in order to sustain a certain level of PD_i.

Expected Shortfall (ES). The risk measure *expected shortfall* at a specified time point and at confidence level $q \in (0, 1)$ is defined as

$$\mathrm{ES}_q(X) = \frac{1}{1-q}\int_q^1 \mathrm{VaR}_u(X)\,\mathrm{d}u.$$

If X is integrable, F_L is continuous and $q \in (0, 1)$ we also have that

$$\mathrm{ES}_q(X) = \mathrm{E}[X \mid X \ge \mathrm{VaR}_q(X)],$$

which means that then the expected shortfall of a portfolio with confidence level q is the same thing as the expected value of X conditional on the event that the outcome is greater or equal to $\mathrm{VaR}_q(X)$ (McNeil et al. 2005, Lemma 2.16).

Thus, if we let X be a random variable with continuous and strictly increasing distribution function that represents the value of the equity of a company and if default occurs when $Y = X - \lambda \le 0$, where $\lambda \ge 0$ is the value of the liabilities of the company, we get that $\mathrm{PD} = F_Y(0)$, $\mathrm{ELGD} = \mathrm{ES}_{F_Y(0)}(-Y)/\lambda$ and $\mathrm{EL} = \mathrm{EAD} \cdot F_Y(0) \cdot \mathrm{ES}_{F_Y(0)}(-Y)/\lambda$. Also, if we add $\mathrm{VaR}_q(-Y)$ in cash to the equity of the company, we get that $\mathrm{PD} = 1 - q$, $\mathrm{ELGD} = \mathrm{ES}_q(-Y)/\lambda$ and $\mathrm{EL} = \mathrm{EAD}\cdot(1-q)\cdot\mathrm{ES}_q(-Y)/\lambda$. These relations are only meant to illustrate the connection between credit quality and risk measures. Reality, of course, is much more complex than in this setting, but hopefully these identities nevertheless can serve instructive purposes.

To sum up, to meet its target PD a bank should, for an appropriate confidence level, calculate $\mathrm{VaR}_q(L)$. To meet its target ELGD it should also, for an appropriate confidence level, calculate $\mathrm{ES}_q(L)$. It may very well happen that the amount of cash to be added in

[9] In some texts UL also refers to other mathematical entities, such as the standard deviation of the loss variable or an outcome of a loss variable that exceeds the expected loss. Sometimes mean-VaR ($\mathrm{VaR}_q^{\mathrm{mean}}$) or *economic capital* (EC) are also used to denote UL, as defined here.

order to meet the demand on PD doesn't yield the same result as the computation of the amount of cash necessary to meet the demand on ELGD. However, this may be resolved by adding other forms of capital than cash to the portfolio. A property of risk measure that often is desirable is *coherence*.

Coherent Risk Measures. The risk measure ρ is *coherent* if it satisfies the following conditions, where X and Y are random variables

Translation invariance: $\rho(X+\lambda) = \rho(X) - \lambda$ for all $\lambda \in \mathbb{R}$.
If ρ measures the amount of cash that is necessary to add to the portfolio in order to attain the desirable risk level then, of course, if we add the amount λ of cash to the portfolio the capital requirement should decrease by the same amount.

Monotonicity: If $X \leq Y$ almost surely then $\rho(X) \leq \rho(Y)$.
If the loss of one portfolio almost surely is greater or equal to another, then that portfolio should also have a capital buffer that is greater or equal to that of the other.

Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
The economical interpretation of this is that a well diversified portfolio needs a smaller capital buffer than a not well diversified portfolio.

Positive homogeneity[10]: For every $\lambda \in \mathbb{R}$ we have that $\rho(\lambda X) = \lambda\rho(X)$.
If we increase the amount invested λ times, the capital buffer should increase by the same amount.

It is possible to construct examples where VaR violates the subadditivity property (Hult et al. 2012, pp.176–178), and hence it is not a coherent risk measure. ES, on the contrary, is a coherent risk measure (Hult et al. 2012, Proposition 6.6). Conceptually, this is because VaR doesn't consider the shape of the tail (for a lucid example, see Hull 2009, pp.451–452), whereas ES does. One could also say that VaR takes PD into account but not ELGD. The lack of the subadditivity property has lead to criticism against VaR. However, it should be noted that VaR is subadditive and coherent in some settings.

Of course, in practice we can only estimate VaR and ES. Therefore it is essential that an analysis of the estimation error is made, e.g., by resampling methods (e.g., bootstrap or jackknife), by considering the dependence structure (e.g., using different copulas) or/and by considering the shape of the tail(s). Thus, the analysis of the estimation error isn't simply to generate a confidence interval but also a qualitative consideration of the *model risk*. Often, it is more difficult to estimate ES than VaR since the statistical error usually is larger for ES. This is however not an argument against the use of ES, it merely says

[10]Sometimes, to reflect *liquidity risk* the property $\rho(\lambda X) > \lambda\rho(X)$ is preferred to positive homogeneity. However, in credit risk we are interested in the cash flows from the obligors, not the market value of the contracts.

that it is difficult to predict large losses. Also VaR and ES are increasingly more difficult to asses as q increases.

Risk measures of different portfolios are often estimated by historical data (this is however not always the case since forward looking data such as implied volatility and bond market prices may be included in the estimations). In practice, however, there are often conceivable events not reflected in historical data, which can be resolved by scenario simulation.

4 The Internal Ratings Based Approach

Banks may use either the Standardized Approach or the IRB approach to compute regulatory capital. To use the IRB approach, banks first have to apply for this at the supervisory authorities. This explains why this approach is more common among large banks. The IRB formula estimates the unexpected loss of the loss portfolio with a one-year horizon and is based on the Vasicek model in MtM mode. To make computations sufficiently tractable for regulatory purposes two assumptions have been made: there is only one risk factor and the portfolio is infinitely fine grained.

4.1 The Asymptotic Single Risk Factor Model

In order for the computation of regulatory capital for credit risk to be sufficiently tractable, portfolio invariance (see Section 1.4) is preferred. This has been resolved by applying the *asymptotic single risk factor* (ASRF) model (BCBS 2005, p.4) that was developed by Gordy (2003). The ASRF framework is based on the following two assumptions:

Assumption 4.1. The credit portfolio is infinitely fine-grained.

Assumption 4.2. There is only one risk factor, Z, and for that risk factor $\mathrm{E}(L \mid Z = z)$ is continuously and strictly monotonously increasing or decreasing in z.

Following the outline of Hibbeln (2010, p.36) an infinitely fine-grained credit portfolio can formally be defined as follows.

Definition 4.1. A credit portfolio is *infinitely fine-grained* if the the portfolio consists of a nearly infinite number of obligors and if the conditions

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{EAD}_i \longrightarrow \infty \quad \text{and} \quad \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{\mathrm{EAD}_j}{\sum_{i=1}^{j} \mathrm{EAD}_i} \right)^2 < \infty$$

are satisfied.

Hibbeln (2010, pp.50-52) shows that for an infinitely granular portfolio we have that

$$\mathrm{P}\left(\lim_{n\to\infty}(L^{(n)} - \mathrm{E}[L^{(n)}\,|\,\boldsymbol{Z}] = 0)\right) = 1,$$

where $L^{(n)} = L = \sum_{i=1}^{n} L_i$. As a result of this we have that (Gordy 2003, p.206)

$$\lim_{n\to\infty} \mathrm{VaR}_q(L^{(n)}) - \mathrm{VaR}_q\left(\mathrm{E}[L^{(n)}\,|\,\boldsymbol{Z}]\right) = 0.$$

If there only is one risk factor, Z, and $\mathrm{E}(L\,|\,Z = z)$ is continuously and strictly monotonously decreasing[11] in z, we have that (Hibbeln 2010, pp.53-54)

$$\mathrm{VaR}_q(\mathrm{E}[L\,|\,Z]) = \mathrm{E}[L\,|\,Z = \mathrm{VaR}_{1-q}(Z)].$$

Thus, if Assumption 4.1 and 4.2 are satisfied it follows that

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{VaR}_q(L^{(n)}) &= \lim_{n\to\infty} \mathrm{E}[L^{(n)}\,|Z = \mathrm{VaR}_{1-q}(Z)] \\
&= \lim_{n\to\infty} \mathrm{E}\left[\sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot D_i \;\middle|\; Z = \mathrm{VaR}_{1-q}(Z)\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{E}[D_i \mid Z = \mathrm{VaR}_{1-q}(Z)].
\end{aligned}$$

This leads to the following definition of $\mathrm{VaR}_q^{\mathrm{ASRF}}(L)$,

$$\mathrm{VaR}_q^{\mathrm{ASRF}}(L) = \mathrm{VaR}_q\big(\mathrm{E}[L|Z]\big) = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{E}[D_i \mid Z = \mathrm{VaR}_{1-q}(Z)].$$

From this we conclude that $\mathrm{VaR}_q^{\mathrm{ASRF}}$ is a portfolio invariant risk measure. As mentioned above, if the portfolio is invariant the summation over obligors may just as well be done over exposures. Another desirable property is that $\mathrm{VaR}_q^{\mathrm{ASRF}}$ is a coherent risk measure. One should, however, bear in mind that Assumption 4.1 and 4.2 rarely are realistic. In the sequel, we will however always assume that Assumption 4.2 is satisfied.

Since the expected loss doesn't depend on how obligors are assigned to the exposures of the portfolio (see section 2.4), the unexpected loss in the ASRF framework is defined as $\mathrm{UL}_q^{\mathrm{ASRF}}(L) = \mathrm{VaR}_q^{\mathrm{ASRF}}(L) - \mathrm{EL}$.

[11]If $\mathrm{E}(L\,|\,Z = z)$ would be continuously and strictly monotonously increasing in z we would instead get that $\mathrm{VaR}_q(\mathrm{E}[L\,|\,Z]) = \mathrm{E}[L\,|\,Z = \mathrm{VaR}_q(Z)]$, which is the case for the CreditRisk$^+$ model.

4.2 The IRB Formula

The IRB formula is basically a computation of the unexpected loss of the credit portfolio in MtM mode at a 99.9 % confidence level and on a one-year time horizon (see BCBS 2005). The model that underpins the IRB formula is based on the Vasicek and ASRF models. If we compute $\mathrm{UL}_{0.999}^{\mathrm{ASRF}}(L)$ for the Vasicek model in default mode we get

$$\begin{aligned} \mathrm{UL}_{0.999}^{\mathrm{ASRF}}(L) &= \mathrm{VaR}_{0.999}^{\mathrm{ASRF}}(L) - \mathrm{EL} \\ &= \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{E}\left[D_i \mid Z = \mathrm{VaR}_{0.001}(Z) \right] - \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{PD}_i \\ &= \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \left(\Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_i) + \sqrt{\rho_i}\,\Phi^{-1}(0.999)}{\sqrt{1-\rho_i}} \right) - \mathrm{PD}_i \right). \end{aligned} \tag{4.1}$$

The IRB formula differs from (4.1) in three ways: downturn LGDs are used instead of ELGDs, the formula is in MtM mode and is multiplied by a constant called the scaling factor.

DLGD_i The *downturn* LGD (DLGD_i) is the expected loss given default conditional on economical downturn conditions (BCBS 2005, p.5). Since the unrealistic assumption that the LGDs are independent of Z has been made, the use of DLGDs when computing $\mathrm{VaR}_q^{\mathrm{ASRF}}(L)$ instead of ELGDs can be seen as an ad hoc adjustment by choosing a conservative view on risk. Somewhat remarkably, however, DLGDs are used in the IRB formula to compute both $\mathrm{VaR}_q^{\mathrm{ASRF}}(L)$ and EL.

Maturity Adjustment In the IRB formula the loss variable of every exposure is multiplied by the *maturity adjustment* that is defined as

$$\frac{1 + (M_i - 2.5)b_i}{1 - 1.5b_i},$$

where $b_i = (0.11852 - 0.05478\ln(\mathrm{PD}_i))^2$ and M_i is the *effective maturity*, defined in BCBS 2006a, p.50. The purpose of the maturity adjustment is to map the formula to MtM mode and has been estimated by regression from observations provided by Monte Carlo simulations (BCBS 2005, p.10). The regression was done in such a way that the maturity adjustment is linear in M_i. The linearization is a rather coarse simplification since the exact maturity adjustment would be strongly concave (BCBS 2006b, p.12). We notice that if the effective maturity is one, the default mode and MtM mode coincides.

Scaling Factor In order to maintain the aggregate level of regulatory capital when Basel II was implemented, the *scaling factor* (1.06) was introduced in the IRB formula (BCBS 2006a, § 14).

Thus, the IRB formula is

$$\mathrm{UL}^{\mathrm{IRB}}(L) = \tag{4.2}$$
$$1.06 \cdot \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{DLGD}_i \left(\Phi \left(\frac{\Phi^{-1}(\mathrm{PD}_i) + \sqrt{\rho_i^{\mathrm{IRB}}}\, \Phi^{-1}(0.999)}{\sqrt{1-\rho_i^{\mathrm{IRB}}}} \right) - \mathrm{PD}_i \right) \cdot \frac{1 + (M_i - 2.5)b_i}{1 - 1.5 b_i}.$$

The parameter ρ_i^{IRB} says how much exposure i depends on the state of the economy, Z, and thus defines the dependence structure in the portfolio. Empirical data show that D_i is more dependent on Z if PD_i is large (BCBS 2005, p.12). This is reflected in the IRB formula where ρ_i^{IRB} is a function of PD_i and increases as PD_i decreases. The definition of ρ_i^{IRB} in the IRB formula differs whether it is computed for a corporate, sovereign or institutional exposure (C,S,I) or a retail exposure. The definition of ρ_i^{IRB} for corporate, sovereign or institutional (C,S,I) exposures that aren't small or medium-sized entities is

$$\rho_i^{(\mathrm{C,S,I})} = 0.12 \cdot \frac{1 - e^{-50 \cdot \mathrm{PD}_i}}{1 - e^{-50}} + 0.24 \left(1 - \frac{1 - e^{-50 \cdot \mathrm{PD}_i}}{1 - e^{-50}} \right). \tag{4.3}$$

The definitions of ρ_i^{IRB} for other kinds of exposures can be found in Hibbeln (2010, pp.41-42). From (4.3) we note that $0.12 < \rho_i^{(\mathrm{C,S,B})} < 0.24$.

The FIRB and AIRB approaches. The difference between the FIRB approach and the AIRB approach lies in which parameters that are provided by the supervisory authorities and which parameters that are estimated by the bank. The parameters in the IRB formula that have to be estimated are the PDs, DLGDs, CCFs, and the Ms. Among these parameters, all but the PDs are provided by supervisory authorities in the FIRB approach for non-retail exposures. In the AIRB approach, all parameters are estimated by the bank. However, for retail exposures there is no difference between the FIRB and AIRB approaches. In the IRB retail approach, the CCFs, PDs and the DLGDs are estimated by the bank. The effective maturity is however provided by supervisory authorities and is $M_i = 1$. Another interpretation of this is that the IRB formula for retail exposures is computed in default mode instead of MtM mode.

Two terms that often are used in connection to regulatory capital are *capital requirement* and *risk-weighted assets*. The *capital requirement*, K, is defined as the regulatory capital as a share of the total EAD of the portfolio and the risk-weighted assets (RWA) of a credit portfolio is defined as $\mathrm{RWA} = 12.5 \cdot K \cdot \sum_{i=1}^{n} \mathrm{EAD}_i$.

5 Name Concentration and Basel II

Regulatory capital has been calibrated for a well-diversified portfolio, typically associated with a large internationally active bank (BCBS 2005, p.4 and BCBS 2006b, p.19). Since the IRB formula has been derived in the ASRF framework it cannot correctly estimate the unexpected loss for banks that differ from this benchmark in respect to sector and name concentration. Thus, banks that differ from this benchmark should address this issue under pillar 2 (BCBS 2005, p.4). However, the benchmark is not well-defined since the data for which the IRB formula was calibrated aren't publicly available (Hibbeln 2010, p.184). This makes it difficult how to assess additional capital for sector concentration risk. For name concentration risk, however, the infinitely granular portfolio is usually used as benchmark (BCBS 2006b, p.16).

One should know that sector and name concentration risk are, in general, not independent. In order to derive an add-on for concentration risk Assumption 4.1 and 4.2 have to be relaxed simultaneously. This has been done by Pykhtin (2004) for the Vasicek model. However, often the assumption of one risk factor and the assumption of an infinitely fine-grained portfolio are relaxed separately, which gives rise to one add-on for sector concentration risk and one add-on for name concentration risk, that together sum up to the total add-on for concentration risk. This approximation seems to be widely used throughout the industry (see BCBS 2006b), e.g., the Swedish Financial Authority suggests a model to compute an add-on for name concentration risk where the assumption of infinite granularity is relaxed separately (Edlund 2009*b*, p.4). Name concentration risk will also be dealt with separately in this thesis and we will therefore in the sequel assume that there only is one risk factor. There are, in principle, three different ways one can approach name concentration risk under pillar 2:

- Monte Carlo simulations
- Regression models
- Model-based approximations

Large banks often make use of vendor software products to analyze credit risk. These are often based on Monte Carlo simulations with several risk factors and make no assumptions regarding the granularity of the portfolio. The Monte Carlo simulations by these software products may be very time consuming. A scenario may take days, or even weeks, to simulate. CreditRisk$^+$, however, does not use Monte Carlo simulations but on the other hand it only provides solutions in default mode. Another drawback of software products is that they are difficult to reconcile with the computations of regulatory capital for credit risk. Other drawbacks are that software products are expensive and that there is no clear way how to allocate the add-on capital.

Regression methods are often based on observations provided by Monte Carlo simulations. The Herfindahl-Hirschman index (HHI) is used in many common regression methods of concentration risk and is defined as

$$\mathrm{HHI}(L) = \sum_{i=1}^{n} s_i^2,$$

where $s_j = \mathrm{EAD}_j / \sum_{i=1}^{n} \mathrm{EAD}_i$. Compared to software products the greatest advantage of regression methods is computational speed. The greatest drawback is accuracy. It is difficult to say for what portfolios the regression method provides a good accuracy and therefore rather often has to be recalibrated. Model-based approximations don't require any software products and in some cases provide an accurate and fast way to compute an add-on for name concentration risk. In BCBS 2006b, p.10 we find the following text:

> The various methodologies, proposed by practitioners and researchers, for dealing with name concentration risk can be generally classified into those that are more ad hoc, based on heuristic measures of risk concentration, and those that are based on more rigorous models of risk. Model-based approaches are strictly preferable, as long as they are feasible to implement.

5.1 Model-Based Approximations

To assess the add-on, a natural starting point is to extend the model that underpins the IRB formula for granular portfolios. However, this is difficult for two reasons. First, the use of DLGDs, the scaling factor and the maturity adjustment make the derivation of the IRB formula opaque. It is difficult to extend the formula when we don't know the exact reasoning behind the calibration of the model. Second, even if the IRB formula would be extended this would not provide a solution on how to include credits to the portfolio for which the Standardized Approach has been applied.

Another way to approach this problem is to notice that the regulatory capital for obligor i is an estimation of $\mathrm{UL}_{0.999}^{\mathrm{ASRF}}(L_i)$ on a one-year time horizon in MtM mode (see BCBS 2005, p.4–5). Thus, we face the problem to estimate $\mathrm{UL}_{0.999}(L) - \mathrm{UL}_{0.999}^{\mathrm{ASRF}}(L)$, i.e., the add-on for name concentration risk. Notice that we are not confined to any particular credit risk model since we don't make any assumptions regarding the model that has been used to derive the IRB formula, only that it has been derived within the ASRF framework. Considering different model-based approximations there are a few things to bear in mind when making the decision of what particular method to use.

- Is the method sufficiently accurate?
- Does the method require more data than required to compute regulatory capital?
- Is the add-on a coherent risk measure?
- Does the method provide an answer in default or MtM mode?

By the accuracy of the method we mean how well the method approximates $\mathrm{UL}_{0.999}(L) - \mathrm{UL}_{0.999}^{\mathrm{ASRF}}(L)$ with respect to the particular credit risk model that is being used. As shown above, the IRB formula is coherent. However, if we leave the ASRF framework this is no longer necessarily the case. Even if we want to approximate $\mathrm{UL}_{0.999}(L)$ and this measure in general isn't coherent it still seems reasonable to choose a model in which $\mathrm{UL}_{0.999}(L)$ is a coherent risk measure.

The reason why default mode at all is considered is that MtM mode entails technical difficulties. There exist several different model-based approximations of which some of are presented in BCBS 2006b and Lütkebohmert (2008). Presumably, the most widely used model-based approximations in the banking industry are those based on the *granularity adjustment*, on which the focus will be in the remaining part of this thesis.

5.2 The Granularity Adjustment

The *granularity adjustment* (GA) for one-factor models is a model-based approximation of the error in the computation of $\mathrm{VaR}_q(L)$ due to Assumption 4.1 in the ASRF model[12], i.e., $\mathrm{GA}_q(L) \approx \mathrm{VaR}_q(L) - \mathrm{VaR}_q^{\mathrm{ASRF}}(L)$. It was first derived by Wilde (2001). Later the derivation was simplified by Martin and Wilde (2002) who used the results of Gourieroux et al. (2000). Pykhtin (2004) generalized the GA to a multi-factor setting. Here, we present the derivation of GA by Martin and Wilde (2002).

For $\varepsilon = 1$ we have

$$\mathrm{VaR}_q(L) - \mathrm{VaR}_q^{\mathrm{ASRF}}(L) = \mathrm{VaR}_q\big(\mathrm{E}[L \,|\, Z] + \varepsilon(L - \mathrm{E}[L \,|\, Z])\big) - \mathrm{VaR}_q(\mathrm{E}[L \,|\, Z)]). \quad (5.1)$$

The granularity adjustment is simply a second-order Taylor approximation of (5.1) around $\varepsilon = 0$.

[12] In this section it will be assumed that $\mathrm{E}[L \,|\, Z = z]$ is continuously and strictly monotonously decreasing in z. The derivation of the case when $\mathrm{E}[L \,|\, Z = z]$ is continuously and strictly monotonously increasing in z is completely analogous. The only difference in (5.8) and (5.9) is that we have $z = \mathrm{VaR}_q(Z)$ instead of $z = \mathrm{VaR}_{1-q}(Z)$.

This gives that

$$\mathrm{GA}_q(L) = \frac{\partial}{\partial\varepsilon}\mathrm{VaR}_q\big(\mathrm{E}[L\,|\,Z] + \varepsilon(L - \mathrm{E}[L\,|\,Z])\big)\big|_{\varepsilon=0} \\ + \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}\mathrm{VaR}_q\big(\mathrm{E}[L\,|\,Z] + \varepsilon(L - \mathrm{E}[L\,|\,Z])\big)\big|_{\varepsilon=0}. \tag{5.2}$$

Gourieroux et al. (2000) showed that if (X, Y) is a bivariate random vector with continuous joint distribution we have that

$$\frac{\partial \mathrm{VaR}_q(X+\varepsilon Y)}{\partial\varepsilon} = \mathrm{E}\big[Y \mid X + \varepsilon Y = \mathrm{VaR}_q(X+\varepsilon Y)\big],$$

$$\frac{\partial^2 \mathrm{VaR}_q(X+\varepsilon Y)}{\partial\varepsilon^2} = -\,\mathrm{V}\big(Y \mid X+\varepsilon Y = \mathrm{VaR}_q(X+\varepsilon Y)\big)\cdot\frac{\partial \ln f_{X+\varepsilon Y}(z)}{\partial z}\big(\mathrm{VaR}_q(X+\varepsilon Y)\big) \\ -\frac{\partial}{\partial z}\mathrm{V}(Y \mid X+\varepsilon Y = z)\bigg|_{z=\mathrm{VaR}_q(X+\varepsilon Y)}.$$

Thus, for $\varepsilon = 0$ we get that

$$\frac{\partial \mathrm{VaR}_q(X+\varepsilon Y)}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \mathrm{E}\big[Y \mid X = \mathrm{VaR}_q(X)\big],$$

$$\frac{\partial^2 \mathrm{VaR}_q(X+\varepsilon Y)}{\partial\varepsilon^2}\bigg|_{\varepsilon=0} = -\left[\mathrm{V}\big(Y|X=x\big)\cdot\frac{\partial \ln f_X(x)}{\partial x} + \frac{\partial}{\partial x}\mathrm{V}(Y|X=x)\right]_{x=\mathrm{VaR}_q(X)} \\ = -\frac{1}{f_X(x)}\frac{\partial}{\partial x}\Big(f_X(x)\mathrm{V}(Y \mid X = x)\Big)\bigg|_{x=\mathrm{VaR}_q(X)}.$$

If we let $X = \mathrm{E}[L\,|\,Z]$ and $Y = L - \mathrm{E}[L\,|\,Z]$ we get that

$$\frac{\partial}{\partial\varepsilon}\mathrm{VaR}_q\Big(\mathrm{E}[L\,|\,Z] + \varepsilon(L - \mathrm{E}[L\,|\,Z])\Big)\bigg|_{\varepsilon=0} = \mathrm{E}\left[L - \mathrm{E}[L|Z]\;\middle|\;\mathrm{E}[L|Z] = \mathrm{VaR}_q(\mathrm{E}[L|Z])\right] \\ = \mathrm{E}\left[L - \mathrm{E}[L|Z]\;\middle|\;Z = \mathrm{VaR}_{1-q}(Z)\right] \\ = \mathrm{E}\big[L\,|Z = \mathrm{VaR}_{1-q}(Z)\big] - \mathrm{E}\big[L\,|Z = \mathrm{VaR}_{1-q}(Z)\big] = 0,$$

where we have used that $\mathrm{E}[L|Z] = \mathrm{VaR}_q\big(\mathrm{E}[L|Z]\big) \iff Z = \mathrm{VaR}_{1-q}(Z)$, which follows from that $\mathrm{E}(L\,|\,Z = z)$ is continuous and strictly monotonously decreasing in z. This gives that

$$\mathrm{GA}_q(L) = -\frac{1}{2} \cdot \frac{1}{f_X(x)} \cdot \frac{\partial}{\partial x}\Big(f_X(x)\mathrm{V}(Y \,|\, X = x)\Big)\Big|_{x=\mathrm{VaR}_q(X)}. \tag{5.3}$$

Now if we make the change of variable $x(z) = \mathrm{E}[L \,|\, Z = z]$ we get that

$$x = \mathrm{VaR}_q(X) \implies z = \mathrm{VaR}_{1-q}(Z), \tag{5.4}$$

since $\mathrm{VaR}_q(X) = \mathrm{VaR}_q\big(\mathrm{E}[L \,|\, Z]\big) = \mathrm{E}\big[L \,|\, \mathrm{VaR}_{1-q}(Z)\big]$. We also get that

$$\begin{aligned}\mathrm{V}(Y \,|\, X = x) &= \mathrm{V}\Big(L - \mathrm{E}[L \,|\, Z] \,\Big|\, E[L \,|\, Z] = x\Big) = \mathrm{V}\Big(L \,\Big|\, \mathrm{E}[L \,|Z] = x\Big)\\ &= \mathrm{V}(L \,|\, Z = z).\end{aligned} \tag{5.5}$$

Furthermore, we have that $q = F_X\big(\mathrm{VaR}_q(X)\big) = F_X\big(\mathrm{VaR}_q(\mathrm{E}[L \,|\, Z])\big) = F_X\big(\mathrm{E}[L \,|\, Z = \mathrm{VaR}_{1-q}(Z)]\big)$ and $1 - q = F_Z\big(\mathrm{VaR}_{1-q}(Z)\big)$. If we let $z = \mathrm{VaR}_{1-q}(Z)$ we get that $F_x\big(\mathrm{E}[L \,|\, Z = z]\big) = 1 - F_Z(\mathrm{VaR}_{1-q}(Z)\big)$. Taking the derivative on both sides of this equation with respect to z, we get that

$$f_X(x) = -\frac{f_Z(z)}{\frac{\partial}{\partial z}\mathrm{E}[L \,|\, Z = z]}. \tag{5.6}$$

If we let $r(\cdot)$ be a differentiable function and use the chain rule on $r\big(x(z)\big)$ we get that

$$\frac{\partial}{\partial x} r(x) = \frac{1}{x'(z)} \cdot \frac{\partial}{\partial z}\mathrm{E}[L \,|\, Z = z]. \tag{5.7}$$

Using (5.4), (5.5), (5.6) and (5.7) we can write (5.3) on the form

$$\mathrm{GA}_q(L) = -\frac{1}{2f_Z(z)} \cdot \frac{\partial}{\partial z}\left(\frac{f_Z(z)\mathrm{V}[L|Z = z]}{\frac{\partial}{\partial z}\mathrm{E}[L|Z = z]}\right)\Bigg|_{z=\mathrm{VaR}_{1-q}(Z)}. \tag{5.8}$$

Formula (5.8) is the formula commonly known as the granularity adjustment[13]. Even though we have derived the granularity adjustment in a VaR setting instead of a UL setting this doesn't matter since we have that

$$\mathrm{UL}_q(L) - \mathrm{UL}_q^{\mathrm{ASRF}}(L) = \mathrm{VaR}_q(L) - \mathrm{EL} - \big(\mathrm{VaR}_q^{\mathrm{ASRF}}(L) - \mathrm{EL}\big) = \mathrm{VaR}_q(L) - \mathrm{VaR}_q^{\mathrm{ASRF}}(L).$$

To ease notation we rewrite (5.8) with the notation $f_Z(z) = f(z)$, $\mathrm{E}[L \,|\, Z = z] = g(z)$ and $\mathrm{V}[L \,|\, Z = z] = h(z)$. With this notation the granularity adjustment can be rewritten as

[13] Another definition of the granularity adjustment is the add-on for name concentration risk proposed in BCBS 2001, § 456. This definition will however not be used here.

$$
\begin{aligned}
\mathrm{GA}_q(L) =& -\frac{1}{2f(z)} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{f(z)h(z)}{g'(z)}\right)\Bigg|_{z=\mathrm{VaR}_{1-q}(Z)} \\
=& -\frac{1}{2f(z)}\left(\frac{1}{g'(z)}\frac{\mathrm{d}}{\mathrm{d}z}(f(z)h(z)) + f(z)h(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{g'(z)}\right)\right)\Bigg|_{z=\mathrm{VaR}_{1-q}(Z)} \\
=& -\frac{1}{2f(z)}\left(\frac{1}{g'(z)}(f'(z)h(z)+f(z)h'(z)) - f(z)h(z)\frac{g''(z)}{(g'(z))^2}\right)\Bigg|_{z=\mathrm{VaR}_{1-q}(Z)} \\
=& -\frac{1}{2}\left(\left(h(z)\frac{f'(z)}{f(z)} + h'(z)\right)\frac{1}{g'(z)} - h(z)\frac{g''(z)}{(g'(z))^2}\right)\Bigg|_{z=\mathrm{VaR}_{1-q}(Z)}. \qquad (5.9)
\end{aligned}
$$

In order to use the granularity adjustment in practice we have to impose a credit risk model. In the following two chapters we will study the granularity adjustment for the Vasicek and CreditRisk$^+$ model.

5.2.1 The Vasicek Model

In this section we consider the granularity adjustment in a single-factor setting for the Vasicek model, which for homogeneous portfolios was first derived by Pykhtin and Dev (2002). Here, we follow the derivation of Hibbeln (2010, Section 4.2.1.2) for heterogeneous portfolios. In the Vasicek model we have that $Z \sim \mathcal{N}(0,1)$ which gives that $f'(z) = -zf(z)$. Using this and (5.9) we get that

$$
\mathrm{GA}_q^{\text{Vasicek}}(L) = \frac{1}{2}\left(\frac{zh(z)-h'(z)}{g'(z)} + h(z)\frac{g''(z)}{(g'(z))^2}\right)\Bigg|_{z=\Phi^{-1}(1-q)}. \qquad (5.10)
$$

If we use (2.3), (2.5) and the notation $u_i(z) = \left(\frac{\Phi^{-1}(\mathrm{PD}_i)-\sqrt{\rho_i}\,z}{\sqrt{1-\rho_i}}\right)$, we get that

$$
\begin{aligned}
g(z) = \mathrm{E}[L \,|\, Z=z] &= \mathrm{E}\left(\sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot D_i \,\middle|\, Z=z\right) \\
&= \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{E}[D_i \,|\, Z=z] = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \Phi\big(u_i(z)\big).
\end{aligned}
$$

Since we have that

$$
\begin{aligned}
\mathrm{V}(\mathrm{LGD}_i \cdot D_i \,|\, Z=z) &= \mathrm{E}\big[(\mathrm{LGD}_i \cdot D_i)^2 \,|\, Z=z\big] - \big(\mathrm{E}[\mathrm{LGD}_i \cdot D_i \,|\, Z=z]\big)^2 \\
&= \mathrm{E}[\mathrm{LGD}_i^2] \cdot \mathrm{E}[D_i^2 \,|\, Z=z] - \mathrm{ELGD}_i^2 \cdot \big(\mathrm{E}[D_i \,|\, Z=z]\big)^2 \\
&= (\mathrm{VLGD}_i + \mathrm{ELGD}_i^2) \cdot \mathrm{E}[D_i^2 \,|\, Z=z] - \mathrm{ELGD}_i^2 \cdot \big(\mathrm{E}[D_i \,|\, Z=z]\big)^2 \\
&= (\mathrm{VLGD}_i + \mathrm{ELGD}_i^2) \cdot \Phi(u_i(z)) - \mathrm{ELGD}_i^2 \cdot \Phi^2\big(u_i(z)\big),
\end{aligned}
$$

we get that

$$\begin{aligned}
h(z) = \mathrm{V}[L \mid Z = z] &= \mathrm{V}\left(\sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot D_i \,\middle|\, Z = z\right) \\
&= \sum_{i=1}^{n} \mathrm{EAD}_i^2 \cdot \mathrm{V}\left(\mathrm{LGD}_i \cdot D_i \,\middle|\, Z = z\right) \\
&= \sum_{i=1}^{n} \mathrm{EAD}_i^2 \cdot \Big((\mathrm{VLGD}_i + \mathrm{ELGD}_i^2) \cdot \Phi\big(u_i(z)\big) - \mathrm{ELGD}_i^2 \cdot \Phi^2\big(u_i(z)\big) \Big). \qquad (5.11)
\end{aligned}$$

In order to compute $h'(z)$, $g'(z)$ and $g''(z)$, we need to know $\frac{\mathrm{d}}{\mathrm{d}z}\Phi\big(u_i(z)\big)$ and $\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Phi\big(u_i(z)\big)$

$$\frac{\mathrm{d}}{\mathrm{d}z}\Phi\big(u_i(z)\big) = -\sqrt{\frac{\rho_i}{1-\rho_i}} \cdot \varphi\big(u_i(z)\big),$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Phi\big(u_i(z)\big) = -\frac{\rho_i}{1-\rho_i} \cdot u_i(z) \cdot \varphi\big(u_i(z)\big).$$

From this we conclude that

$$h'(z) = -\sum_{i=1}^{n} \mathrm{EAD}_i^2 \cdot \sqrt{\frac{\rho_i}{1-\rho_i}} \cdot \varphi\big(u_i(z)\big)\Big(\mathrm{VLGD}_i + \mathrm{ELGD}_i^2\big(1 - 2 \cdot \Phi\big(u_i(z)\big)\big)\Big), \qquad (5.12)$$

$$g'(z) = -\sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \sqrt{\frac{\rho_i}{1-\rho_i}} \cdot \varphi\big(u_i(z)\big), \qquad (5.13)$$

$$g''(z) = -\sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \frac{\rho_i}{1-\rho_i} \cdot u_i(z) \cdot \varphi\big(u_i(z)\big). \qquad (5.14)$$

Thus, the granularity adjustment for the Vasicek model in a one-factor setting is given by (5.10) with $h(z)$, $h'(z)$, $g'(z)$ and $g''(z)$ as in (5.11), (5.12), (5.13) and (5.14), respectively. A derivation of the second order granularity adjustment for the Vasicek model can be found in (Hibbeln 2010, Section 4.2.1.4).

The granularity adjustment presented by Emmer and Tasche (2005, (2.15)) is the same granularity adjustment presented in this section with the additional assumption that the LGDs are constants and thereby neglecting the variance of the LGDs.

5.2.2 The CreditRisk$^+$ Model

In the one-factor setting of the CreditRisk$^+$ model, following the notation of Section 2.3, we have that the a posteriori distribution is

$$P(D_i = k \,|\, Z = z) = \frac{\big(\mathrm{PD}_i(1 - w_i + w_i z)\big)^k}{k!} \cdot e^{-\big(\mathrm{PD}_i(1-w_i+w_i z)\big)} \quad \text{for } k \in \{0, 1, 2, ...\}.$$

Since $Z \sim \Gamma(1/\sigma^2, \sigma^2)$, we have that

$$f(z) = \frac{z^{1/\sigma^2 - 1}}{\Gamma(1/\sigma^2)} \cdot \frac{e^{-z/\sigma^2}}{\sigma^{2/\sigma^2}}, \tag{5.15}$$

$$f'(z) = \frac{e^{-z/\sigma^2} z^{1/\sigma^2 - 1}}{\Gamma(1/\sigma^2)\sigma^{2(1/\sigma^2+1)}} \cdot \left(\frac{1-\sigma^2}{z} - 1\right), \tag{5.16}$$

$$g(z) = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{PD}_i(1 - w_i + w_i z),$$

$$g'(z) = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{PD}_i \cdot w_i, \tag{5.17}$$

$$, g''(z) = 0, \tag{5.18}$$

$$h(z) = \sum_{i=1}^{n} \mathrm{EAD}_i^2 \cdot \mathrm{PD}_i \cdot \big(1 - w_i + w_i z\big)\Big(\mathrm{ELGD}_i^2 + \mathrm{VLGD}_i\big(1 + \mathrm{PD}_i(1 - w_i + w_i z)\big)\Big), \tag{5.19}$$

$$h'(z) = \sum_{i=1}^{n} \mathrm{EAD}_i^2 \cdot \mathrm{PD}_i \cdot w_i \Big(\mathrm{ELGD}_i^2 + \mathrm{VLGD}_i\big(1 + 2 \cdot \mathrm{PD}_i(1 - w_i + w_i z)\big)\Big). \tag{5.20}$$

Thus, the granularity adjustment in a single-factor setting for the CreditRisk$^+$ model is given by[14] (5.9) with with $f(z)$, $f'(z)$, $g'(z)$, $g''(z)$, $h(z)$ and $h'(z)$ defined as in (5.15), (5.16), (5.17), (5.18), (5.19) and (5.20), respectively, i.e.,

$$\mathrm{GA}_q^{\mathrm{CreditRisk}^+}(L) = \tag{5.21}$$

$$\frac{\big(\frac{1}{\sigma^2} - \frac{1}{z}(\frac{1}{\sigma^2} - 1)\big) \sum_{i=1}^{n} \mathrm{EAD}_i^2 x_i (\mathrm{ELGD}_i^2 + \mathrm{VLGD}_i(1 + x_i)) - \sum_{i=1}^{n} \mathrm{EAD}_i^2 \cdot \mathrm{PD}_i \cdot w_i (\mathrm{ELGD}_i^2 + \mathrm{VLGD}_i(1 + 2x))}{2 \cdot \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{PD}_i \cdot w_i},$$

where $x = \mathrm{PD}_i(1 - w_i + w_i z)$, $z = \mathrm{VaR}_q(Z)$ and $Z \sim \Gamma(1/\sigma^2, \sigma^2)$.

[14] In (5.9) we have q instead of $1 - q$.

5.2.3 Pillar 2 Compliance

The regulatory capital for credit risk estimates $\mathrm{UL}^{\mathrm{ASRF}}_{0.999}(L_i)$ in MtM mode for $i = 1, \dots, n$. With this as a starting point, under pillar 2, we address the question how to estimate $\mathrm{UL}_{0.999}(L) - \mathrm{UL}^{\mathrm{ASRF}}_{0.999}(L)$. We also want the models to be reconciled in the sense that if we assume that the portfolio is infinitely fine grained, then we arrive at the solution that $\mathrm{UL}_{0.999}(L)$ is equal to the regulatory capital for credit risk.

This task seems difficult to accomplish for MtM mode. The situation is complicated by the fact that we lack the data for which the maturity adjustment has been calibrated. However, if we have access to vendor software products we can calibrate the granularity adjustment to compute an add-on for MtM mode, as shown by Gordy and Marrone (2012). Here, we follow another approach that doesn't require vendor software products. Instead we make the erroneous assumption that $\mathrm{UL}^{\mathrm{IRB}}(L_i)$ is an estimation of $\mathrm{UL}^{\mathrm{ASRF}}_{0.999}(L_{i,})$ in default mode. From this assumption we can use the regulatory capital to obtain necessary parameters (ρ_i and w_i) to compute the granularity adjustment for the Vasicek and CreditRisk$^+$ model. This approach has been used by Gordy and Lütkebohmert (2007) for the CreditRisk$^+$ model.

At first, it may seem natural to define ρ_i as ρ_i^{IRB} for the Vasicek model. This is however not feasible since, as has been pointed out above, we do not make any assumptions about the underlying model for regulatory capital. Thus, for the Vasicek model we solve the following equation for ρ_i:

$$\begin{aligned}\mathrm{UL}^{\mathrm{ASRF}}_{q}(L_i) &= \mathrm{EAD}_i \cdot \mathrm{ELGD}_i\left(\Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_i) + \sqrt{\rho_i}\Phi^{-1}(q)}{\sqrt{1-\rho_i}}\right) - \mathrm{PD}_i\right)\\ &= \mathrm{UL}^{\mathrm{IRB}}_{q}(L_i),\end{aligned} \tag{5.22}$$

where $q = 0.999$. Solving equation (5.22) by completing the square we arrive at

$$\sqrt{\rho_i} = \frac{1}{b^2 + c_i^2}\left(- a_i b \pm c_i\sqrt{b^2 + c_i^2 - a_i^2}\right), \tag{5.23}$$

where

$$a_i = \Phi^{-1}(\mathrm{PD}_i), \quad b = \Phi^{-1}(q) \quad \text{and} \quad c_i = \Phi^{-1}\left(\mathrm{PD}_i + \frac{\mathrm{UL}^{\mathrm{IRB}}_{q}(L_i)}{\mathrm{EAD}_i \cdot \mathrm{ELGD}_i}\right).$$

However, (5.23) does not always yield a unique answer, e.g., if an obligor has the characteristics $\mathrm{EAD} = 10^4$, $\mathrm{ELGD} = 0.17$, $\mathrm{PD} = 0.0003$, $\rho^{\mathrm{IRB}} = 0.2$ and $M = 1$ this implies that $\sqrt{\rho}$ in the Vasicek model is either 0.4518 or 0.9856. A way around this problem is to use ρ_i^{IRB} as a proxy for ρ_i in the Vasicek model but this, of course, entails a loss of fidelity.

Since $\mathrm{E}[X] = \mu$ if $X \sim \mathrm{Po}(\mu)$, in the CreditRisk$^+$ model we solve the following equation for w_i:

$$\begin{aligned}\mathrm{UL}_q^{\mathrm{ASRF}}(L_i) &= \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{E}\big[D_i \,|\, Z = \mathrm{VaR}_q(Z)\big] - \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{PD}_i \\ &= \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \Big(\mathrm{PD}_i\big(1 - w_i + w_i \mathrm{VaR}_q(Z)\big) - \mathrm{PD}_i\Big) \\ &= \mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{PD}_i \cdot w_i\big(\mathrm{VaR}_q(Z) - 1\big) = \mathrm{UL}_q^{\mathrm{IRB}}(L_i).\end{aligned}$$

Thus, we have that

$$w_i = \frac{\mathrm{UL}_q^{\mathrm{IRB}}(L_i)}{\mathrm{EAD}_i \cdot \mathrm{ELGD}_i \cdot \mathrm{PD}_i \cdot \big(\mathrm{VaR}_q(Z) - 1\big)}. \tag{5.24}$$

From this we conclude that there always exists a unique solution. This definition of w_i together with (5.21) is the formula derived by Gordy and Lütkebohmert (2007) and will be discussed in the next section.

5.2.4 The Granularity Adjustment of Gordy and Lütkebohmert

The formula of Gordy and Lütkebohmert (2007) is equivalent to (5.21) with w_i defined as in (5.24). The resulting explicit formula for the granularity adjustment is

$$\begin{aligned}\mathrm{GA}_q(L) = \frac{1}{2\sum_{i=1}^{n} \mathrm{UL}_i} &\left(\delta \sum_{i=1}^{n} \left(\gamma_i\big(\mathrm{UL}_i + \mathrm{EL}_i\big) + \big(\mathrm{UL}_i + \mathrm{EL}_i\big)^2 \cdot \frac{\mathrm{VLGD}_i}{\mathrm{ELGD}_i^2}\right)\right. \\ &\left. - \sum_{i=1}^{n} \mathrm{UL}_i \left(\gamma_i + 2\big(\mathrm{UL}_i + \mathrm{EL}_i\big)\frac{\mathrm{VLGD}_i}{\mathrm{ELGD}_i^2}\right)\right),\end{aligned} \tag{5.25}$$

where UL_i is short notation for $\mathrm{UL}^{\mathrm{IRB}}(L_i)$, EL_i is defined as in Section 2.4 and

$$\delta = \big(\mathrm{VaR}_q(Z) - 1\big)\left(\frac{1}{\sigma^2} - \frac{1}{\mathrm{VaR}_q(Z)}\left(\frac{1}{\sigma^2} - 1\right)\right),$$

$$\gamma_i = \frac{\mathrm{EAD}_i}{\mathrm{ELGD}_i} \cdot (\mathrm{ELGD}_i^2 + \mathrm{VLGD}_i).$$

Gordy and Lütkebohmert (2007) also present an approximation of (5.25) that simplifies the formula. If we factor out EAD_i in the sums in (5.25) and assume that $\mathrm{UL}_i/\mathrm{EAD}_i$ and $\mathrm{EL}_i/\mathrm{EAD}_i$ are small we can approximate the formula by setting terms that are products of these expressions to zero, which gives that

$$\mathrm{GA}_q(L) \approx \mathrm{GA}_q^{\mathrm{Approx.}}(L) = \frac{1}{2\sum\limits_{i=1}^{n} \mathrm{UL}_i} \cdot \sum_{i=1}^{n} \gamma_i \Big(\delta \big(\mathrm{UL}_i + \mathrm{EL}_i \big) - \mathrm{UL}_i \Big). \tag{5.26}$$

The accuracy of this approximation is discussed in Gordy and Lütkebohmert (2007, Section 5). In order to use the granularity adjustment together with the CreditRisk$^+$ model it is not sufficient to specify w_i, we also have to specify σ^2. In BCBS 2001, § 445, the value $\sigma^2 = 4$ is used, which together with the value of $\mathrm{VaR}_{0.999}(Z)$ yields the value $\delta = 4.83$. Alternative values of σ^2 are discussed in Gordy and Lütkebohmert (2007, pp.21–22). The Swedish Financial Authority suggests banks with IRB permission to use (5.26) together with $\delta = 4.83$ to compute the add-on for name concentration risk (Edlund 2009*b*, p.4).

Note that the connection between (5.25) and (5.26) is

$$\mathrm{GA}_q(L) = \mathrm{GA}_q^{\mathrm{Approx.}}(L) + \frac{1}{2\sum\limits_{i=1}^{n} \mathrm{UL}_i} \cdot \sum_{i=1}^{n} (\mathrm{UL}_i + \mathrm{EL}_i) \cdot \frac{\mathrm{VLGD}_i}{\mathrm{ELGD}_i^2} \cdot \big(\mathrm{UL}_i(\delta - 2) + \delta \mathrm{EL}_i \big), \tag{5.27}$$

which implies that the approximation (5.26) always is smaller than (5.25) if $\delta > 2$.

5.2.5 Coherence

From (4.1) we note that $\mathrm{UL}_q^{\mathrm{ASRF}}(L)$ is a coherent measure but this isn't necessarily true for $\mathrm{UL}_q^{\mathrm{ASRF}}(L) + \mathrm{GA}_q(L)$. In fact, the risk measure we want to approximate, i.e., $\mathrm{UL}_q(L)$ is in general not subadditive. Still, coherent risk measures are often preferable. The aim of this section is to determine whether $\mathrm{UL}_q^{\mathrm{ASRF}}(L) + \mathrm{GA}_q(L)$ is a coherent risk measure within the CreditRisk$^+$ and Vasicek models. One can readily see that $\mathrm{GA}_q(L)$ satisfies the monotonicity and positive homogeneity properties and that $\mathrm{UL}_q^{\mathrm{ASRF}}(L) + \mathrm{GA}_q(L)$ satisfies the translation invariance property for the Vasicek and CreditRisk$^+$ models. From this we conclude that $\mathrm{UL}_q^{\mathrm{ASRF}}(L) + \mathrm{GA}_q(L)$, for the considered models, is a coherent risk measure iff $\mathrm{GA}_q(L)$ is subadditive. It is easy to show that (5.25) is subbaditive. If we write the numerator as $\sum_{i=1}^{n} a_i$, the denominator as $\sum_{i=1}^{n} b_i$ and notice that $a_i, b_i > 0$, we get

$$\begin{aligned} \mathrm{GA}_q(L_1 + L_2) &= \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = \frac{\sum_{i=1}^{u} a_i + \sum_{i=u+1}^{n} a_i}{\sum_{i=1}^{u} b_i + \sum_{i=u+1}^{n} b_i} \\ &= \frac{\sum_{i=1}^{u} a_i}{\sum_{i=1}^{u} b_i + \sum_{i=u+1}^{n} b_i} + \frac{\sum_{i=u+1}^{n} a_i}{\sum_{i=1}^{u} b_i + \sum_{i=u+1}^{n} b_i} \end{aligned}$$

$$\leq \frac{\sum_{i=1}^{u} a_i}{\sum_{i=1}^{u} b_i} + \frac{\sum_{i=u+1}^{n} a_i}{\sum_{i=u+1}^{n} b_i} = \mathrm{GA}_q(L_1) + \mathrm{GA}_q(L_2),$$

where u and $n-u$ are the numbers of obligors in L_1 and L_2, respectively. Thus, we conclude that $\mathrm{UL}_q^{\mathrm{ASRF}}(L) + \mathrm{GA}_q(L)$, together with $\mathrm{GA}_q(L)$ defined as in Gordy and Lütkebohmert (2007), is a coherent risk measure. In the same fashion, it can be shown that $\mathrm{UL}_q^{\mathrm{ASRF}}(L) + \mathrm{GA}_q(L)$ is a coherent risk measure if $\mathrm{GA}_q(L)$ is defined as in (5.26). The granularity adjustment for the Vasicek model, however, is not subadditive. In fact, the granularity adjustment together with the Vasicek model may result in a negative add-on, which of course is an unwanted property. An example of this is a homogeneous portfolio with ELGD $= 0.45$, PD $= 0.2$, $\rho = 0.7$ and VLGD defined as in Section 2. The formula for the granularity adjustment with the Vasicek model can for homogeneous portfolios explicitly be expressed as[15]

$$\begin{aligned}\mathrm{GA}_q(L) =& \frac{\mathrm{EAD}}{2}\left(\frac{\mathrm{ELGD}^2 + \mathrm{VLGD}}{\mathrm{ELGD}}\left(\frac{\Phi(x)}{\varphi(x)} \cdot \frac{\Phi^{-1}(\alpha)(1-2\rho) - \Phi^{-1}(\mathrm{PD})\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 1\right)\right.\\ &\left. - \mathrm{ELGD} \cdot \Phi(x)\left(\frac{\Phi(x)}{\varphi(x)} \cdot \frac{\Phi^{-1}(\alpha)(1-2\rho) - \Phi^{-1}(\mathrm{PD})\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 2\right)\right),\end{aligned}$$

where $x = \left(\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho}\Phi^{-1}(q)\right)/\sqrt{1-\rho}$ and we have suppressed the obligor index i in the notation. A derivation of this expression can be found in[16] Hibbeln (2010, Section 4.5.5). From this we conclude that we could have the case where $\mathrm{GA}_q(L) = a \cdot \mathrm{EAD}$ and $a < 0$. Let this be the case for L_1 and L_2 that are identical portfolios but have different obligors. Then we get that

$$\mathrm{GA}_q(L_1 + L_2) = a \cdot \mathrm{EAD} > 2a \cdot \mathrm{EAD} = \mathrm{GA}_q(L_1) + \mathrm{GA}_q(L_2).$$

Thus, the granularity adjustment for the Vasicek model is not a coherent risk measure. However, it is difficult to say if this has any practical significance or if it only considers stylized portfolios. A way around this problem would be to use a formula that is derived in a similar fashion to the granularity adjustment but for ES_q instead of VaR_q. For the Vasicek model this results not only in a coherent risk measure, but also in a simpler formula. This approach is considered in Hibbeln (2010, Section 4.3) but will not be considered here.

[15] Note that this formula for the add-on motivates the use of HHI. The formula can be expressed as $\mathrm{GA}_q(L) = \mathrm{EAD} \cdot a$, where a depends on ELGD, VLGD, PD and ρ. Also, if HHI is multiplied by total EAD and by a constant, b, we get that HHI $= \mathrm{EAD} \cdot b$, where b is a parameter to be estimated from observations, usually provided by Monte Carlo simulations.

[16] Note that it seems like two of the plus signs should be replaced by minus signs after the last equal sign in the derivation.

5.2.6 The Retail Portfolio

Credits can be divided into four categories: corporate, sovereign, institutional and retail credits. Risk assessment is often preferable from a holistic point of view. In practice, however, name concentration in retail portfolios is usually considered separately due to practical reasons. Banks are allowed to pool exposures with similar characteristics (Hibbeln 2010, p.42) in order to compute regulatory capital for retail credits. Because of this it is often difficult to retrieve obligor specific information for retail exposures, which is necessary to compute an add-on for name concentration risk. It is however usually assumed that the retail portfolio does not have any name concentration risk. In BCBS 2001, § 427 we find the following text:

> 427. The granularity adjustment should, in principle, be applied at the most aggregate portfolio level possible. As a practical matter, we propose that it should be applied to the non-retail portion of total bank exposure. By its very nature, retail business is highly unlikely ever to worsen the granularity of a bank portfolio. Unless a bank has a very high proportion of its portfolio in retail loans, neither is it likely that the retail portion of the portfolio would greatly reduce the measured granularity of the total portfolio. The proposed treatment of granularity, therefore, is a conservative approach, but one that we believe is reasonable for the vast number of banks with well-managed risk management systems.

The method proposed by the Swedish Financial Authorities, (5.26), also excludes the retail portfolio from the computations (Edlund 2009*b*, p.4). Thus, the add-on for name concentration risk is computed as the sum of the add-on for the retail portfolio and the add-on for the remaining part of the credit portfolio, i.e., $\mathrm{GA}_q(L_{(\mathrm{C,S,I})}) + \mathrm{GA}_q(L_{\mathrm{Retail}}) = \mathrm{GA}_q(L_{(\mathrm{C,S,I})}) + 0$. However, as we will see, if we assume that the add-on for the retail portfolio is negligible, then it is indeed a simple matter to compute an add-on with the granularity adjustment for the CreditRisk$^+$ model with respect to the total credit portfolio. We also show that if the add-on is computed as in (5.25) or (5.26), then including the retail portfolio always results in a smaller add-on.

Let the number of obligors in the corporate, sovereign and institutional portfolio be u and the number of obligors in the total portfolio to be n. Considering (5.21), we notice that if we write the numerator as $\sum_{i=1}^{n} \alpha_i$ and the denominator as $\sum_{i=1}^{n} \beta_i$ the granularity adjustment for the total portfolio with the CreditRisk$^+$ model can be written as

$$\mathrm{GA}_q(L_{(\mathrm{C,S,I})} + L_{\mathrm{Retail}}) = \frac{\sum\limits_{i=1}^{n} \alpha_i}{\sum\limits_{i=1}^{n} \beta_i} = \frac{\sum\limits_{i=1}^{u} \alpha_i}{\sum\limits_{i=1}^{n} \beta_i} + \frac{\sum\limits_{i=u+1}^{n} \alpha_i}{\sum\limits_{i=1}^{u} \beta_i + \sum\limits_{i=u+1}^{n} \beta_n}. \tag{5.28}$$

The assumption that the add-on for the retail portfolio is negligible can be formalized as

$$0 = \mathrm{UL}_q(L_{\mathrm{Retail}}) - \mathrm{UL}_q^{\mathrm{ASRF}}(L_{\mathrm{Retail}}) \approx \mathrm{GA}_q(L_{\mathrm{Retail}}) = \frac{\sum\limits_{i=u+1}^{n} \alpha_i}{\sum\limits_{i=u+1}^{n} \beta_i}.$$

This gives that

$$\left| \frac{\sum\limits_{i=u+1}^{n} \alpha_i}{\sum\limits_{i=1}^{u} \beta_i + \sum\limits_{i=u+1}^{n} \beta_n} \right| \leq \left| \frac{\sum\limits_{i=u+1}^{n} \alpha_i}{\sum\limits_{i=u+1}^{n} \beta_n} \right| = 0, \tag{5.29}$$

where we also use that $\beta_i > 0$ for $i = 1, \dots, n$. Thus, if we make the assumption that $\sum_{i=u+1}^{n} \alpha_i / \sum_{i=u+1}^{n} \beta_i = 0$, use (5.28) and (5.29) we get that

$$\mathrm{GA}_q(L_{(\mathrm{C,S,I})} + L_{\mathrm{Retail}}) = \frac{\sum\limits_{i=1}^{u} \alpha_i}{\sum\limits_{i=1}^{n} \beta_i}.$$

Thus, the only difference to when the retail portfolio is excluded is that we include all obligors in the summation in the denominator. Also, since it doesn't matter on what level the retail exposures are aggregated the pooling of obligors does not constitute a problem. For (5.25) and (5.26) we also have that $\alpha_i > 0$, which means that in this case the add-on always will be smaller if the retail portfolio is included.

6 Capital Allocation

The profitability of a decision made in a particular business unit may depend on decisions made in other business units of the same company. This gives rise to the problem of how to make business units work in a coordinated way in order to maximize the utility of the shareholders. Performance measures, such as RAROC, the Sharpe ratio, different risk measures and Jensen's alpha can in different ways be used to define preference among portfolios and evaluate performance of business units. However, since we already have defined the risk implicitly as a cost (see Section 3) we define the optimal portfolio to be the portfolio that maximizes the profit of the bank. A common way to solve this problem is to allocate capital (profits, costs, buffer capital, etc.) within the company in such a way that if all business units maximize their own profits, then the business units also maximize the profit of the company. In this respect, capital allocation of regulatory capital for credit risk is unproblematic thanks to the property of portfolio invariance.

However, there is no obvious way how to allocate the add-on for name concentration risk.

Capital Allocation Principle Consider the case where we have $L_1, \dots, L_n$ investment opportunities. Let the vector $\boldsymbol{\alpha}$ represent how much is invested in each investment opportunity and let ρ be a risk or performance measure. To ease notation, let ρ take $\boldsymbol{\alpha}$ as an argument instead of $\boldsymbol{\alpha}^\top \boldsymbol{L}$. With this notation we have that $\rho(\alpha_i \,|\, \boldsymbol{\alpha})$ for $i = 1, \dots, n$ is a *capital allocation principle* if it satisfies the *full allocation property*, i.e.,

$$\rho(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \rho(\alpha_i \,|\, \boldsymbol{\alpha}).$$

First Approach Let $t_0 < t_1 < t_2 \dots$ be time points and consider a company that at t_j holds the portfolio $\boldsymbol{\alpha}_{t_j}$ and that at each time point is given an investment opportunity, $\boldsymbol{x}_{i,t_j} = x_{i,t_j}\boldsymbol{e_i}$ where $\boldsymbol{e_i}$ is a standard basis vector, which it may accept or reject. If the company accepts the deal we get that $\boldsymbol{\alpha}_{t_{j+1}} = \boldsymbol{\alpha}_{t_j} + \boldsymbol{x}_{i,t_j}$ and if it rejects the deal, $\boldsymbol{\alpha}_{t_{j+1}} = \boldsymbol{\alpha}_{t_j}$. In this setting, it would be natural to define the capital allocation principle as the marginal risk contribution, i.e.,

$$\rho_{\mathrm{marg}}(\alpha_i \,|\, \boldsymbol{\alpha}) = \rho(\boldsymbol{\alpha}) - \rho(\boldsymbol{\alpha} - \alpha_i \boldsymbol{e}_i).$$

However, if ρ is positive homogeneous, subadditive and differentiable Tasche (2004, Proposition 2) showed that

$$\rho(\boldsymbol{\alpha}) \geq \sum_{i=1}^{n} \rho_{\mathrm{marg}}(\alpha_i \,|\, \boldsymbol{\alpha}).$$

With equality iff ρ is exactly additive. Thus, the marginal contribution capital allocation principle does not satisfy the full allocation property for coherent risk measures.

Second Approach As in the first approach, let $t_0 < t_1 < t_2 \dots$ be time points and $\boldsymbol{\alpha}_{t_j}$ denote the portfolio of a company at time t_j. The company rebalances its portfolio at every time point such that $\boldsymbol{\alpha}_{t_{j+1}} = \boldsymbol{\alpha}_{t_j} + \boldsymbol{\Delta x}_{t_j}$ where $\boldsymbol{\Delta x}_i = (\Delta x_{1,t_j}, \dots, \Delta x_{n,t_j})^\top$ and $|\boldsymbol{\Delta x_{t_j}}| \leq \varepsilon$, where ε is small. If we assume that the portfolio is positive homogeneous we have that

$$\rho(\boldsymbol{\alpha}_j) = \boldsymbol{\alpha}_j^\top \nabla \rho(\boldsymbol{\alpha}_j). \tag{6.1}$$

Thus, if ρ is positive homogeneous and we linearize ρ around $\boldsymbol{\alpha}_j$ we get

$$\begin{aligned}\rho(\boldsymbol{\alpha}_{t_{j+1}}) &= \rho(\boldsymbol{\alpha}_{t_j} + \boldsymbol{\Delta x}_{t_j}) \approx \rho(\boldsymbol{\alpha}_{t_j}) + \boldsymbol{\Delta x}^\top \nabla \rho(\boldsymbol{\alpha}_{t_j}) = \boldsymbol{\alpha}_{t_j}^\top \nabla \rho(\boldsymbol{\alpha}_{t_j}) + \boldsymbol{\Delta x}^\top \nabla \rho(\boldsymbol{\alpha}_{t_j}) \\ &= (\boldsymbol{\alpha}_j + \boldsymbol{\Delta x}_{t_j})^\top \nabla \rho(\boldsymbol{\alpha}_{t_j}) = \boldsymbol{\alpha}_{t_{j+1}} \nabla \rho(\boldsymbol{a}_{t_j}).\end{aligned}$$

This means that if we use the capital allocation principle

$$\rho(\alpha_i \,|\, \boldsymbol{\alpha}) = \alpha_i \frac{\partial \rho}{\partial \alpha_i}(\boldsymbol{\alpha}),$$

it is easy find out much how much $\rho(\boldsymbol{\alpha})$ will change if we change α_i with a small amount. The full allocation property follows directly from (6.1) and the capital allocation principle is called *Euler capital allocation principle.* Other motivations for the Euler principle can be found in Tasche (2008, p.5).

Since the granularity adjustment with the Vasicek and CreditRisk$^+$ models are positive homogeneous it is possible to use the Euler principle as a capital allocation principle. In this case, using the Euler principle implies that it is easy to determine how much the add-on for name concentration risk changes if we change EAD_i with a small amount.

Euler allocation and the Granularity Adjustment for the CreditRisk$^+$ model If we apply the Euler capital allocation principle to the granularity adjustment for the CreditRisk$^+$ model and use the notation of section 5.2.4 we get

$$\frac{\partial \text{GA}_q(L)}{\partial x_j} = \frac{2a_j x_j \sum\limits_{i=1}^{n} b_i x_i - b_j \sum\limits_{i=1}^{n} a_i x_i^2}{\left(\sum\limits_{i=1}^{n} b_i x_i\right)^2},$$

where $x_i = \text{EAD}_i, \;\; b_i x_i = 2\text{UL}_i$ and

$$a_i x_i^2 = \delta\left(\gamma_i(\text{UL}_i + \text{EL}_i) + (\text{UL}_i + \text{EL}_i)^2 \cdot \frac{\text{VLGD}_i}{\text{ELGD}_i^2}\right) - \text{UL}_i\left(\gamma_i + 2(\text{UL}_i + \text{EL}_i) \cdot \frac{\text{VLGD}_i}{\text{ELGD}_i}\right).$$

If we use the approximation (5.26) we get a different value for a_i

$$a_i x_i^2 = \gamma_i\Big(\delta(\text{UL}_i + \text{EL}_i) - \text{UL}_i\Big).$$

Note that a_i and b_i are independent of x_i. However, Euler allocation cannot be used if we include the retail portfolio since the granularity adjustment then isn't positive homogeneous.

Euler allocation and the Granularity Adjustment for the Vasicek model If we apply the Euler capital allocation principle to the granularity adjustment for the Vasicek model and the notation $x_i = \text{EAD}_i$, we get that

$$\frac{\partial \text{GA}_q(L)}{\partial x_j} = \frac{x_i\big(a_i g(x_i) - c_i v(x_i)\big) - \frac{1}{2}\big(b_i f(x_i) - u(x_i)\big)}{\big(g(x_i)\big)^2} + \frac{b_i u(x_i) v(x_i)}{\big(g(x_i)\big)^3},$$

where

$$f(x_j) = \sum_{i=1}^{n} a_i x_i^2, \quad g(x_j) = \sum_{i=1}^{n} b_i x_i, \quad u(x_j) = \sum_{i=1}^{n} c_i x_i^2, \quad v(x_j) = \sum_{i=1}^{n} d_i x_i,$$

$$a_i = c_i \Phi^{-1}(q) - \varphi(s_i)\sqrt{\frac{\rho_i}{1-\rho_i}}\left(\mathrm{VLGD}_i + \mathrm{ELGD}_i^2\big(1 - 2\Phi(s_i)\big)\right),$$

$$b_i = \varphi(s_i)\mathrm{ELGD}_i\sqrt{\frac{\rho_i}{1-\rho_i}},$$

$$c_i = \Phi(s_i)\big(\mathrm{ELGD}_i^2 + \mathrm{VLGD}_i\big) - \Phi^2(s_i)\mathrm{ELGD}_i^2,$$

$$d_i = \varphi(s_i)s_i\mathrm{ELGD}_i \cdot \frac{\rho_i}{1-\rho_i},$$

$$s_i = \frac{\Phi^{-1}(\mathrm{PD}_i) + \sqrt{\rho_i}\Phi^{-1}(q)}{\sqrt{1-\rho_i}}.$$

New capital allocation principles are preferably implemented gradually in order to avoid too drastic changes in the incentives of the business units. This can be achieved simply by choosing a linear combination of the present and the new allocation. It is easy to see, indeed, that this satisfies the full allocation property.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the present and the new allocations, respectively, and let both allocations satisfy the full allocation property, i.e. $\boldsymbol{a}^\top \mathbf{1} = \boldsymbol{b}^\top \mathbf{1} = \gamma$, where $\mathbf{1}$ is an $n \times 1$ vector of ones and γ the add-on for name concentration risk. Then, if $x \in [0, 1]$ we have that

$$x\boldsymbol{a}^\top \mathbf{1} + (1-x)\boldsymbol{b}^\top \mathbf{1} = x\gamma + (1-x)\gamma = \gamma.$$

7 Results

In this section discussed models are applied to the credit portfolio of Nordea for a specific time point. First, assumptions and simplifications made in the data retrieval step are explained. Next, in Table 1, add-ons for name concentration risk under pillar 2 in Basel II are computed with the granularity adjustment for the Vasicek and CreditRisk$^+$ models. For the CreditRisk$^+$ model, the retail has also been included. Finally, in Figure 1 and Figure 2, the add-ons have been allocated to obligors with Euler capital allocation. Data have been processed in SAS.

7.1 Data retrieval

In order to make an analysis of the credit portfolio, obligor specific data first had to be retrieved and exposures aggregated to obligors. To accomplish this, a number of simplifications and assumptions had to be made:

- All obligors are assigned to a super-obligor, which gives rise to a partition of the set of obligors. It is, however, not clear whether obligors can default independently within the sets of the partition. Therefore, analyses have been made on two portfolios, one which is divided in sub-obligors and one which is divided super-obligors. Since name concentration reaches its minimum and maximum in these cases, these portfolios are denoted by *Min* and *Max*.

- DLGDs have been used instead of ELGDs. However, DLGDs might actually be preferred to ELGDs in order to meet pillar 2 requirements since DLGDs are used in the IRB formula (see section 4.2).

- Since we have that PD $= 0$ for sovereign exposures in the Basel II framework, these do not affect the loss distribution and therefore are excluded from computations. Thus, corporate, institutional and retail credit data are retrieved in the first step. For corporate and institutional data, exposures are aggregated to obligors, whereas retail data are aggregated to large groups with similar characteristics.

- All retail exposures have been assigned $\rho^{\mathrm{IRB}} = 0.15$. However, this is only correct for residential mortgage exposures (Hibbeln 2010, p.42), but since most exposures are of this kind this is unlikely to affect the result significantly.

7.2 Add-on

The following simplifications and assumptions were made to compute the add-ons:

- It is assumed that $\mathrm{VLGD}_i = 0.25 \cdot \mathrm{ELGD}_i(1 - \mathrm{ELGD}_i)$ (see section 2).

- ρ_i^{IRB} has been used in the Vasicek model (see section 5.2.3).

Name concentration add-ons for pillar 2 in Basel II have been computed with the granularity adjustment for the Vasicek and CreditRisk$^+$ models. For the CreditRisk$^+$ model, computations have also been made when the retail portfolio is included (see Section 5.2.6). We note that the Vasicek model results in the smallest add-on. The difference can be explained by different model calibrations, how σ and ρ_i have been chosen (see Section 5.2.3). In line with (5.27), we have that the simplified formula of Gordy and Lütkebohmert results in a slightly smaller add-on than the non-simplified version. In contradiction to the

	Max		Min	
	NR	R	NR	R
Vasicek	100		80	
Gordy & Lütkebohmert	142	113	114	90
Gordy & Lütkebohmert, simplified	140	112	112	89

Table 1: Name concentration add-on in percent of the add-on for the Vasicek model. For *Max* computations have been made on a super-obligor level and for *Min* computations have been made on a sub-obligor level. NR denotes *no retail* and R denotes *retail*, i.e., whether or not the retail portfolio has been included in the computations (see section 5.2.6). *Vasicek* has been computed as in section 5.2.1 with $\rho_i = \rho_i^{\mathrm{IRB}}$, *Gordy & Lütkebohmert* has been computed as in (5.25) and *Gordy & Lütkebohmert, simplified* as in (5.26).

excerpt in section 5.2.6 from BCBS 2001, § 427, the add-on is substantially reduced if the retail portfolio is included in the computations.

7.3 Capital Allocation

In order to analyze different capital allocations we define snc_i (single-name concentration factor) for $i = 1, \ldots, n$ as

$$(\text{allocated capital of add-on for name concentration risk})_i = \mathrm{snc}_i \cdot \mathrm{UL}^{\mathrm{IRB}}(L_i). \quad (7.1)$$

The difference between Euler allocation for the simplified and non-simplified formulae of Gordy and Lütkebohmert is barely discernible in Figure 1 and Figure 2. As seen in Table 3, if we want to decrease the add-on by changing the outstanding amount to an obligor (exposure at default of obligor i, EAD_i) by a small amount, then EAD_i should be decreased only for around 0.5 % of the obligors whereas EAD_i should be increased for remaining obligors. However, this small fraction of obligors constitute around 45 % of

	$\max(\mathrm{snc}_1, \ldots, \mathrm{snc}_n)$		$\min(\mathrm{snc}_1, \ldots, \mathrm{snc}_n)$	
	Max	Min	Max	Min
Vasicek	100	102	-5.6	-4.4
Gordy & Lütkebohmert	152	132	-7.4	-6
Gordy & Lütkebohmert, simplified	150	128	-7.4	-5.8

Table 2: Smallest and largest snc_i in Figure 1 and Figure 2 in percent of the largest snc_i for the Vasicek model in Figure 1.

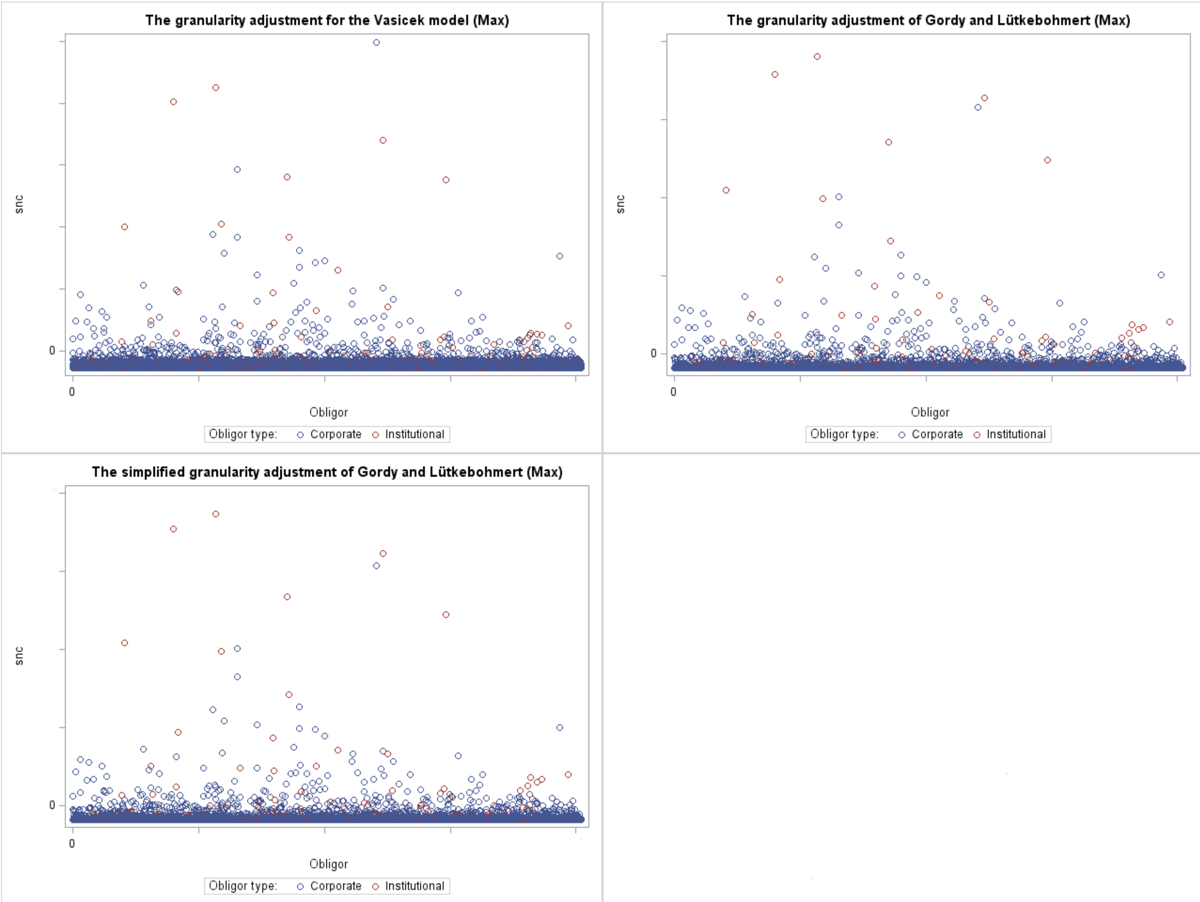

Figure 1: The order of obligors has been randomized but is the same for the plots above. Euler allocation has been applied for the plots (see section 6). Computations have been done on a super-obligor level.

total EAD. Thus, the capital allocation of the add-on will be most noticeable for a few and very large obligors. From Table 2, we note that the magnitude of snc_i can be much greater for positive allocations than for negative allocations.

	$100 \cdot \#\{\mathrm{snc}_i > 0\}/n$		$100 \cdot \sum_{i=1}^{n} \mathrm{EAD} \cdot I_{\mathrm{snc}_i>0} / \sum_{i=1}^{n} \mathrm{EAD}$	
	Max	Min	Max	Min
V	0.58 (15)	0.39 (20)	48 (40)	41 (43)
GL	0.59 (17)	0.41 (22)	49 (40)	42 (44)
GL s.	0.58 (17)	0.41 (23)	48 (40)	42 (44)

Table 3: Share in percent of number and EAD of obligors with snc_i above zero in Figure 1 and Figure 2. Numbers in parentheses denote the share of institutional obligors in percent.

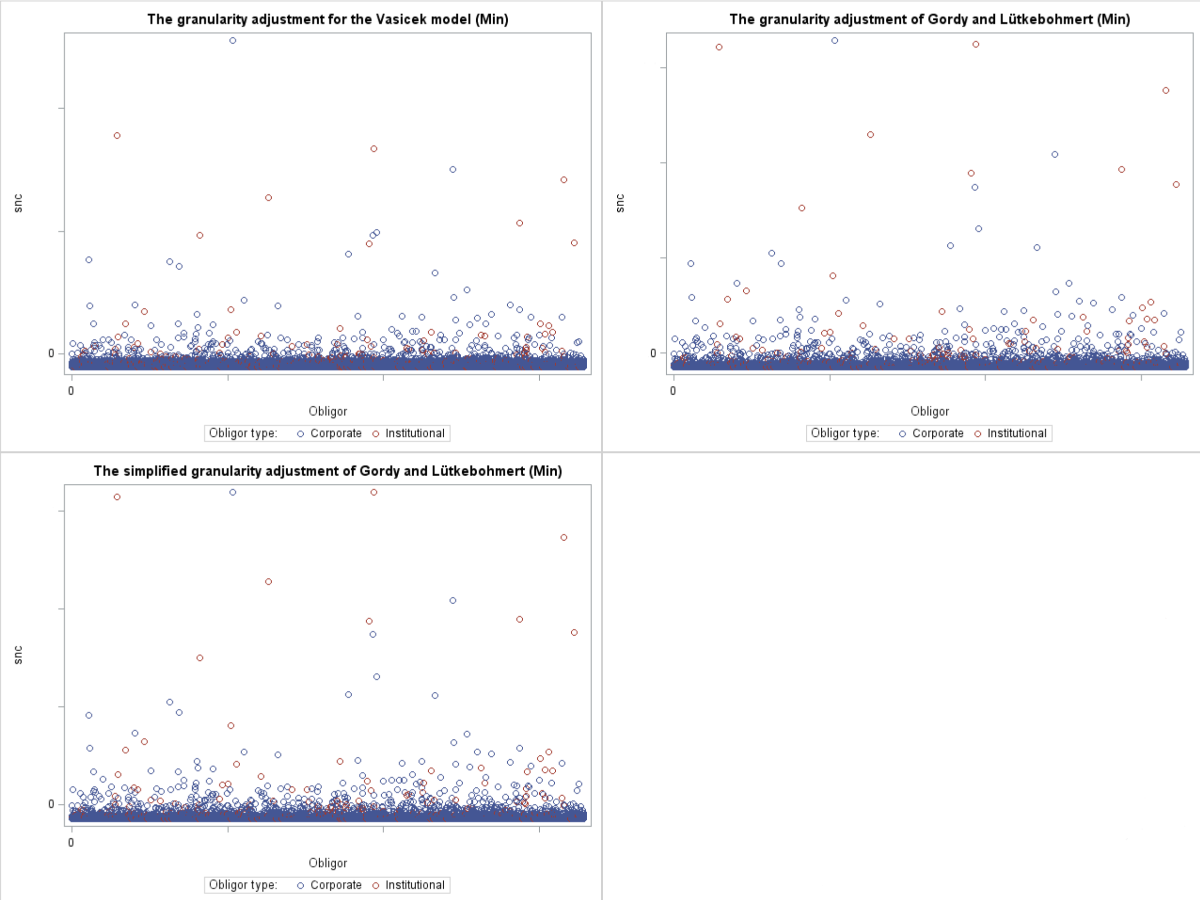

Figure 2: Same result as in Figure 1 but on a sub-obligor level.

8 Conclusion

Banks are concerned with the risk of large losses primarily for two reasons: to meet regulatory demands and to obtain a target level of credit quality, i.e., the trade-off between borrowing costs and capital buffer size that maximizes profits. In many cases, banks prefer a unified view on risk, which means that the bank's and the supervisory authorities' risk perspectives are reconciled.

In Basel II, the capital charge for credit risk under pillar 1 (regulatory capital for credit risk) is portfolio invariant, i.e., regulatory capital increases by the same amount independently to what portfolio a credit exposure is added. To achieve this property, the regulatory capital of a credit portfolio is an estimation of the unexpected loss (the difference between expected loss and Value at Risk) for a particular diversification among geographical areas, industries and sizes of outstanding amounts to obligors. If a bank deviates from this benchmark, this is to be addressed under pillar 2 and the potential add-on is called concentration risk. Diversification among geographical areas and industries (sector concentration) and diversification among sizes of outstanding amounts to obligors (name concentration) are often treated independently for simplicity, i.e., independent add-ons are obtained by relaxing the assumptions of sector and name concentration separately. The infinite granular portfolio is often used as a benchmark to assess an add-on for name concentration risk.

In principle, there exist three approaches to assess an add-on for name concentration risk: Monte-Carlo simulations, model-based approximations and regression methods. Monte-Carlo simulations are often time consuming, difficult to reconcile with the regulatory framework and often require expensive software. Regression methods require Monte-Carlo simulations for calibration and re-calibration and are often not as accurate as model-based approximations.

A model-based approximation that has gained great publicity is the granularity adjustment, which is a second order Taylor approximation around the infinite granular portfolio. In order to compute an add-on with the granularity adjustment it has to be done w.r.t. a credit risk model. Here we have considered two models: the CreditRisk$^+$ model and the Vasicek model. In Section 5.2.5 we have shown that the former is coherent when parameters are specified as in (5.25) (the Gordy and Lütkebohmert formula) and (5.26) (the simplified Gordy and Lütkebohmert formula), whereas the latter isn't. We have also shown how the retail portfolio can be included in the computations for the CreditRisk$^+$ model and that this, when parameters are specified as in in (5.25) and (5.26), always results in a smaller add-on (see Section 5.2.6). See Gordy and Lütkebohmert (2007) and Hibbeln (2010) for information on the accuracy of the models.

Applying the discussed methods on a real credit portfolio we note that the Vasicek model results in the smallest add-on and that the difference between the simplified and non-simplified formula of Gordy and Lütkebohmert is small. We also note that the simplified formula results in a smaller add-on than the non-simplified formula, which is in line with (5.27). If the retail portfolio is included in the computations of the Gordy and Lütkebohmert formulae, the add-on is substantially smaller.

Capital charges for credit risk are often allocated to exposures. For regulatory capital this is natural since it is portfolio invariant. Name concentration risk, however, is not portfolio invariant and there doesn't exist any obvious way to allocate the add-on to obligors. Here we have chosen to look closer on Euler allocation, which has the property that if all sub-units maximize their profits, then the profit of the bank is maximized as well. If the size of the outstanding amount for an obligor is changed by a small enough proportion then the capital allocated to that obligor changes by the same proportion and the full allocation property is preserved, i.e., the allocated capital still sums up to the add-on.

An interesting result is that if we apply Euler allocation for the two mentioned models on a real credit portfolio, almost all obligors will have a negative capital allocated to them for name concentration risk. Only around 0.5 % of the obligors will have a positive capital allocation. However, these obligors together constitute around 45 % of the total outstanding amount.

Implementing new allocation principles it is important to prevent large sudden changes in the business operations. One way to solve this is to implement a linear combination of the present allocation and the new allocation and then gradually progress towards the new allocation.

To sum up, we note that in order to meet pillar 2 requirements there are several reasons to prefer the formulae of Gordy and Lütkebohmert, (5.25) and (5.26), to the granularity adjustment for the Vasicek model. First, the formulae of Gordy and Lütkebohmert are coherent, whereas this is not true for the Vasicek model. Also, it is not clear how to specify the parameter ρ_i in the Vasicek model. Even if the Vasicek model results in a considerably more complex formula it is still very easy to implement. However, the complexity makes the formula more opaque and less fitted for qualitative reasoning. Another interesting approach, which hasn't been considered here, is to use ES instead of VaR. More information of this approach with the Vasicek model can be found in Hibbeln (2010, Section 4.3).

Since financial risk always should be considered from a holistic perspective it seems reasonable to include the retail portfolio in the computations, which for the formulae of Gordy and Lütkebohmert always implies a smaller add-on. However, this does not seem to be a widely used approach in the industry.

It is not at all clear how to allocate the add-on to exposures. Euler allocation provides a natural approach only if the scenario in Section 6 corresponds to reality. Also, if Euler allocation is to be implemented one difficulty is to decide how often the allocation is to be updated. How to allocate the add-on if the retail portfolio is included in the computations remains an unsolved problem. If reality doesn't correspond to the scenario in Section 6 it might in fact be impossible to optimize the credit portfolio by capital allocation, due to the full allocation property. However, this can be solved by evaluating business units with other performance measures than capital allocation.

A few suggestions on further research topics are:

- Extend the granularity adjustment for the Vasicek model to include the retail portfolio.
- Determine conditions for existence and uniqueness of the solution to (5.22). Also, even if there in some cases exist two solutions two this equation, only one seems to be reasonable. Is this always the case?

- Develop an allocation principle for the case when the retail portfolio is included in the computations of the add-on.
- Evaluate other performance measures than capital allocation.
- Evaluate what difference it makes to use DLGDs instead of ELGDs in the computations.

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