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Pricing Interest Rate Derivatives in the Multi-Curve Framework with a Stochastic Basis

by

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Abstract

The financial crisis of 2007/2008 has brought about a lot of changes in the interest rate market in particular, as it has forced to review and modify the former pricing procedures and methodologies. As a consequence, the Multi-Curve framework has been adopted to deal with the inconsistencies of the frameworks used so far, namely the single-curve method.

We propose to study this new framework in details by focusing on a set of interest rate derivatives such as deposits, swaps and caplets, then we explore a stochastic approach to model the Libor-OIS basis spread, which has appeared since the beginning of the crisis and is now the quantity of interest to which a lot of researchers dedicate their work (F.Mercurio, M.Bianchetti and others).

A discussion follows this study to set the light on the challenges and difficulties related to the modeling of basis spread.

Sammanfattning

Den stora finanskris som inträffade 2007/2008 har visat att nya värderingsmetoder för räntederivat är nödvändiga. Den metod baserat på multipla räntekurvor som introducerats som lösning på de problem som finanskrisen synliggjort, speciellt gällande räntespread, har givit upphov till nya utmaningar och bekymmer.

I detta arbete utforskas den nya metoden baserat på multipla räntekurvor samt en stokastisk modell för räntespread. Slutsatserna och diskussionen om resultaten som presenteras tydliggör kvarvarande utmaningar vid modellering av räntespread.

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Overview

The financial crisis that started in 2007 has had big repercussions on the way the market of interest rate derivatives used to be apprehended. While the valuation of these instruments was straightforward before then, it has now become a challenge for most financial institutions to consolidate their strategies and quantitative methods by taking into account the market changes and the regulations being issued.

Before the crisis, number of transactions weren't risk free but counterparties didn't mind it because the credit and liquidity risk underlying them were negligible in that context. After the crisis, things have changed and more banks and institutions tend to enter collateralized transactions under what is known as the Credit Support Annex (**CSA**).

The **CSA** ensures a riskless scenario for both counterparties involved in any transaction as it allows to mitigate the credit and liquidity risks by agreeing on a collateral payment. This payment could consist of cash and/or assets that compensate the mark-to-market in order to nullify the risk of default of any of the counterparties.
We propose in this work, to study the modern multi-curve framework that has been introduced after the crisis in the pricing of some interest rate derivatives in OTC markets.

In this work, we will only focus on the EUR market. Also, since the majority of financial institutions have started to enter only collateralized (secured) transactions, we will focus on a setting where the collateral is taken into account.

First, we introduce the multi-curve framework starting with the pricing of swaps and deposits that are commonly used in the construction of yield curves. The yield curve construction procedure is then presented.

The significant widening in basis spreads has forced to introduce a stochastic spread to try capture its volatile aspect in interest rate markets.

For this purpose and after going through relevant literature on the multi-curve framework and the modeling of the basis spread, we will select a model and assess its efficiency on recent market data to discuss its validity and limitations. This study will focus on the EUR caplet market where we will use caplet data to calibrate the chosen model and then compare the outcomes with the deterministic spread case.

Definitions and Key Words

Abbreviations and Key Words:

CDS	Credit Default Swap
CSA	Credit Support Annex
D	Day
DCL	Dexia Crédit Local
EONIA	Euro OverNight Index Average
IRS	Interest Rate Swap(s)
M	Month
mkt	Market
NPV	Net Present Value
OIS	Overnight Index Swap(s)
ON	Over Night
OTC	Over-The-Counter
SABR	Stochastic Alpha Beta Rho
W	Week
XIBOR	X InterBank Offered Rate (X = {LIBOR, EURO, ...})
Y	Year

Definitions:

In this section, we define some notions that are going to be used throughout this report. Other notions will be defined when used.

Definition 1: IBOR

IBOR stands for InterBank Offered Rate and it is the rate at which prime banks offer loans to each other. It is the reference rate for the OTC transactions. In the Euro area, the EURIBOR is the principal benchmark for short term interest rates around the world.
This reference rate is computed as the average of the rates submitted by a panel of prime banks called the "Contribution Panel". For the USD currency, the reference rate is the LIBOR.
The IBOR exists in the overnight (ON) maturity, 1W maturity and 1M, 2M until 12M maturities, but the most traded ones are of maturity 3M (LIBOR) and 6M (EURIBOR), used mainly in the swaps market.

Definition 2: Tenor

The time left for the repayment of a loan or a contract, also referring to the maturity or time to maturity when dealing with instruments such as bonds.

Definition 3: OIS Rates

As will be presented, the OIS rate is the fair swap rate of the Overnight Index Swaps, which are plain vanilla swaps whose underlying rate is the overnight rate (Eonia for the Euro area). In the market, the OIS rates are tied to the overnight rates as it will be seen in the pricing formula of the OIS swaps in section 1.4.1 and thus to compare Eurbior rates to the overnight ones, OIS rates are used because they are quoted in different tenors as the Euribor.

Definition 4: EONIA

EONIA stands for Euro OverNight Index Average and is the second main money benchmark in the Euro area. It is the rate used in the overnight transactions (one-day-maturity deposits). It is computed as the average of all the overnight unsecured transactions weighted by their volumes in the Euro market. These rates are introduced by the contribution panel formed by the world's leading banks.
Since these rates have a daily tenor, the credit and liquidity risks reflected on them are negligible, which makes them the best available proxies for the risk-free rates respectively in each currency.

Definition 5: Basis Spread

It corresponds to the interest rate difference between two different floating rates such as different maturity Libor rates or between different types of rates like Euribor and OIS rates. It is usually computed in basis points (bps) where 1bp = 10^{-2}%.

N.B: The dates are written in the French way.

Goals and Recurrent Notations

The aim of my master thesis is to perform the pricing of some plain vanilla interest rate derivatives (deposits, swaps and caplets) in the multi-curve framework after constructing the yield curves properly, then study models for a stochastic basis spread and choose one specific model to implement in the pricing library of the DCL and assess its efficiency in generating prices that are consistent with the market. I have divided the work into subtasks as follows:

1. Explore the literature on the multi-curve framework,
2. Implement the pricing of plain vanilla interest rate swaps and basis swaps in the multi-curve setting,
3. Propose a model for a stochastic basis spread and implement it,
4. Perform the pricing of plain vanilla interest rate caplets and compare the results with market data,
5. Conclude on the validity and accuracy of the model and discuss its outcomes and eventual limitations.

In addition to the notations that are going to be explained throughout the report, we will use the following ones recurrently:

- $P_x(t,T)$ is the discount factor (the zero coupon price at time t) taken from the curve C_x, which is whether the discount curve C_d or the forward curve C_m.
$$P_x(t,T) = \mathbb{E}_t^Q[e^{-\int_t^T r_x dx}],$$
where Q is the risk neutral measure and r_x is the short rate.
- $\mathcal{F}_t$ to designate the natural filtration associated to a stochastic process. It is mainly used in the no-arbitrage pricing formula, in the conditional expectations, and we will, from now on, use $\mathcal{F}_t$ for every stochastic process, for simplicity, but it doesn't mean that it is the same filtration for all of them. It represents the information generated by the corresponding process up until time t.
- $\mathbb{E}_t^{Q_x^T}$ to designate the conditional expectation $\mathbb{E}^{Q_x^T}[.|\mathcal{F}_t]$ taken under **the measure Q_x^T, which is defined when considering $T \to P_x(t,T)$ to be the numeraire**. In this report x will designate d for the discount curve and m for the forward curve of tenor m.
- $\gamma_m(T_{i-1}, T_i)$ is the year fraction corresponding to the period $[T_{i-1}^m, T_i^m]$ for a term structure that is homogeneous with a tenor m, i.e. $T_{i-1}^m - T_i^m = m$.

1 The Multi-Curve Framework

The OTC interest rate derivatives market has for a long time been mastered before the financial crisis of 2007, in terms of valuation. Not only has the crisis brought about a drastic change in the behaviour of interest rates, particularly the Libor rate, but it also has forced practitioners to rethink their evaluation strategies.

The multi-curve paradigm has since then been incorporated and is now adopted as the modern framework.

Our aim in this section is to emphasize this transition as well as its whys and wherefores.

1.1 Interest Rate Spread Increase

Before the financial crisis, the interest rates were mutually consistent, meaning, their differences were considered negligible as their spreads were very small in terms of basis points.

The figure below shows a history of the EURIBOR-3M (of tenor 3M) and EONIA (OIS-3M) before and after august 2007 as well as the corresponding spread.

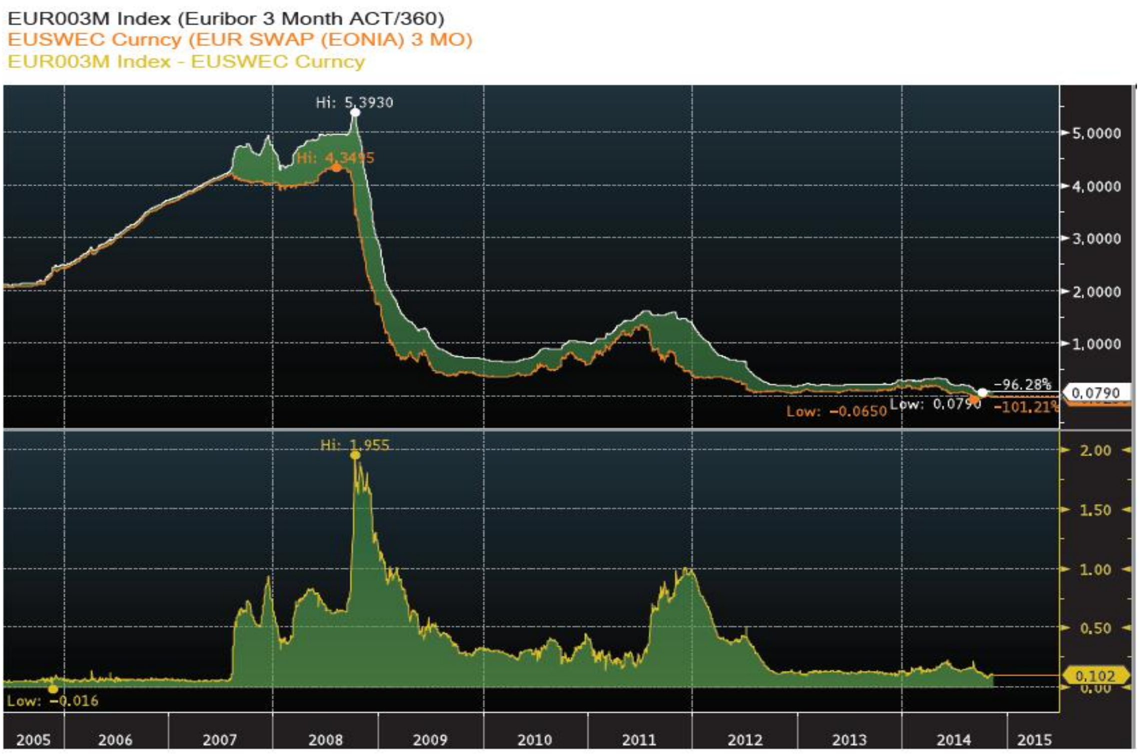


Figure 1: OIS - EURIBOR 3M spread in %, EURIBOR 3M in white, OIS 3M in orange and spread in yellow (lower graph) - Bloomberg data.

The rates and spread are quoted in percent.

Before the third quarter of 2007, the two rates were matching almost perfectly as their spread hovered around 0.
We observe an obvious increase of the spread from the third quarter of 2007 (August 2007) from -1.6 bps (-0.016 %: the spread was exceptionally negative at one date at the end of 2005, but always positive since then) to 195.5 bps (1.955 %) in October 2008 and then stabilizes from then up until the second quarter of 2011. The peak in 2008 is mainly due to the Lehman Brothers collapse.
The end of 2011 was characterized by a new explosion of the EURIBOR 3M - OIS 3M spread due to the interest rate remarkable decrease especially in the greek market.
From the third quarter of 2012 until now, the spread seemed to have stabilized again at around 10 bps.

This spread explosion is a consequence of the difference in the credit and liquidity risk intrinsic to both Euribor and Eonia rates.
As a matter of fact, the level of credit and liquidity risk in the Euribor rate is higher than in Eonia. Eonia being an overnight rate calculated from the overnight transactions, it has a low level of risk, because these transactions can be considered risk-free as changes during an overnight period are relatively weak, and the probability of default of one of the counterparties (or both) engaged in such transactions tends to zero.
That's why OIS rates are considered to be the best proxies for risk-free rates, for the time being.
On the other side, the credit and liquidity risk components of the Ibor (in general) have increased during the crisis, which was also observable throught the explosion of the Credit Default Swaps (CDS) describing a general increase in the default risk (credit crunch). These CDS swaps allowed counterparties to protect themselves from the default of their counterparties.

The reason the spread increased is that banks among other institutions started having less confidence in proposing loans to each other as some of them became less creditworthy, which has led them to increase their loan interest rates amongst which, the Libor.

To understand this, we present a scenario where a bank A sells a bond to bank B. The bond has a probability p of defaulting. Bank B enters a CDS (buys the CDS from A) with bank A, which insures for B to receive the face value of the bond from A in case of default in exchange for the bond, which is worthless for B (see figure 2). The CDS, in case of default, pays the face value of the bond to B and nothing if no default occurs.

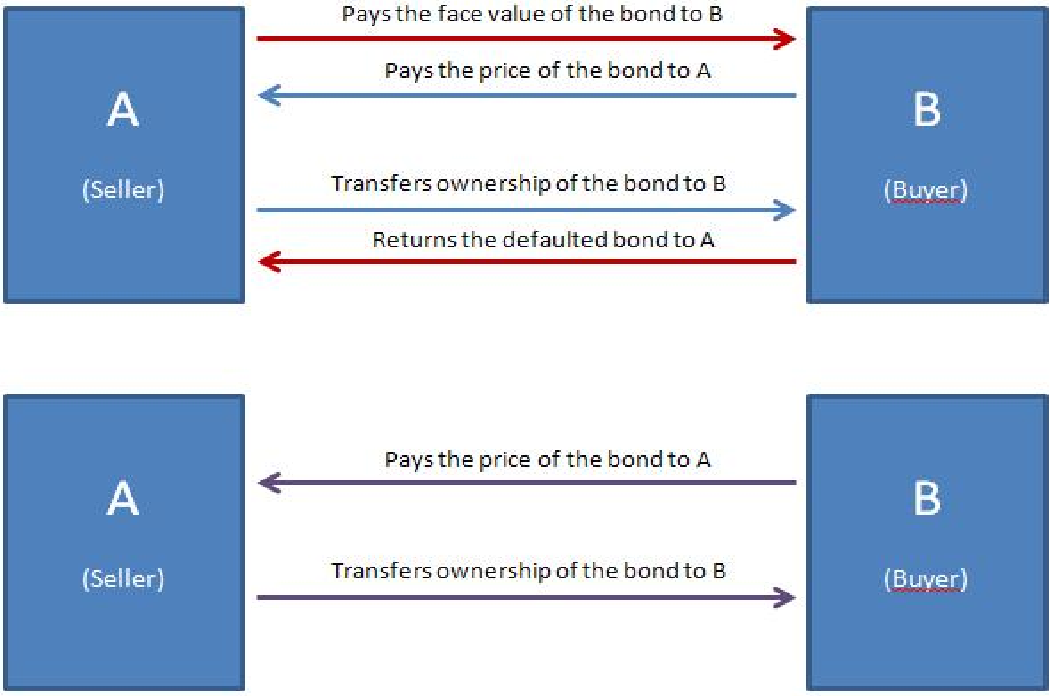


Figure 2: Up: Scenario of a bond exchange between A (seller) and B (buyer), where B also buys a CDS from A to protect itself from an eventual default.
Down: Scenario of a bond exchange between A (seller) and B (buyer), without CDS.
Red arrows correspond to the event of default (probability p) and the blue ones to the event of non default (probability $1-p$).

In our example, the bank giving the loan is bank B (B buys the bond by paying an amount which is equivalent to the loan at first and receiving interests), so if A defaults (or if the bond defaults), B won't receive its interest payments and thus will lose out of this transaction. If B enters a CDS with A then it can protect itself from A's default. This scenario was common during the period of the crisis and CDS swaps increased remarkably, expressing the low level of confidence between counterparties. As a consequence loans' interest rates increased as well to try compensate the high risk of default.

1.2 Market Segmentation

The divergence of the interest rate have led to a segmentation of the market into sub-areas corresponding to interest rates of different tenors. The value of the spread after the crisis doesn't allow to consider using a unique yield curve indifferently for all tenors as a correct practice.

As a matter of fact the idea of assigning a distinct yield curve to each different tenor happened to be wiser and more realistic according to the current context of the market.
Before this segmentation, the pricing of interest rate derivatives only needed one unique curve both for computing the forward rates and the discount factors used in the computation of the net present value (NPV) of the cash flows generated by the derivative of interest. This formerly

common practice cannot be used anymore due to the inconsistencies between interest rates and the fact that the pricing formulae should take the market segmentation into account.

The latter is what led to introduce the multi-curve framework.

Now, it is crucial to build one curve for the discounting of the future cash flows and distinct curves corresponding to the different tenors used for computing the forward rates. From now on, the discounting curve and the forwarding (also called fixing or funding) curves will be noted C_d and C_m respectively, where $m \in \{m_1, m_2, ..., m_k\}$ the different relevant tenors.

Before we get to the curve construction, we start by introducing the multi-curve framework and the modifications it brings to the pricing of interest rate derivatives.

1.3 Pricing of Interest Rate Derivatives in the Multi-Curve Framework

Recall that the principle of the multi-curve framework is to define a unique curve dedicated to discounting the future cash flows of a derivative and multiple curves for funding, each corresponding to a different rate tenor. Let's start by presenting the fundamental quantity of the multi-curve setting.

1.3.1 The FRA rate

The FRA rate is the rate of a Forward Rate Agreement, which is a contract between two counterparties in which they agree on applying a certain rate during a certain period of time starting from a predetermined date in the future.
Let T_1 and T_2 define a future period $[T_1, T_2]$ and t be the present date. The FRA rate, $FRA(t; T_1, T_2)$, can be seen as the fixed rate to be exchanged at T_2 for the Libor rate $L(T_1, T_2)$ so that the t-value of the corresponding "one cash flow" swap is zero.

Denote by $\Phi_{Swap}(t)$ the t-price of this swap. Under the $Q_d^{T_2}$ measure (whose numeraire is $P_d(t, T_2)$ the discount factor from t to T_2 related to the discount curve) the process $\{\Phi_{Swap}(t)/P_d(t, T_2)\}$ is a martingale, so the condition above is expressed by:

$$\frac{\Phi_{Swap}(t)}{P_d(t, T_2)} = \mathbb{E}_t^{Q_d^{T_2}}\left[\frac{\Phi_{Swap}(T_2)}{P_d(T_2, T_2)}\right] = 0$$

where $\Phi_{Swap}(T_2)$ is the payoff of the swap at T_2 i.e. $\Phi_{Swap}(T_2) = N\omega(L(T_1, T_2) - FRA(t; T_1, T_2))$ and $P_d(T_2, T_2) = 1$.
So

$$N\omega\mathbb{E}_t^{Q_d^{T_2}}\left[L(T_1, T_2) - FRA(t; T_1, T_2)\right] = 0,$$

where N is the notional amount of the swap and $\omega = \pm 1$ depending on the counterparty side.

or

$$\mathbb{E}_t^{Q_d^{T_2}}\left[L(T_1,T_2)\right] - \mathbb{E}_t^{Q_d^{T_2}}\left[FRA(t;T_1,T_2)\right] = 0,$$

and since $FRA(t;T_1,T_2)$ is known at time t, it is then $\mathcal{F}_t$-measurable so,

$$FRA(t;T_1,T_2) = \mathbb{E}_t^{Q_d^{T_2}}\left[L(T_1,T_2)\right].$$

The Libor rate $L(T_1,T_2)$ is defined by:

$$L(T_1,T_2) = \frac{1}{T_2-T_1}\left(\frac{1}{P_d(T_1,T_2)} - 1\right). \tag{1}$$

In the single-curve framework where only one curve was used for all purposes (discounting and funding), the forward rate $F_d(t;T_1,T_2) = \frac{1}{T_2-T_1}\left(\frac{P_d(t,T_1)}{P_d(t,T_2)} - 1\right)$ corresponding to the period $[T_1,T_2]$ used to coincide with the FRA rate $FRA(t;T_1,T_2)$ because:

$$FRA(t;T_1,T_2) = \mathbb{E}_t^{Q_d^{T_2}}\left[L(T_1,T_2)\right] = \mathbb{E}_t^{Q_d^{T_2}}\left[F_d(T_1;T_1,T_2)\right] = F_d(t;T_1,T_2),$$

since $F_d(t;T_1,T_2)$ is a martingale under $Q_d^{T_2}$.

In the multi-curve framework $L(T_1,T_2) \neq F_d(T_1;T_1,T_2)$ since the discount factors in (1) are not the ones corresponding to the discount curve. The discount factors to be used will now depend on the underlying tenor of the Libor and thus the fixings will be calculated from the corresponding forward curve and not from the discount one.

In this new framework, the equality $FRA(t;T_1,T_2) = F_d(t;T_1,T_2)$ won't hold anymore but a spread appears between the two quantities.

Furthermore we have:

$$\mathbb{E}_t^{Q_d^{T_2}}\left[FRA(T_1;T_1,T_2)\right] = \mathbb{E}_t^{Q_d^{T_2}}\left[\mathbb{E}^{Q_d^{T_2}}\left[L(T_1,T_2)|\mathcal{F}_{T_1}\right]\right]$$

and since $L(T_1,T_2)$ is known at time T_1 then

$$\mathbb{E}_t^{Q_d^{T_2}}\left[FRA(T_1;T_1,T_2)\right] = \mathbb{E}_t^{Q_d^{T_2}}\left[L(T_1,T_2)\right] = FRA(t;T_1,T_2).$$

This means that under $Q_d^{T_2}$, $FRA(t;T_1,T_2)$ is a martingale.
This is an important result that will be of precious use in the pricing of interest rate derivatives in the multi-curve framework.

1.3.2 The No-Arbitrage Pricing Formula

We have just shown that the FRA rate is a martingale under the forward measure.
Let $\Phi(T)$ the payoff of an interest rate derivative at a maturity time T. The arbitrage-free price of the derivative at time t is given by:

$$\Phi(t) = P_d(t,T)\mathbb{E}_t^{Q_d^T}[\Phi(T)], \tag{2}$$

where $P_d(t,T) = \mathbb{E}_t^{Q}[e^{-\int_t^T r_u du}]$ (Q is the risk-neutral measure) and $P_d(t,T) = e^{-\int_t^T r_u du}$ if the short collateral rate r_u is deterministic.

The payoff $\Phi(T)$ is generally a function of the Libor rate L, so the expectation of the Libor rate will lead to the appearance of the FRA rate in the pricing formula.

We consider the collateral rate to be the OIS rate when dealing with collateralized instruments, so the discounting will be performed using the OIS discount curve: $P_d(t,T) = e^{-\int_t^T r_{OIS}(u)du}$.

1.3.3 Pricing Formulae for Plain Vanilla Instruments

In this section, we derive the pricing formulae for the plain vanilla interest rate derivatives used in the construction of yield curves in the multi-curve framework, namely deposits and swaps.

Remark:
Nowadays, the instruments traded in the market are mostly collateralized, meaning that the transaction between two counterparties is written under collateral agreement (CSA) as explained in the "Overview". For that purpose, the discounting is mostly performed using the OIS curve whose construction is explained in section 1.4.1. Our aim is not to discuss the collateral management, that is why we position ourselves in a market where all trades are made under perfect collateral (Section (1.4.1)).

- **Deposits:**

  A deposit is a zero coupon contract where a counterparty A (lender) lends a nominal N at T_0 to a counterparty B (borrower), which at maturity T, pays the notional amount back to the lender as well as the interest accrued over the period $[T_0, T]$ at a simply compounded rate $R_m(T_0, T)$ fixed at a date $T_F \leq T_0$ and of tenor m corresponding to the time interval $[T_0, T]$ (i.e. $m = T - T_0$).
  Remark: A deposit is not a collateralized contract, so the discounting doesn't involve the OIS curve and instead uses the forward curve of tenor m.

  The payoff of the deposit at maturity T from the lender's point view is given by:
  $$\Phi_{Deposit}(T) = N(1 + R_m(T_0, T)\gamma(T_0, T)),$$
  where $\gamma(T_0, T)$ is the day count fraction corresponding to the period $[T_0, T]$.
  The price at a time $t \in [T_F, T]$ is obtained by using the no-arbitrage argument and the fact that under the probability measure Q_m^T whose numeraire is $P_m(t, T)$ (discount factor based on the same rate tenor m as the deposit rate's), the process $\{\frac{\Phi_{Deposit}(t)}{P_m(t,T)}\}$ is a martingale. So:
  $$\frac{\Phi_{Deposit}(t)}{P_m(t,T)} = \mathbb{E}_t^{Q_m^T}\left[\frac{\Phi_{Deposit}(T)}{P_m(T,T)}\right]$$
  and because the rate R_m is already fixed and known at time t, $\{\Phi_{Deposit}(T)\}$ is $\mathcal{F}_t-$measurable so
  $$\Phi_{Deposit}(t) = P_m(t,T)\mathbb{E}_t^{Q_m^T}\left[\Phi_{Deposit}(T)\right] = NP_m(t,T)(1 + R_m(T_0,T)\gamma(T_0,T)).$$
  Moreover, as the initial amount invested in the deposit is N, by no-arbitrage agrument, the following relation holds:
  $$\Phi_{Deposit}(t) = N = NP_m(t,T)(1 + R_m(T_0,T)\gamma(T_0,T)). \tag{3}$$

- **Interest Rate Swaps (IRS):**

  A swap is a contract where two counterparties agree to exchange two sets of cash flows (generated by two legs) in the same currency, typically one of the two legs is indexed (tied to) a floating rate, particularly Euribor and the second one is a fixed rate K determined at the agreement date.
  Let $\Omega_T = \{T_0, T_1, ..., T_n\}$ be the cash flow schedule of the floating leg and $\Omega_S = \{S_0, S_1, ..., S_{n'}\}$ that of the fixed leg (with rate K), where $S_0 = T_0$ is the settlement date of the contract and also the date where the first Euribor rate $L_m(T_0, T_1)$ (m is the tenor of the underlying Euribor) is fixed but payed only at T_1. n and n' are the numbers of floating and fixed payments of the swap, respectively.
  Figure (3) shows the swap cash flow diagram:

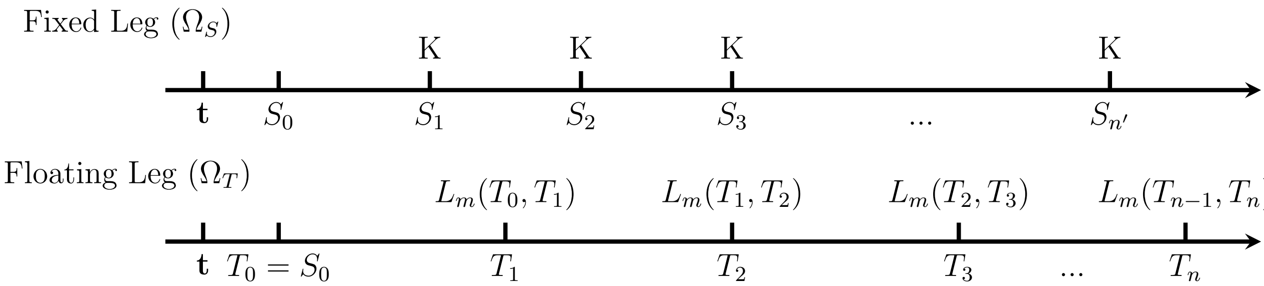


Figure 3: IRS cash flow diagram.

  The price of the IRS at t, $T_0 \le t \le T$ (T is the maturity of the swap) is given by (ref. equation (2)):

$$\Phi_{IRS}(t) = N\omega \left[\sum_{i=1}^{n} P_d(t, T_i)\gamma_{float}(T_{i-1}, T_i)\mathbb{E}_t^{Q_d^{T_i}}\left[L_m(T_{i-1}, T_i)\right] - \sum_{j=1}^{n'} K P_d(t, S_j)\gamma_K(S_{j-1}, S_j) \right]$$

  Where each payment of each leg is discounted at time t. So

$$\Phi_{IRS}(t) = N\omega \left[R_{IRS}(t, \Omega_T, \Omega_S) - K\right] A_d(t, \Omega_S), \tag{4}$$

  where

$$R^{IRS}(t, \Omega_T, \Omega_S) = \frac{\sum_{i=1}^{n} P_d(t, T_i) FRA_m(t, T_{i-1}, T_i)\gamma_{float}(T_{i-1}, T_i)}{A_d(t, \Omega_S)}, \tag{5}$$

$$A_d(t, \Omega_S) = \sum_{j=1}^{n'} P_d(t, S_j)\gamma_K(S_{j-1}, S_j),$$

  Here $P_d(t, T_i)$ is the discount factor.
  N is the notional, $\omega = 1$ if the fixed leg is payed and $\omega = -1$ if the floating leg is payed, $FRA_m(t, T_{i-1}, T_i) = \mathbb{E}_t^{Q_d^{T_i}}\left[L_m(T_{i-1}, T_i)\right]$ is the FRA rate of tenor m corresponding to $[T_{i-1}, T_i]$, $\gamma_{float}(T_{i-1}, T_i)$ and $\gamma_K(S_{j-1}, S_j)$ are the year fractions corresponding to the floating and fixed payment periods $[T_{i-1}, T_i]$ ($i \in \{0, 1, ..., n\}$) and $[S_{j-1}, S_j]$ ($j \in \{0, 1, ..., n'\}$), respectively.

Remark: The calculation of the discount factors P_d appearing in the formulae above is explained in the following section.

1.4 Curve Construction in the Multi-Curve Framework

First, we present the algorithm of the pricing procedure in the multi-curve framework, then we explain the yield curve construction procedure for the discounting and forward curves:

1. Construct C_d by selecting a range of most tradable and more liquid vanilla interest rate instruments with different underlying tenors (i.e. underlying interest rates of different tenors), then use a bootstrapping procedure to compute the discount factors,

2. Construct the C_m's curves using proper selections of vanilla interest rate derivatives whose underlying XIBOR rate belong to a specific tenor category and a bootstrapping procedure. Typically, we cannot build a forwarding curve using instruments with different underlying rate tenors,

3. Use each forwarding curve C_m to compute the FRA rates with tenor m for each coupon $i \in \{1, ..., n\}$:

$$FRA_m(t; T_{i-1}, T_i) = \frac{1}{\gamma_m(T_{i-1}, T_i)} \left(\frac{P_m(t, T_{i-1})}{P_m(t, T_i)} - 1 \right), t \in [T_{i-1}, T_i)$$

   where P_m's are the discount factors from the forward curve of tenor m and $\gamma_m(T_{i-1}, T_i)$ is the day count fraction between T_{i-1} and T_i.

4. Compute the cash flows c_i as the expectation of the corresponding coupon payoff Φ_i with respect to the discounting T_i-forward measure $Q_d^{T_i}$ associated to the numeraire $P_d(t, T_i)$ (t-zero coupon price maturing at time T_i or discount factor), taken from the discount curve:

$$c_i(t, T_i) = \mathbb{E}_t^{Q_d^{T_i}}[\Phi_i],$$

5. Use C_d to compute the discount factor $P_d(t, T_i)$,

6. Compute the t price $\phi(t)$ of the derivative of interest such that:

$$\phi(t) = \sum_{i=1}^{m} P_d(t, T_i) \mathbb{E}_t^{Q_d^{T_i}}[\Phi_i].$$

The bootstrapping of the forwarding curves used in the Multi-Curve framework is also called **dual-bootstrapping** since it involves the discount factors computed from the discount curve i.e. each forwarding curve is built using the unique discount curve.

As mentioned in the algorithm above, a set of plain vanilla interest rate derivatives called **stripping instruments** is selected to construct the yield curves.

The choice of plain vanilla instruments is crucial because of the their simplicity in terms of pricing and bootstrapping, plus they usually are very liquid and widely traded in the interest rate market.

Depending on the range of maturities (pillars) of the curve to construct we use:

- **Deposits for pillars up to 1Y:**
  For curve construction, we use deposit rate quotations depending on the nature of the curve we would like to create. For example for an IBOR discount curve, we use the IBOR rate (Euribor, Libor, etc ...) such that $R_m(T_0, T_i) = L_m(T_0, T_i)$ to compute the discount factor corresponding to the pillar T_i of the curve. From equation (3), one gets:
  $$P_m(T_0, T_i) = \frac{1}{1 + R_m(T_0, T_i)\gamma(T_0, T_i)}.$$
  Then from these discount factors, one can easily compute the FRA rates using the formula mentioned in step 3 of the Multi-Curve algorithm above.

- **Interest rate swaps (IRS) for different tenors, from 1Y up to higher ones such as 30Y or more:**
  Equation (5) in the previous section represents a bootstrapping equation from which the discount factors $P_d(t, T_i)$ can be computed recursively. In the example of the OIS discount curve proposed below, we detail the procedure.

The choice of the instruments used in constructing the forwarding curves must be done such that their underlying tenor corresponds to that of the curve i.e. to build C_f^{3M}, only instruments with underlying rate tenor 3M must be used (stripped).

The bootstrapping procedure consists of deducing the rates for each maturity (or pillar) from the market quotes (deposit rate or swap rate) of the instruments used. Sometimes some maturities are missing as the corresponding rates are not quoted in the market, so to complete the curve construction, one performs an interpolation to obtain the rates for the missing maturities.

In the bootstrapping of the forwarding curves, the discount factors taken from the discount curve are used. This is why this procedure is called **dual bootstrapping** and it is in fact the most important feature of the Multi-Curve framework.

1.4.1 Example: The OIS Discount Curve

A crucial starting point for the Multi-Curve framework, is the discount curve. This curve has to be correctly constructed because not only is it used to discount the future cash flows generated by the derivative but it intervenes in the construction of the forward curves as well. Since we are interested in transactions written under a collateral agreement or in other words, in riskless transactions, and since we have adopted the overnight rate (Eonia for EUR) as the risk-free rate, we will construct the discount curve from plain vanilla interest rate instruments indexed on the overnight rate. Such a curve is called **OIS discount curve**.

The principal instrument used for the bootstrapping of the OIS discount curve is the **"Overnight Index Swap"**.

The OIS is a plain vanilla interest rate swap exchanging a fixed rate and the daily compounded overnight rate. Let $\Omega_T = \{T_0, T_1, ..., T_n\}$ be the cash flow schedule of the floating leg and $\Omega_S = \{S_0, S_1, ..., S_{n'}\}$ that of the fixed rate K, where $S_0 = T_0$ is the settlement date of the contract. The frequency of the floating leg is not daily but is considered to be the same as the frequency of the fixed leg. n and n' are the numbers of floating and fixed payments of the OIS, respectively.

The schedule of the floating leg is decomposed into daily periods i.e.

$$[T_{i-1}, T_i] = \bigcup_{k=1}^{n_i} (T_{i,k-1}, T_{i,k}],$$

where n_i is the number of daily overnight payments in the period $[T_{i-1}, T_i]$. Let $R_{ON}(T_i, \mathbf{T_i})$ ($\mathbf{T_i} = \{T_{i,0}, ..., T_{i,n_i}\}$ the overnight payment schedule for the i-th period) be the coupon rate compounded from overnight rates over the i-th period $[T_{i-1}, T_i]$.

Figure 4 displays the diagram of the OIS cash flows.

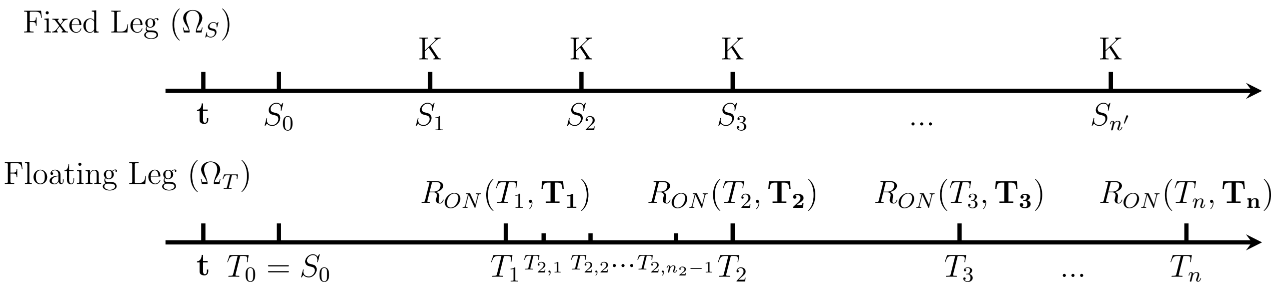


Figure 4: OIS cash flow diagram.

We have:

$$1 + R_{ON}(T_i, \mathbf{T_i})\gamma_{ON}(T_{i-1}, T_i) = \prod_{k=1}^{n_i}[1 + R_{ON}(T_{i,k-1}, T_{i,k})\gamma_{ON}(T_{i,k-1}, T_{i,k})],$$

or

$$R_{ON}(T_i, \mathbf{T_i}) = \frac{1}{\gamma_{ON}(T_{i-1}, T_i)} \left\{ \prod_{k=1}^{n_i}[1 + R_{ON}(T_{i,k-1}, T_{i,k})\gamma_{ON}(T_{i,k-1}, T_{i,k})] - 1 \right\}$$

The OIS price (supposing the fixed leg being payed) at t, $T_0 \leq t \leq T_{OIS}$ (T_{OIS} is the maturity of the OIS) is given by:

$$\Phi_{OIS}(t) = N\omega \left(R^{OIS}(t, \Omega_T, \Omega_S) - K \right) A_c(t, \Omega_S),$$

where

$$R^{OIS}(t, \Omega_T, \Omega_S) = \frac{\sum_{i=1}^{n} P_c(t, T_i) R_{ON}(t, \mathbf{T_i}) \gamma_{ON}(T_{i-1}, T_i)}{A_c(t, \Omega_S)}, \tag{6}$$

$$A_c(t, \Omega_S) = \sum_{j=1}^{n'} P_c(t, S_j) \gamma_K(S_{j-1}, S_j),$$

Here $P_c(t,T_i)$ is the discount factor under collateral using the risk free Eonia rate. N is the notional in EUR (for an OIS on underlying Eonia), $R_{ON}(t,\mathbf{T_i})$ is the FRA overnight rate Eonia corresponding to the i-th period $[T_{i-1},T_i]$, $\gamma_{ON}(T_{i-1},T_i)$ and $\gamma_K(S_{j-1},S_j)$ are the year fractions corresponding to the floating and fixed payment periods $[T_{i-1},T_i]$ $(i\in\{0,1,...,n\})$ and $[S_{j-1},S_j]$ $(j\in\{0,1,...,n'\})$, respectively.

Moreover the FRA overnight rate can be rewritten as follows:

$$
\begin{aligned}
R_{ON}(t,\mathbf{T_i}) &= \mathbb{E}_t^{Q_d^{T_i}}\left[R_{ON}(T_i,\mathbf{T_i})\right] \\
&= \frac{1}{\gamma_{ON}(T_{i-1},T_i)}\mathbb{E}_t^{Q_d^{T_i}}\left\{\prod_{k=1}^{n_i}[1+R_{ON}(T_{i,k-1},T_{i,k})\gamma_{ON}(T_{i,k-1},T_{i,k})]-1\right\} \\
&= \frac{1}{\gamma_{ON}(T_{i-1},T_i)}\left\{\prod_{k=1}^{n_i}\left[1+\mathbb{E}_t^{Q_d^{T_{i,k}}}[R_{ON}(T_{i,k-1},T_{i,k})]\gamma_{ON}(T_{i,k-1},T_{i,k})\right]-1\right\} \\
&= \frac{1}{\gamma_{ON}(T_{i-1},T_i)}\left\{\prod_{k=1}^{n_i}\left[1+\underbrace{R_{ON}(t;T_{i,k-1},T_{i,k})}_{=\mathbb{E}_t^{Q_d^{T_{i,k}}}[R_{ON}(T_{i,k-1},T_{i,k})]}\gamma_{ON}(T_{i,k-1},T_{i,k})\right]-1\right\}
\end{aligned}
$$

Where we used the tower property for the conditioned expectations.
We consider a scenario where the contracts considered are under **perfect collateral**, which means the FRA overnight rates $R_{ON}(t,T_{i,k-1},T_{i,k})$ can be replicated using the collateral zero coupon bond prices $P_c(t,.)$, i.e.

$$R_{ON}(t,T_{i,k-1},T_{i,k}) = \frac{1}{\gamma_{ON}(T_{i,k-1},T_{i,k})}\left(\frac{P_c(t,T_{i,k-1})}{P_c(t,T_{i,k})}-1\right).$$

Thus we have,

$$R_{ON}(t,\mathbf{T_i}) = \frac{1}{\gamma_{ON}(T_{i-1},T_i)}\left[\frac{P_c(t,T_{i-1})}{P_c(t,T_i)}-1\right]$$

so

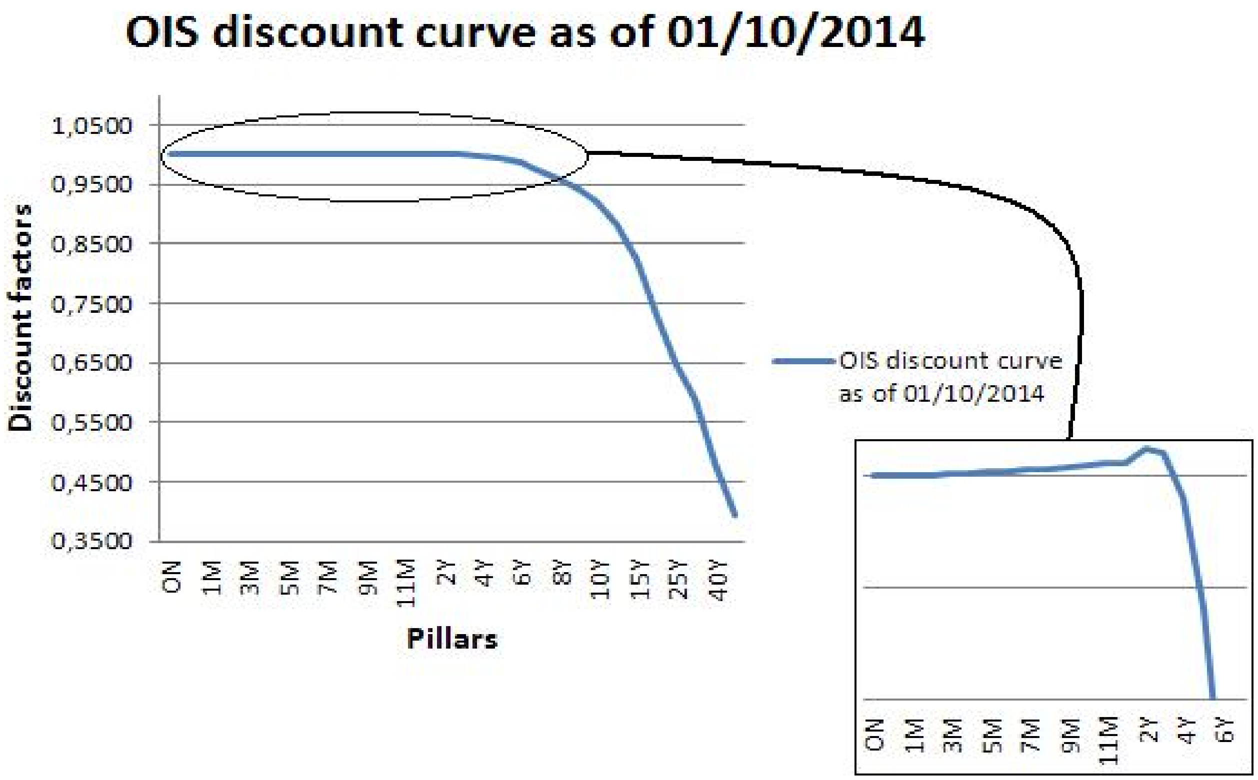
$$P_c(t,T_i)R_{ON}(t,\mathbf{T_i})\gamma_{ON}(T_{i-1},T_i) = P_c(t,T_{i-1})-P_c(t,T_i),$$

so the nominator of (6) is then a simple telescopic sum, which simplifies into

$$P_c(t,T_0)-P_c(t,T_n).$$

Using market quotes for $R^{OIS}(t,\Omega_T,\Omega_S)$ and equation (6) above, we obtain a recursive equation for the discount factors P_c. Computing all of them allows to create the OIS discount curve for a reference date $t\geq T_0$.

Figure 5 shows the curve obtained after performing the bootstrapping in the library using DCL market quotes for pillars (ON until 10Y, 12Y, 15Y, 20Y, 25Y, 30Y, 40Y and 50Y) and then using linear interpolation on the zero rates for the missing pillars, where the reference date chosen is 01/10/2014. The instruments used in the bootstrapping of this OIS curve are OIS swaps for maturities from 1M to 50Y and an overnight deposit for maturity ON.



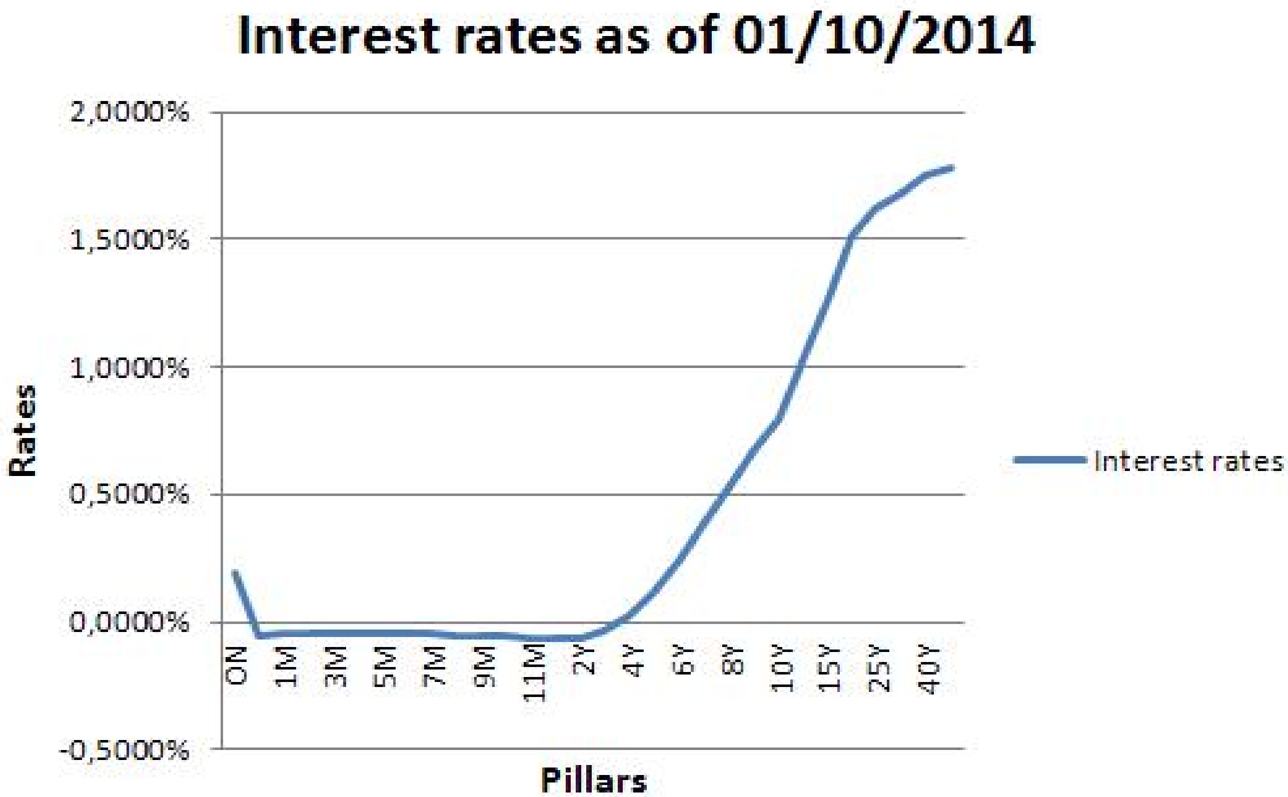


Figure 5: Up: OIS discount curve as of 01/10/2014, linear interpolation on the zero rates, down: Quoted interest rates.

1.4.2 Example: Pricing of an IRS in the Multi-Curve Setting

We propose to detail the pricing procedure of an interest rate swap where we apply the derived formula (4).
We propose to price the swap whose details are given in table 1.
The pricing date chosen is 30/09/2014 and it is also the reference date for the yield curves.
The detailed calculations are given in table 2, where of course only the cash flows occuring after 30/09/2014 are taken into account. The cash flows are computed following formula (4).

Start Date	30/09/2008
End Date	30/03/2019
RECEIVED FIXED LEG	
Frequency	6M
Accrual Basis	30E/360
Currency	EUR
Fixed Rate	4.58%
Notional (actual)	6.000.000 EUR (amortized)
Discount Curve	**OIS**
PAYED FLOATING LEG	
Frequency	3M
Accrual Basis	A360
Currency	EUR
Margin	0.24%
Notional	6.000.000 EUR
Discount Curve	**OIS**
Projection Curve	**EURIBOR FWD 3M**

Table 1: Characteristics of the swap to be priced.

Floating Leg					
Payment Dates	**Notional Base**	**Rate**	**Cash flow**	**Discount Factor**	**NPV**
30/09/2014	- 4 981 115.04	0.2090%	- 5 715.55	1.0000000	- 5 715.55
30/12/2014	- 4 875 282.57	0.0820%	- 3 968.21	0.9998090	- 3 967.45
30/03/2015	- 4 875 282.57	0.0875%	- 3 877.20	0.9995860	- 3 875.59
30/06/2015	- 4 767 026.54	0.0801%	- 3 741.71	0.9993850	- 3 739.41
30/09/2015	- 4 767 026.54	0.0818%	- 3 689.02	0.9991700	- 3 685.96
30/12/2015	- 4 656 291.45	0.0935%	- 3 690.47	0.9989330	- 3 686.53
30/03/2016	- 4 656 291.45	0.1176%	- 3 811.97	0.9986360	- 3 806.77
30/06/2016	- 4 543 020.52	0.1385%	- 4 120.37	0.9982830	- 4 113.30
30/09/2016	- 4 543 020.52	0.1668%	- 4 351.49	0.9978580	- 4 342.17
30/12/2016	- 4 427 155.69	0.2098%	- 4 818.15	0.9973290	- 4 805.28
30/03/2017	- 4 427 155.69	0.2336%	- 5 034.12	0.9967470	- 5 017.74
30/06/2017	- 4 308 637.56	0.2567%	- 5 263.24	0.9960930	- 5 242.68
29/09/2017	- 4 308 637.56	0.2800%	- 5 459.65	0.9953890	- 5 434.47
29/12/2017	- 4 187 405.36	0.3372%	- 6 699.09	0.9945410	- 6 662.52
29/03/2018	- 4 187 405.36	0.3666%	- 7 081.60	0.9936300	- 7 036.49
29/06/2018	- 4 063 396.94	0.3952%	- 7 444.03	0.9926280	- 7 389.15
28/09/2018	- 4 063 396.94	0.4239%	- 7 780.23	0.9915650	- 7 714.61
31/12/2018	- 3 936 548.73	0.5173%	- 9 407.57	0.9902260	- 9 315.62
29/03/2019	- 3 936 548.73	0.5573%	- 9 392.08	0.9888790	- 9 287.63
Total					**- 104 838.92**

Fixed Leg					
Payment Dates	**Notional Base**	**Rate**	**Cash flow**	**Discount Factor**	**NPV**
30/09/2014	4 981 115.04	4.58%	114 067.53	1.0000000	114 067.53
30/03/2015	4 875 282.57	4.58%	111 643.97	0.9995860	111 597.75
30/09/2015	4 767 026.54	4.58%	109 164.91	0.9991700	109 074.25
30/03/2016	4 656 291.45	4.58%	106 629.07	0.9986360	106 483.65
30/09/2016	4 543 020.52	4.58%	104 035.17	0.9978580	103 812.28
30/03/2017	4 427 155.69	4.58%	101 381.87	0.9967470	101 052.08
02/10/2017	4 308 637.56	4.58%	98 667.80	0.9953640	98 210.42
03/04/2018	4 187 405.36	4.58%	95 891.58	0.9935780	95 275.75
01/10/2018	4 063 396.94	4.58%	93 051.79	0.9915290	92 263.55
01/04/2019	3 936 548.73	4.58%	90 146.97	0.9888310	89 140.15
Total					**1 020 977.41**
		NPV	**916 138.49**		

Table 2: Pricing of the swap as of 30/09/2014.

The "Accrual Basis" gives the convention used in computing the year fractions (day count convention) of the payment periods. Here the convention is A360 (Actual/360) for the floating leg, meaning for a period $[T_1, T_2]$ for example, the corresponding year fraction is computed as: "actual number of days between date T_1 and T_2"/360.
The convention for the fixed leg is "30E/360", meaning that for a period $[T_1, T_2]$, such that date T_i corresponds to Y_i years, M_i months and D_i days for $i \in \{1, 2\}$, the corresponding year fraction is given by:

$$\gamma(T_1, T_2) = \frac{360 \times (Y_2 - Y_1) + 30 \times (M_2 - M_1) + (D_2 - D_1)}{360},$$

and if $D_1 = 31$ or $D_2 = 31$ then we change them to 30.
The "Margin" adds up to the FRA rate in the computation of the cash flows, i.e. at each payment date, the rate payed is the FRA rate corresponding to this date plus the margin.

Here, we have considered a swap whose notional is amortized. It doesn't change anything in the pricing procedure but at each payment, we have a different notional amount. For the floating leg which is tied to a Euribor 3M, the column "Rate" contains the FRA rates computed from the 3M Euribor forward curve constructed as of 30/09/2014 as explained in step 3 of the algorithm detailed in section 1.4. Also the discount factors correspond to the OIS discount curve as of 30/09/2014.

2 Modeling the Stochastic Basis Spread

As previously introduced, the current market situation has forced practitionners to rethink the pricing procedures of interest rate derivatives, which now should take into account the widening of spreads between interest rates, mainly the Libor - OIS spread.

In this study, we focus on plain vanilla interest rate caplets, which are widely liquid instruments most commonly used for model calibration because of their simplicity in terms of structure and pricing i.e. sometimes, even when considering complicated scenarios, for example, in case of a stochastic spread, one can derive closed formulae for these plain vanilla instruments.

Our goal in this section, is to price the previously mentioned derivatives in a multi-curve setting with a stochastic basis spread. Depending on the choice we make for the model, we can either derive a closed formula or if this happens to be too complicated, use Monte Carlo simulations to compute the price.
The reason we chose plain vanilla instruments is to be able to collect market data to benchmark our model. In other words, the market prices of these instruments will allow us to assess the efficiency and accuracy of the model used. Moreover, even though the instruments considered are simple ones, we might have lenghty formulae. The advantage of having closed formulae for plain vanilla interest rate derivatives is that it allows to easily calibrate richer models to price more complicated products such as exotic derivatives. Again the choice of simple instruments for calibration is justified by their liquidity in the market and thus the availability of data to use for the process.

First, we start by introducing some important features of the basis spread.

When dealing with modeling problems, one has to first grasp the characteristics of the quantity to be modelled. One should favor simple and tractable models over complicated ones. A model should allow flexibility and the possibility to be modified afterwards because the market is in constant change and this forces to constently update the models.
In our case, we would like to choose a model that allows to replicate (or approaches) the behaviour of the spread that is viewed in the market.

2.1 The positivity of the Libor - OIS spread

As introduced in section 1.1, the Libor-OIS spread appeared from august 2007 and increased considerably to reach its peak in October 2008 to then stabilize at a lower but non negligible level.

Historically, the Libor-OIS spread has (almost) always been positive.
This can be explained by the fact that ever since the credit crunch, the Libor rate moved from being a riskless interet rate to holding a high risk feature, as opposed to the OIS rate, which is tied to the riskless Eonia rate. This divergence in terms of credit risk between the two rates that used to hover around each other before the crisis, is mainly due to the increase of the level

of the Libor rate right after the crisis, enough to separate from the OIS rate. Then after its peak in 2008, the Libor rate decreased to stabilize at a lower level but still has never matched the OIS rate back again. When comparing two interest rates in terms of their inherent risk, the riskier rate is generally higher.

Summing up, in terms of credit risk, the spread Libor-OIS is in fact supposed to be positive but in terms of liquidity risk, we cannot say that for sure but historically speaking, the spread has almost always been positive.

2.2 Choice of The Model for the Stochastic Basis Spread

As mentioned in the begining of this section, our aim is to select a model for the spread that can easily lead to closed formulae and if not, can easily be used in simulations.

In this study, we have chosen to focus on a model for the spread that also has the advantage of insuring its positivity.

The model we have selected is the Libor Market Model (LMM), which was extended to include a stochastic basis spread by F.Mercurio [6].
The reason we chose this model is because it allows to derive closed formulae for caplets that are not very complicated to implement and use for calibration. Moreover, in his article, F.Mercurio emphasizes the efficiency and tractability of this model and he managed to obtain good results for the set of data he used (2010). The idea here is to evaluate the validity of the model for recent data.

2.3 Context and Notations

Let us consider a tenor m for the Ibor and the corresponding term structure $\{T_0^m, T_1^m, ..., T_N^m\}$, such that $\forall k \in \{1, ..., N\}, T_k^m - T_{k-1}^m = m$.
In the multi-curve framework, $FRA(t, T_{k-1}^m, T_k^m) \neq F_d(t, T_{k-1}^m, T_k^m)$, where we note:

$$F_k^m(t) = F_d(t, T_{k-1}^m, T_k^m) = \frac{1}{\gamma_m(T_{k-1}^m, T_k^m)} \left(\frac{P_d(t, T_{k-1}^m)}{P_d(t, T_k^m)} - 1 \right),$$

and

$$L_k^m(t) = FRA(t, T_{k-1}^m, T_k^m).$$

In our case the forward rate $F_d(t, T_{k-1}^m, T_k^m)$ corresponds to OIS as we consider a discounting using the OIS curve.
The basis spread is then defined as the difference between the previous two quantities, which used to coincide before the crisis:

$$\forall t \geq T_0^m, S_k^m(t) = L_k^m(t) - F_k^m(t). \tag{7}$$

The Libor Market Model (LMM), which aimed at modeling the joint evolution of a set of consecutive forward Libor rates in its single-curve version, is now going to be extended to the multi-curve case by modeling the joint evolution of FRA rates for different tenors (L_k^m's) and

forward rates that correspond to the OIS discount curve (F_k^m's). From equation (7), we have three ways of extending the LMM to the multi-curve framework:
Either by modeling the joint dynamics of F_k^m and S_k^m, or F_k^m and L_k^m or L_k^m and S_k^m.
Since, we would like to capture a realistic behavior for the spread, which has historically almost always been positive, it seems natural to directly model the spread by choosing a suitable model ensuring its positivity. Moreover, in practice, it is common to build the Libor curves from the OIS one adding up a spread to it.
We will then model the evolution of S_k^m and F_k^m and obtain the joint evolution of the Libor rates L_k^m.

The processes S_k^m and F_k^m are assumed to be independent for simplicity, but a correlation can be introduced between the two processes by modeling them with correlated Brownian motions.

Under the measure $Q_d^{T_k^m}$ whose numeraire is the zero coupon bond price $P_d(t, T_k^m)$ calculated from the OIS discount curve, F_k^m is a martingale and we have previously shown that the FRA rate L_k^m is a martingale under the same measure. Thus, S_k^m is a martingale under $Q_d^{T_k^m}$.

Before assuming any model for the spread and the forward OIS rates, we start by focusing on the pricing of interest rate caplets in this new framework.

2.4 Caplet Pricing

In this section, we propose to derive the pricing formula for caplets under the previous assumptions as proposed by F.Mercurio in his article [6], "Libor Market Models with Stochastic Basis". The advantage of choosing this version of the LMM model is that it allows to obtain a closed form expression for caplet price before even making a choice upon the dynamics of the spread and the forward OIS rates.

A **caplet** of maturity T_{k-1}^m is a call option indexed on a Libor rate of tenor m, $L_m(T_{k-1}^m, T_k^m)$, struck at K (with a strike K). $L_m(T_{k-1}^m, T_k^m)$ is fixed (determined) at T_{k-1}^m and payed at T_k^m.
The payoff of the caplet at maturity T_{k-1}^m is:
$$\gamma_m(T_{k-1}^m, T_k^m)\left[L_m(T_{k-1}^m, T_k^m) - K\right]^+. \tag{8}$$
See figure 6 below.

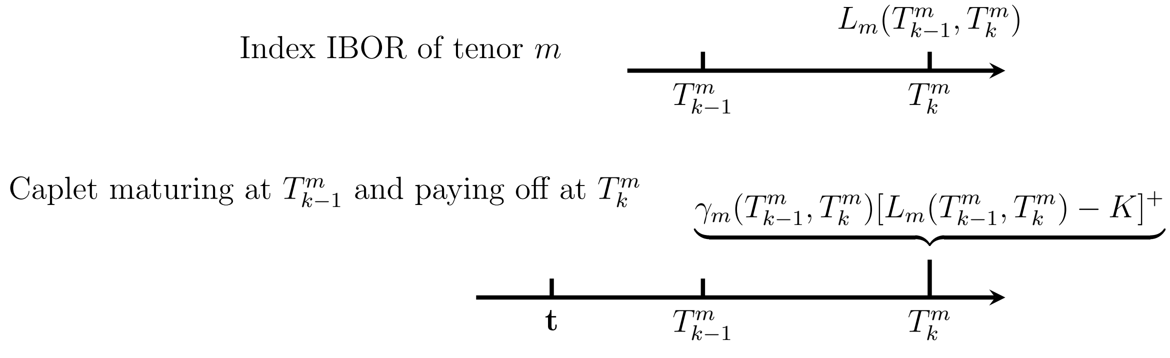


Figure 6: Caplet maturing at T_{k-1}^m and paying off at T_k^m.

Actually, the libor rate $L_m(T_{k-1}^m, T_k^m)$ is fixed two days prior to T_{k-1}^m, but **we have set the fixing lag to 0** instead of 2 for simplicity. It is important to make the distinction here between the maturity date of the caplet, which corresponds to the date T_{k-1}^m where the Libor is fixed and the payment date of the caplet. In fact, the caplet pays off at its maturity date T_{k-1}^m but the payment is conventionally made at the end of the rate period T_k^m.
We denote by $\Phi_C(t, K; T_{k-1}^m, T_k^m)$ the price of the caplet at time $t \leq T_{k-1}^m$.

$$\begin{aligned}
\Phi_C(t, K; T_{k-1}^m, T_k^m) &= \gamma_m(T_{k-1}^m, T_k^m) P_d(t, T_k^m) \mathbb{E}_t^{Q^{T_k^m}} \left[(L_m(T_{k-1}^m, T_k^m) - K)^+ \right] \\
&= \gamma_m(T_{k-1}^m, T_k^m) P_d(t, T_k^m) \mathbb{E}_t^{Q^{T_k^m}} \left[(L_k^m(T_{k-1}^m) - K)^+ \right] \\
&= \gamma_m(T_{k-1}^m, T_k) P_d(t, T_k^m) \mathbb{E}_t^{Q^{T_k^m}} \left[(S_k^m(T_{k-1}^m) + F_k^m(T_{k-1}^m) - K)^+ \right] \\
&= \gamma_m(T_{k-1}^m, T_k^m) P_d(t, T_k^m) \mathbb{E}_t^{Q^{T_k^m}} \left[\left(S_k^m(T_{k-1}^m) - (K - F_k^m(T_{k-1}^m)) \right)^+ \right] \\
&= \gamma_m(T_{k-1}^m, T_k^m) P_d(t, T_k^m) \mathbb{E}_t^{Q^{T_k^m}} \left\{ \mathbb{E}^{Q^{T_k^m}} \left[\left(S_k^m(T_{k-1}^m) - (K - F_k^m(T_{k-1}^m)) \right)^+ | \mathcal{F}_t \cup \mathcal{G} \right] \right\}
\end{aligned}$$

where $\mathcal{G}$ is the sigma algebra generated by $F_k^m(T_{k-1}^m)$ and we have $\mathcal{F}_t \subseteq \mathcal{F}_t \cup \mathcal{G}$.

Since F_k^m and S_k^m are independent, then:
$\mathbb{E}^{Q^{T_k^m}} \left[\left(S_k^m(T_{k-1}^m) - (K - F_k^m(T_{k-1}^m)) \right)^+ | \mathcal{F}_t \cup \mathcal{G} \right] = \mathbb{E}_t^{Q^{T_k^m}} \left[\left(S_k^m(T_{k-1}^m) - (K - F_k^m(T_{k-1}^m)) \right)^+ \right]$,
so:

$$\Phi_C(t, K; T_{k-1}^m, T_k^m) = \gamma_m(T_{k-1}^m, T_k^m) P_d(t, T_k^m) \int_{-\infty}^{+\infty} \mathbb{E}_t^{Q^{T_k^m}} \left[\left(S_k^m(T_{k-1}^m) - (K - y) \right)^+ \right] f_{F_k^m(T_{k-1}^m)}(y) dy,$$

where $f_{F_k^m(T_{k-1}^m)}$ is the density function of $F_k^m(T_{k-1}^m)$.

$$\begin{aligned}
\Phi_C(t, K; T_{k-1}^m, T_k^m) \quad &= \quad \gamma_m(T_{k-1}^m, T_k^m) P_d(t, T_k^m) \left\{ \int_{-\infty}^{K} \mathbb{E}_t^{Q^{T_k^m}} \left[\left(S_k^m(T_{k-1}^m) - (K - y) \right)^+ \right] f_{F_k^m(T_{k-1}^m)}(y) dy \right. \\
&+ \quad \left. \int_{K}^{+\infty} \mathbb{E}_t^{Q^{T_k^m}} \left[\left(S_k^m(T_{k-1}^m) - (K - y) \right)^+ \right] f_{F_k^m(T_{k-1}^m)}(y) dy \right\},
\end{aligned}$$

Here, we will make an assumption on the sign of S_k^m, which we suppose to be positive as it is the case historically. We will make sure to choose a positive stochastic process to represent the spread afterwards.

We have:

$$\begin{aligned}
\int_{K}^{+\infty} \mathbb{E}_t^{Q^{T_k^m}} \left[\left(S_k^m(T_{k-1}^m) - \underbrace{(K - y)}_{\leq 0} \right)^+ \right] f_{F_k^m(T_{k-1}^m)}(y) dy &= \int_{K}^{+\infty} \mathbb{E}_t^{Q^{T_k^m}} \left[S_k^m(T_{k-1}^m) - (K - y) \right] f_{F_k^m(T_{k-1}^m)}(y) dy \\
&= \int_{K}^{+\infty} \left[S_k^m(t) - (K - y) \right] f_{F_k^m(T_{k-1}^m)}(y) dy.
\end{aligned}$$

We set:

$$\Phi_C^S(t, K, T_{k-1}^m, T_k^m) = \gamma_m(T_{k-1}^m, T_k^m) P_d(t, T_k^m) \mathbb{E}_t^{Q^{T_k^m}} \left[(S_k^m(T_{k-1}^m) - K)^+ \right].$$

Hence,

$$\begin{aligned}\Phi_C(t,K;T^m_{k-1},T^m_k) &= \int_{-\infty}^{K} \Phi_c^S(t,K-y;T^m_{k-1},T^m_k) f_{F^m_k(T^m_{k-1})}(y)dy \\ &+ \gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\left\{(S^m_k(t)-K)\int_K^{+\infty} f_{F^m_k(T^m_{k-1})}(y)dy + \int_K^{+\infty} y f_{F^m_k(T^m_{k-1})}(y)dy\right\} \\ &= \int_{-\infty}^{K} \Phi_c^S(t,K-y;T^m_{k-1},T^m_k) f_{F^m_k(T^m_{k-1})}(y)dy \\ &+ \gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\left\{S^m_k(t)\int_K^{+\infty} f_{F^m_k(T^m_{k-1})}(y)dy + \int_K^{+\infty} [y-K] f_{F^m_k(T^m_{k-1})}(y)dy\right\}.\end{aligned}$$

We denote by $\Phi_C^F(t,K;T^m_{k-1},T^m_k)$ the price at time t of a caplet struck at K indexed to the forward OIS rate F^m. So:

$$\Phi_C^F(t,K;T^m_{k-1},T^m_k) = \gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\mathbb{E}_t^{Q^{T^m_k}}[(F^m_k(T^m_{k-1})-K)^+] = \int_{-\infty}^{+\infty} [y-K]^+ f_{F^m_k(T^m_{k-1})}(y)dy$$

Deriving with respect to K leads to:

$$\frac{\partial}{\partial K}\Phi_C^F(t,K;T^m_{k-1},T^m_k) = \gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\frac{\partial}{\partial K}\left\{\int_K^{+\infty}[y-K]f_{F^m_k(T^m_{k-1})}(y)dy\right\} = -\gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\int_K^{+\infty} f_{F^m_k(T^m_{k-1})}(y)dy \tag{9}$$

The demonstration of the last equality can be found in the annex.
Finally, we obtain:

$$\boxed{\Phi_C(t,K;T^m_{k-1},T^m_k) = \int_{-\infty}^{K} \Phi_C^S(t,K-y;T^m_{k-1},T^m_k) f_{F^m_k(T^m_{k-1})}(y)dy - S^m_k(t)\frac{\partial}{\partial K}\Phi_C^F(t,K;T^m_{k-1},T^m_k) + \Phi_C^F(t,K;T^m_{k-1},T^m_k) \quad (10)}$$

Remark:

If the spread is deterministic i.e. $S^m_k(T^m_{k-1}) = S^m_k(t) = s$, where s is a constant then the price of the caplet (8) becomes:

$$\begin{aligned}\Phi_C(t,K;T^m_{k-1},T^m_k) &= \gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\mathbb{E}_t^{Q^{T^m_k}}[(L_m(T^m_{k-1},T^m_k)-K)^+] \\ &= \gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\mathbb{E}_t^{Q^{T^m_k}}[(L^m_k(T^m_{k-1})-K)^+] \\ &= \gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\mathbb{E}_t^{Q^{T^m_k}}[(\underbrace{S^m_k(T^m_{k-1})}_{=s}+F^m_k(T^m_{k-1})-K)^+] \\ &= \gamma_m(T^m_{k-1},T^m_k)P_d(t,T^m_k)\mathbb{E}_t^{Q^{T^m_k}}\left[\left(F^m_k(T^m_{k-1})-(K-s)\right)^+\right].\end{aligned}$$

So

$$\Phi_C(t,K;T^m_{k-1},T^m_k) = \Phi_C^F(t,K-s;T^m_{k-1},T^m_k). \tag{11}$$

3 Case Study

In this section, we propose to study a specific example of interest rate caplet pricing where two specific dynamics are chosen for the evolution of the forward OIS rates and the spread. We propose to illustrate the example introduced by F.Mercurio [6], which assumes a shifted log-normal model for the forward OIS rates and SABR dynamics for the spread. It is important to emphasize that this choice is not random as it ensures the positivity S^m and the fact the forward OIS rates F^m can be negative, which is possible with a shifted log-normal model.

Using these models will allow to explicite the caplet pricing formula (10) further. The resulting formula will then be implemented and used for the calibration of the overall model using market data.

Remark: The data used is provided by Bloomberg and corresponds to caps' implied volatilities (for different strikes and maturities) and is provided in the annex. A bootstrapping procedure allows to compute the underlying caplet implied volatilities and thus the corresponding prices, as it will be shown later.

A Monte Carlo method will then be presented and used for a comparison purpose, in order to assess the accuracy of the formula, since the choice we will make on the spread dynamics can lead us to obtain an approximation of the price Φ_C^S only.
Finally, a calibration will be conducted to test how well the chosen model can represent market prices.

3.1 Dynamics for the Forwards OIS Rates and the Spread

As introduced above, the following dynamics will be considered for the $\{F_k^m(t)\}$ and $\{S_k^m(t)\}$ processes:

- **OIS forward rates: Shifted Log-Normal**

$$dF_k^m(t) = \sigma_k^m \left(F_k^m(t) + \frac{1}{\gamma_k^m} \right) dZ_k^m(t), \tag{12}$$

where γ_k^m is the year fraction corresponding to the period $[T_{k-1}^m, T_k^m]$, $\sigma_k^m > 0$ depends on the tenor and $\{Z_k^m\}_{t\geq 0}$ is a standard Wiener process under the $Q_d^{T_k^m}$ measure.

This SDE can easily be solved by defining a new process $\{G_k^m\}$ such that:

$$G_k^m(t) = F_k^m(t) + \frac{1}{\gamma_k^m}.$$

We then have:

$$dG_k^m(t) = dF_k^m(t),$$

or

$$dG_k^m(t) = \sigma_k^m G_k^m(t) dZ_k^m(t).$$

The latter is the SDE of a log-normal process and can easily be solved using Itô's lemma. As a matter of fact:

$$\begin{aligned} d(\ln(G_k^m(t))) &= \frac{dG_k^m(t)}{G_k^m(t)} - \frac{1}{2G_k^m(t)} \underbrace{(dG_k^m(t))^2}_{=(\sigma_k^m)^2 G_k^m(t)^2 dt}, \\ &= -\frac{(\sigma_k^m)^2}{2} dt + \sigma_k^m dZ_k^m(t). \end{aligned}$$

We integrate the last equation between t and T_{k-1}^m and obtain:

$$G_k^m(T_{k-1}^m) = G_k^m(t) e^{\left\{-\frac{(\sigma_k^m)^2}{2}(T_{k-1}^m - t) + \sigma_k^m \left(Z_k^m(T_{k-1}^m) - Z_k^m(t)\right)\right\}},$$

So $\{G_k^m(T_{k-1}^m)\}$ is a lognormmal process whose logarithm has a normal distribution with mean $\ln(G_k^m(t)) - \frac{(\sigma_k^m)^2}{2}(T_{k-1}^m - t)$ and standard deviation $\sigma_k^m \sqrt{(T_{k-1}^m - t)}$.

Finally,

$$F_k^m(T_{k-1}^m) + \frac{1}{\gamma_k^m} = \left[F_k^m(t) + \frac{1}{\gamma_k^m}\right] e^{\left\{-\frac{(\sigma_k^m)^2}{2}(T_{k-1}^m - t) + \sigma_k^m \left(Z_k^m(T_{k-1}^m) - Z_k^m(t)\right)\right\}}. \tag{13}$$

This formula will prove to be useful in the Monte Carlo simulation but will also allow to obtain the expression of the density function of F_k^m, which intervenes in the caplet pricing formula (10).

In fact we have for any positive z:

$$f_{F_k^m(T_{k-1}^m)}(z) = \frac{dF_{F_k^m(T_{k-1}^m)}(z)}{dz},$$

where $F_{F_k^m(T_{k-1}^m)}$ is the cumulative distribution function of $F_k^m(T_{k-1}^m)$ and we have:

$$\begin{aligned} F_{F_k^m(T_{k-1}^m)}(z) &= \mathbb{P}(F_k^m(T_{k-1}^m) \leq z) \\ &= \mathbb{P}(G_k^m(T_{k-1}^m) - \frac{1}{\gamma_k^m} \leq z) \\ &= F_{G_k^m(T_{k-1}^m)}\left(z + \frac{1}{\gamma_k^m}\right), \end{aligned}$$

So

$$f_{F_k^m(T_{k-1}^m)}(z) = f_{G_k^m(T_{k-1}^m)}\left(z + \frac{1}{\gamma_k^m}\right),$$

and since $\{G_k^m(T_{k-1}^m)\} \sim LN\left(\ln(G_k^m(t)) - \frac{(\sigma_k^m)^2}{2}(T_{k-1}^m - t), \sigma_k^m \sqrt{(T_{k-1}^m - t)}\right)$ (log-normal) i.e. $\ln\{(G_k^m(T_{k-1}^m)\} \sim N\left(\ln(G_k^m(t)) - \frac{(\sigma_k^m)^2}{2}(T_{k-1}^m - t), \sigma_k^m \sqrt{(T_{k-1}^m - t)}\right)$.

We have:

$$f_{F_k^m(T_{k-1}^m)}(z) = \frac{1}{\sigma_k^m\left(z+\frac{1}{\gamma_k^m}\right)\sqrt{2\pi(T_{k-1}^m - t)}} \exp\left\{-\frac{\left[\ln\left(\frac{z+\frac{1}{\gamma_k^m}}{F_k^m(t)+\frac{1}{\gamma_k^m}}\right)+\frac{1}{2}(\sigma_k^m)^2(T_{k-1}^m - t)\right]^2}{2(\sigma_k^m)^2(T_{k-1}^m - t)}\right\}. \tag{14}$$

- **The basis spread: SABR**

$$\begin{aligned} dS_k^m(t) &= [S_k^m(t)]^{\beta_k} V_k(t) dX_k(t), & (15)\\ dV_k(t) &= \varepsilon_k V_k(t) dY_k(t), V_k(0) = \alpha_k, & (16)\end{aligned}$$

where $\{X_k\}_{t\geq 0}$ and $\{Y_k\}_{t\geq 0}$ are correlated standard Wiener processes under the $Q_d^{T_k^m}$ measure, i.e. $dX_k(t)dY_k(t) = \rho_k dt$ with $\rho_k \in [-1, 1)$ and $\alpha_k > 0$, $\varepsilon_k > 0$, $\beta_k \in (0, 1]$ are constants.

Here the SABR parameters have a k subscript because they depend on the $[T_{k-1}^m, T_k^m]$ rate period.
The wiener process Z_k^m is independent from X_k and Y_k to ensure the independence between the processes F_k^m and S_k^m.

3.2 Caplet Pricing Formula

Recall the formula derived for interest rate caplets in the general case of a positively distributed forward OIS process:

$$\Phi_C(t, K; T_{k-1}^m, T_k^m) = \int_{-\infty}^{K} \Phi_C^S(t, K-y; T_{k-1}, T_k) f_{F_k^m(T_{k-1}^m)}(y)dy - S_k^m(t)\frac{\partial}{\partial K}\Phi_C^F(t, K; T_{k-1}^m, T_k^m) + \Phi_C^F(t, K; T_{k-1}, T_k).$$

We need to compute the prices Φ^F where F_k^m is the underlying of the call and Φ^S where S_k^m is the underlying of the call i.e.

$$\Phi_C^F(t, K, T_{k-1}^m, T_k^m) = \gamma_k^m P_d(t, T_k^m)\mathbb{E}_t^{Q^{T_k^m}}\left[(F_k^m(T_{k-1}^m) - K)^+\right]$$

and

$$\Phi_C^S(t, K, T_{k-1}^m, T_k^m) = \gamma_k^m P_d(t, T_k^m)\mathbb{E}_t^{Q^{T_k^m}}\left[(S_k^m(T_{k-1}^m) - K)^+\right]$$

Since S_k^m is, in our case, following SABR dynamics then by using Black's formula, we have:

$$\Phi_C^S(t, K-z, T_{k-1}^m, T_k^m) \simeq \gamma_m(T_{k-1}^m, T_k^m)P_d(t, T_k^m)\left(S_k^m(t)N(d_1^S) - (K-z)N(d_2^S)\right), \tag{17}$$

where N is the cumulative standard normal distribution and

$$d_1^S = \frac{\ln\left(\frac{S_k^m(t)}{K-z}\right) + \frac{1}{2}\sigma_S(K-z, S_k^m(t))^2(T_{k-1}^m - t)}{\sigma_S(K-z, S_k^m(t))\sqrt{T_{k-1}^m - t}},$$

$$d_2^S = d_1^S - \sigma_S(K-z, S_k^m(t))\sqrt{T_{k-1}^m - t}.$$

Where σ_S is Hagan's approximated implied volatility given by (see [7]):

$$\sigma_S(K,F) = \frac{\varepsilon_k \ln(F/K) \left\{ 1 + \left[\frac{(\alpha_k(1-\beta_k))^2}{24(FK)^{1-\beta_k}} + \frac{\alpha_k \beta_k \rho_k \varepsilon_k}{4(FK)^{\frac{1-\beta_k}{2}}} + \varepsilon_k^2 \frac{2-3\rho_k^2}{24} \right] T_{k-1}^m + \ldots \right\}}{x(\zeta) \left[1 + \frac{((1-\beta_k)\ln(F/K))^2}{24} + \frac{((1-\beta_k)\ln(F/K))^4}{1920} + \ldots \right]},$$

where

$$\zeta = \frac{\varepsilon_k}{\alpha_k}(FK)^{\frac{1-\beta_k}{2}} \ln(F/K),$$
$$x(\zeta) = \ln\left\{ \frac{\sqrt{1 - 2\rho_k\zeta + \zeta^2} + \zeta - \rho_k}{1-\rho_k} \right\}.$$

Moreover, we have:

$$\begin{aligned}
\Phi_C^F(t,K,T_{k-1}^m,T_k^m) &= \gamma_m(T_{k-1}^m,T_k^m) P_d(t,T_k^m) \mathbb{E}_t^{Q^{T_k^m}} \left[(F_k^m(T_{k-1}^m) - K)^+ \right] \\
&= \gamma_m(T_{k-1}^m,T_k^m) P_d(t,T_k^m) \mathbb{E}_t^{Q^{T_k^m}} \left[\left(G_k^m(T_{k-1}^m) - \left(K + \frac{1}{\gamma_k^m} \right) \right)^+ \right] \\
&= \Phi_C^G \left(t, K + \frac{1}{\gamma_k^m}, T_{k-1}^m, T_k^m \right) \\
&= \gamma_k^m P_d(t,T_{k-1}^m) \mathbb{E}_t^{Q^{T_k^m}} [(G_k^m(T_{k-1}^m) - K)^+],
\end{aligned}$$

and since $G_k^m(T_{k-1}^m)$ is log-normally distributed, then:

$$\Phi_C^F(t,K,T_{k-1}^m,T_k^m) = \Phi_C^G\left(t, K + \frac{1}{\gamma_k^m}, T_{k-1}^m, T_k^m\right) = \gamma_k^m P_d(t,T_k^m) \left(G_k^m(t) N(d_1^G) - \left(K + \frac{1}{\gamma_k^m}\right) N(d_2^G) \right),$$

or

$$\Phi_C^F(t,K,T_{k-1}^m,T_k^m) = \gamma_k^m P_d(t,T_k^m) \left[\left(F_k^m(t) + \frac{1}{\gamma_k^m} \right) N(d_1^G) - \left(K + \frac{1}{\gamma_k^m} \right) N(d_2^G) \right]. \tag{18}$$

where

$$d_1^G = \frac{\ln\left(\frac{F_k^m(t) + \frac{1}{\gamma_k^m}}{K + \frac{1}{\gamma_k^m}} \right) + \frac{1}{2}(\sigma_k^m)^2 (T_{k-1}^m - t)}{\sigma_k^m \sqrt{T_{k-1}^m - t}},$$
$$d_2^G = d_1^G - \sigma_k^m \sqrt{T_{k-1}^m - t}.$$

We differentiate (18) with respect to K:

$$\frac{\partial \Phi_C^F}{\partial K}(t,K,T_{k-1}^m,T_k^m) = \gamma_k^m P_d(t,T_k^m) \left(G_k^m(t) \frac{\partial N(d_1^G)}{\partial K} - N(d_2^G) - \left(K + \frac{1}{\gamma_k^m} \right) \frac{\partial N(d_2^G)}{\partial K} \right) \tag{19}$$

where

$$\frac{\partial N(d_1^G)}{\partial K} = \frac{\partial N(d1^G)}{\partial d_1^G} \frac{\partial d_1^G}{\partial K} = N'(d_1^G) \frac{\partial d_1^G}{\partial K},$$
$$\frac{\partial N(d_2^G)}{\partial K} = N'(d_2^G) \frac{\partial d_2^G}{\partial K},$$

with N' the density function of the standard normal distribution.

We have:

$$\frac{\partial d_1^G}{\partial K} = \frac{\partial d_2^G}{\partial K} = \frac{-1}{\sigma_k^m \sqrt{T_{k-1}^m - t}\left(K + \frac{1}{\gamma_k^m}\right)}.$$

We also have that:

$$\frac{\partial N(d_2^G)}{\partial K} = \left(\frac{F_k^m(t) + \frac{1}{\gamma_k^m}}{K + \frac{1}{\gamma_k^m}}\right)\frac{\partial N(d_1^G)}{\partial K}. \tag{20}$$

(The proof can be found in the annex.)

(20) becomes:

$$\begin{aligned}
\frac{\partial \Phi_C^F}{\partial K}(t, K, T_{k-1}^m, T_k^m) &= \gamma_k^m P_d(t, T_k^m)\left(\left(F_k^m(t) + \frac{1}{\gamma_k^m}\right)\frac{\partial N(d_1^G)}{\partial K} - N(d_2^G) - \left(K + \frac{1}{\gamma_k^m}\right)\left(\frac{F_k^m(t) + \frac{1}{\gamma_k^m}}{K + \frac{1}{\gamma_k^m}}\right)\frac{\partial N(d_1^G)}{\partial K}\right) \\
&= -\gamma_k^m P_d(t, T_k^m) N(d_2^G) \\
\Phi_C^F(t, K, T_{k-1}^m, T_k^m) &= \gamma_k^m P_d(t, T_k^m)\left(\left(F_k^m(t) + \frac{1}{\gamma_k^m}\right)N(d_1^G) - \left(K + \frac{1}{\gamma_k^m}\right)N(d_2^G)\right)
\end{aligned}$$

By substituting the left hand side quantities with their respective expressions in equation (10) as well as using the expression of $f_{F_k^m(T_{k-1}^m)}$, we obtain:

$$\boxed{\begin{aligned}
\Phi_C(t, K; T_{k-1}^m, T_k^m) &\simeq \int_0^K \frac{\Phi_C^S(t, K - z; T_{k-1}^m, T_k^m)}{\sigma_k^m\left(z + \frac{1}{\gamma_k^m}\right)\sqrt{2\pi(T_{k-1}^m - t)}} \exp\left\{-\frac{\left[\ln\left(\frac{z + \frac{1}{\gamma_k^m}}{F_k^m(t) + \frac{1}{\gamma_k^m}}\right) + \frac{1}{2}(\sigma_k^m)^2(T_{k-1}^m - t)\right]^2}{2(\sigma_k^m)^2(T_{k-1}^m - t)}\right\} dz \\
&+ \gamma_k^m P_d(t, T_k^m)\left[\left(F_k^m(t) + \frac{1}{\gamma_k^m}\right)N(d_1^G) + \left(S_k^m(t) - K - \frac{1}{\gamma_k^m}\right)N(d_2^G)\right] \\
&\simeq \int_{\frac{1}{\gamma_k^m}}^{K + \frac{1}{\gamma_k^m}} \frac{\Phi_C^S(t, K + \frac{1}{\gamma_k^m} - z; T_{k-1}^m, T_k^m)}{\sigma_k^m z\sqrt{2\pi(T_{k-1}^m - t)}} \exp\left\{-\frac{\left[\ln\left(\frac{z}{F_k^m(t) + \frac{1}{\gamma_k^m}}\right) + \frac{1}{2}(\sigma_k^m)^2(T_{k-1}^m - t)\right]^2}{2(\sigma_k^m)^2(T_{k-1}^m - t)}\right\} dz \\
&+ \gamma_k^m P_d(t, T_k^m)\left[\left(F_k^m(t) + \frac{1}{\gamma_k^m}\right)N(d_1^G) + \left(S_k^m(t) - K - \frac{1}{\gamma_k^m}\right)N(d_2^G)\right].
\end{aligned}}$$

Remark:

If the spread is deterministic (recall (11)), then the price formula of the caplet becomes:

$$\begin{aligned}\Phi_C(t,K;T_{k-1}^m,T_k^m) &= \gamma_k^m P_d(t,T_k^m)\left[\left(F_k^m(t)+\frac{1}{\gamma_k^m}\right)N\left(\frac{\ln\left(\frac{F_k^m(t)+\frac{1}{\gamma_k^m}}{K+s+\frac{1}{\gamma_k^m}}\right)+\frac{1}{2}(\sigma_k^m)^2(T_{k-1}^m-t)}{\sigma_k^m\sqrt{T_{k-1}^m-t}}\right)\right.\\ &\quad \left. - \left(K+s+\frac{1}{\gamma_k^m}\right)N\left(\frac{\ln\left(\frac{F_k^m(t)+\frac{1}{\gamma_k^m}}{K+s+\frac{1}{\gamma_k^m}}\right)-\frac{1}{2}(\sigma_k^m)^2(T_{k-1}^m-t)}{\sigma_k^m\sqrt{T_{k-1}^m-t}}\right)\right].\end{aligned}$$

At this stage, we have derived the pricing formula for caplets under SABR dynamics for the spread and shifted log-normal for the forward OIS rates, then we derived the formula in case of a deterministic spread but keeping the same shifted log-normal dynamics for the forward OIS rates.

The pricing formula when the spread is stochastic is an approximation because it involves knowing the quantity Φ_C^S (17), which is approximated by a Black price which uses the Hagan approximated implied volatility.

To assess the accuracy of the pricing formula obtained, we propose to compare it to the outcomes of the Monte Carlo Method presented in the following section.

3.3 Monte Carlo Method

In this section, we propose to perform the caplet pricing using a Monte Carlo Method with an antithetic variance reduction method.
We detail the procedure below:

1. We choose a number M of simulations and at each simulation we repeat the following steps.

2. We use a simple forward Euler scheme to compute $S_k^m(T_{k-1}^m)$:
We first discretize the time interval $[t, T_{k-1}^m]$ into N small intervals of length $\Delta t = \frac{T_{k-1}^m - t}{N}$, thus defining a time grid $t_0 = t < t_1 < ... < t_N = T_{k-1}^m$; $\forall i \in \{1, 2, ..., n\}, t_i - t_{i-1} = \Delta t$.
The Forward Euler scheme allows to write:
$$V_k(t_i) = V_k(t_{i-1}) \exp\left\{ -\frac{\varepsilon_k^2}{2}\Delta t + \varepsilon_k \sqrt{\Delta t} Y \right\},$$
$$S_k^m(t_i) = S_k^m(t_{i-1}) + [S_k^m(t_{i-1})]^{\beta_k} V_k(t_{i-1})\sqrt{\Delta t} X,$$
where X and Y are realizations of two ρ_k-correlated wiener processes generated using a Mersen-Twister algorithm for example. We simulate N times to obtain an approximation of
$$S_k^m(T_{k-1}^m) \simeq \sum_{i=1}^{n} (S_k^m(t_{i-1}))^{\beta_k} V_k(t_{i-1})\sqrt{\Delta t} X(t_{i-1}).$$

3. We compute $F_k^m(T_{k-1}^m)$ by simulating:
$$F_k^m(T_{k-1}^m) = -\frac{1}{\gamma_k^m} + \left[F_k^m(t) + \frac{1}{\gamma_k^m} \right] e^{\left\{ -\frac{(\sigma_k^m)^2}{2} N\Delta t + \sigma_k^m \sqrt{N \Delta t} Z \right\}},$$
where Z is a realization of a wiener process that is independent from X and Y.

4. Compute the payoff of the caplet:
$$\gamma_k^m [F_k^m(T_{k-1}^m) + S_k^m(T_{k-1}^m) - K]^+.$$

5. For each simulation, we repeat steps 2, 3 and 4 by considering the opposite of all random numbers, as it must be done in an antithetic variance reduction method.

6. Compute the average payoff of the two methods (the one using the random numbers generated and the one using their opposites).

7. After running all M simulations, we compute the average of the M payoffs and discount them using the right discount factor $(P_d(t, T_k^m))$ obtained from the OIS discount curve already created as of t (the pricing date).

This Monte Carlo method allowed to check the accuracy of the caplet pricing formula and as tables (4, 5 and 6) show, with around 10000 simulations and 10000 time steps, we obtain the same values analytically and using the Monte Carlo method (a difference of order 10^{-8}).

3.4 Model Calibration to Market Caplet Data

After deriving a closed formula for the price of a caplet in the multi-curve setting with a stochastic basis spread, it is now possible to consider calibrating the using market data.
I have used Bloomberg to collect relevant data for this purpose.
The model resulting from choosing the dynamics (12) and (15)-(16) has 5 parameters (for each k): α_k, β_k, ρ_k, ε_k and σ_k^m.
The calibration process was conducted following the steps below:

1. Collect the Euro market caps' implied volatilities for a specific date. We chose 01/10/2014 (October 1st 2014). The underlying tenor is 6M in the EUR market.

2. Perform a bootstrapping procedure to obtain the caplet market prices from the quoted cap volatilities.

3. Fix the parameter β_k and for a specific maturity (i.e. for each specific T_{k-1}^m), solve a least squares minimization problem to obtain the values of the remaining parameters fo which the model prices match the market ones the most.
   As commonly done by practitionners, we chose $\beta_k = 0.5$ and avoided taking $\beta_k = 0$ (purely normal process) or $\beta_k = 1$ (purely log-normal process). Using this value of β_k happened to give better results. In this study, our aim is not to go into further details regarding the SABR model.

In this section, we detail these three steps.

1. We fix a reference date for the pricing (01/10/2014). The yield curves used are also created as of 01/10/2014. We use the market quotes given by Bloomberg to create the OIS discount curve and the Euribor 6M forward curve (as explained in section (reference creation yield curves)).

2. The data collected corresponds to market cap volatilities and not that of caplets, hence the need to perform a bootstrap to compute the underlying caplet market prices.
   In Bloomberg, the quoted cap maturities are 1Y, 18M, 2Y - 10Y, 12Y, 15Y, 20Y, 25Y and 30Y. For each of these maturities, we have quotes for the following strikes: 1%, 1.5%, 2%, 2.5%, 3%, 4%, 5%, 10%.

   We denote by δ the tenor ($\delta = 6M$).
   A cap of strike K and maturity T that is tied to an Euribor 6M index is a sum of caplets whose maturities are multiples of the tenor from δ up to $T - \delta$.
   More formally, we have:

$$\Phi_{cap}^{mkt}(t, K; T, \sigma_T) = \sum_{i=1}^{n-1} \Phi_{caplet}^{mkt}(t, K; T - i\delta, \sigma_{T-i\delta}), \tag{21}$$

   where $\Phi_{cap}^{mkt}(t, K; T)$ is the price at time t of the cap and $\Phi_{caplet}^{mkt}(t, K; T - i\delta)$ is the price at time t of the caplet. $n = \frac{T}{\delta}$, knowing that the quoted maturities are multiples of the tenor, and $T \geq 1Y$ so $n \geq 2$. This allows us to perform the bootstrapping.
   We consider maturities up to 10Y.

As a matter of fact, equation (21) rewrites into:

$$\Phi^{mkt}_{caplet}(t,K;\delta,\sigma_{2\delta}) = \Phi^{mkt}_{cap}(t,K;2\delta,\sigma_{2\delta}), (n=2)$$
$$\Phi^{mkt}_{caplet}(t,K;2\delta,\sigma_{3\delta}) = \Phi^{mkt}_{cap}(t,K;3\delta,\sigma_{3\delta}) - \Phi^{mkt}_{cap}(t,K;2\delta,\sigma_{2\delta}), (n=3)$$
$$\Phi^{mkt}_{caplet}(t,K;3\delta,\sigma_{4\delta}) = \Phi^{mkt}_{cap}(t,K;4\delta,\sigma_{4\delta}) - \Phi^{mkt}_{cap}(t,K;3\delta,\sigma_{3\delta}), (n=4)$$

then for $3 \leq i \leq 10$,

$$\Phi^{mkt}_{caplet}(t,K;(2i-1)\delta,\sigma) + \Phi^{mkt}_{caplet}(t,K;2(i-1)\delta,\sigma) = \underbrace{\Phi^{mkt}_{cap}(t,K;2i\delta,\sigma_{2i\delta}) - \Phi^{mkt}_{cap}(t,K;2(i-1)\delta,\sigma_{2(i-1)\delta})}_{\theta}$$

Here both caplets on the left hand side have the same volatility, which we are interested in finding. For that, we define a function h of σ as the sum of two black formulae representing the prices of the two caplets on the left hand side i.e.:
For $3 \leq i \leq 10$,

$$L_1 = FRA(t,(2i-1)\delta,2i\delta), L_2 = FRA(t,2(i-1)\delta,(2i-1)\delta),$$
$$h(\sigma) = L_1 N\left(\frac{\ln(L_1/K)+\sigma^2/2}{\sigma}\right) - KN\left(\frac{\ln(L_1/K)-\sigma^2/2}{\sigma}\right) + L_2 N\left(\frac{\ln(L_2/K)+\sigma^2/2}{\sigma}\right) - KN\left(\frac{\ln(L_2/K)-\sigma^2/2}{\sigma}\right)$$

The implied volatility of both caplets is the solution of the equation $h(\sigma) = \theta$.

Once all the caplet volatilities are computed, we can use Black's formula again for each one of them to deduce their market prices.

3. Now that the relevant data is available, we can obtain the model parameters $\hat{\alpha_k}$, $\hat{\rho_k}$, $\hat{\varepsilon_k}$ and $\hat{\sigma^\delta_k}$ that allow to match caplet market prices the most i.e.:

$$(\hat{\alpha_k},\hat{\rho_k},\hat{\varepsilon_k},\hat{\sigma^\delta_k}) = \underset{\alpha_k,\rho_k,\varepsilon_k,\sigma^\delta_k}{argmin} \sum_i \left[\Phi^{Model}_C(t,K_i,T) - \Phi^{mkt}_C(t,K_i,T)\right]^2$$

for a given maturity $T \in \{\delta, 2\delta, ..., 20\delta\}$.

3.5 Numercial Results and Discussion

In this section, we present our results in terms of pricing and calibration.
We have chosen to calibrate our model using EUR caplet data as of 01/10/2014. We perform the calibration for maturities 3Y, 4Y and 6Y.
We will only consider data corresponding to strikes up to 3% because the rates as of 01/10/2010 are weak and the behaviour of the caplet prices for much higher strikes cannot be controlled.
Table 3 below presents the calibration results for these three maturities.

We notice that for the 6Y maturity, the calibration doesn't seem to work properly. Furthermore, the correlation ρ_k varies drastically from a maturity to another, which can be explained by the fact that the influence of this parameter is not high enough to affect the model prices. We could as well fix it to zero (which is the convention insuring that no matter the measure change i.e. no matter the forward measure, we have the same volatility dynamics) and calibrate the other parameters. This would lead to the same results approximately. The rather low level of the spread (not exceeding 0.35% for its initial value) can explain the weak influence of rho_k as the contribution of the SABR dynamics in the caplet pricing formula is lower than that of the forward OIS dynamics, to which the prices are highly sensitive (parameter σ^m_k).

	Caplet 3Y	Caplet 4Y	Caplet 6Y
Expiration date	02/10/2017	01/10/2018	01/10/2020
$F_k^{6M}(t)$	0.16873%	0.43822%	1.15707%
$L_k^{6M}(t)$	0.50133%	0.78447%	1.46506%
$S_k^{6M}(t)$	0.33260%	0.34625%	0.30799%
α_k	2.27%	2.05%	1.98%
β_k	0.5	0.5	0.5
ρ_k	0.99	-0.99	0.99
ε_k	40.00%	0.35%	1.00%
σ_k^{6M}	0.1871%	0.236%	0.295%
Calibration error range	[0.001382%; 0.16543%]	[0.25999%; 2.12981%]	[4.43970%; 50.67061%]

Table 3: Results of the calibration as of 01/10/2014.

Tables 4, 5 and 6 provide a comparison between market caplet prices and model prices obtained on the one hand via the analytical formula of caplets (derived in section 3.2) and on the other hand via the Monte Carlo method presented in section 3.3, with $M = 10000$ and $N = 10000$.

Strike	Model Formula Price	Model MC Price	Market Price	Difference MC/Formula (bps)	Difference Model/Market (bps)
1,00%	0,05141540%	0,05141540%	0,05141469%	1.13E-08	7,11E-05
1,50%	0,01662938%	0,01662938%	0,01662914%	5.17E-08	2,47E-05
2,00%	0,00591765%	0,00591765%	0,00592568%	2.00E-08	8,02E-04
2,50%	0,00266705%	0,00266705%	0,00266555%	3.00E-08	1,51E-04
3,00%	0,00151899%	0,00151899%	0,00151648%	6.74E-08	2,51E-04

Table 4: Pricing of the 3Y maturity caplet as of 01/10/2014 (comparison between analytical and Monte Carlo results): Model prices $\simeq$ Market prices.

Strike	Model Formula Price	Model MC Price	Market Price	Difference MC/Formula (bps)	Difference Model/Market (bps)
1,00%	0,14612490%	0,14612490%	0,14570939%	7.47E-08	4,16E-02
1,50%	0,06624542%	0,06624542%	0,06486394%	1.51E-08	1,38E-01
2,00%	0,02517027%	0,02517027%	0,02498899%	3.19E-08	1,81E-02
2,50%	0,00789758%	0,00789758%	0,00778089%	1.43E-08	1,17E-02
3,00%	0,00202370%	0,00202370%	0,00201845%	5.90E-08	5,25E-04

Table 5: Pricing of the 4Y maturity caplet as of 01/10/2014 (comparison between analytical and Monte Carlo results): Model prices $\simeq$ Market prices (less accurate than for 3Y).

Strike	Model Formula Price	Model MC Price	Market Price	Difference MC/Formula (bps)	Difference Model/Market (bps)
1,00%	0,41980511%	0,41980511%	0,46080412%	1.33E-07	4,10
1,50%	0,28223469%	0,28223469%	0,29534723%	8.53E-08	1,31
2,00%	0,17781946%	0,17781946%	0,16198569%	6.90E-08	1,58
2,50%	0,10440318%	0,10440318%	0,08405746%	7.32E-08	2,03
3,00%	0,05684376%	0,05684376%	0,03772717%	2.24E-08	1,91

Table 6: Pricing of the 6Y maturity caplet as of 01/10/2014 (comparison between analytical and Monte Carlo results): Model prices $\neq$ Market prices.

Figure 7 displays the comparison between market prices and model prices (analytical and MC with 10000 simulations). We notice that the analytical formula and the MC method provide the same results in terms of pricing, which proves that the formula derived for the caplet price is correct and that the SABR approximation (17) is accurate.

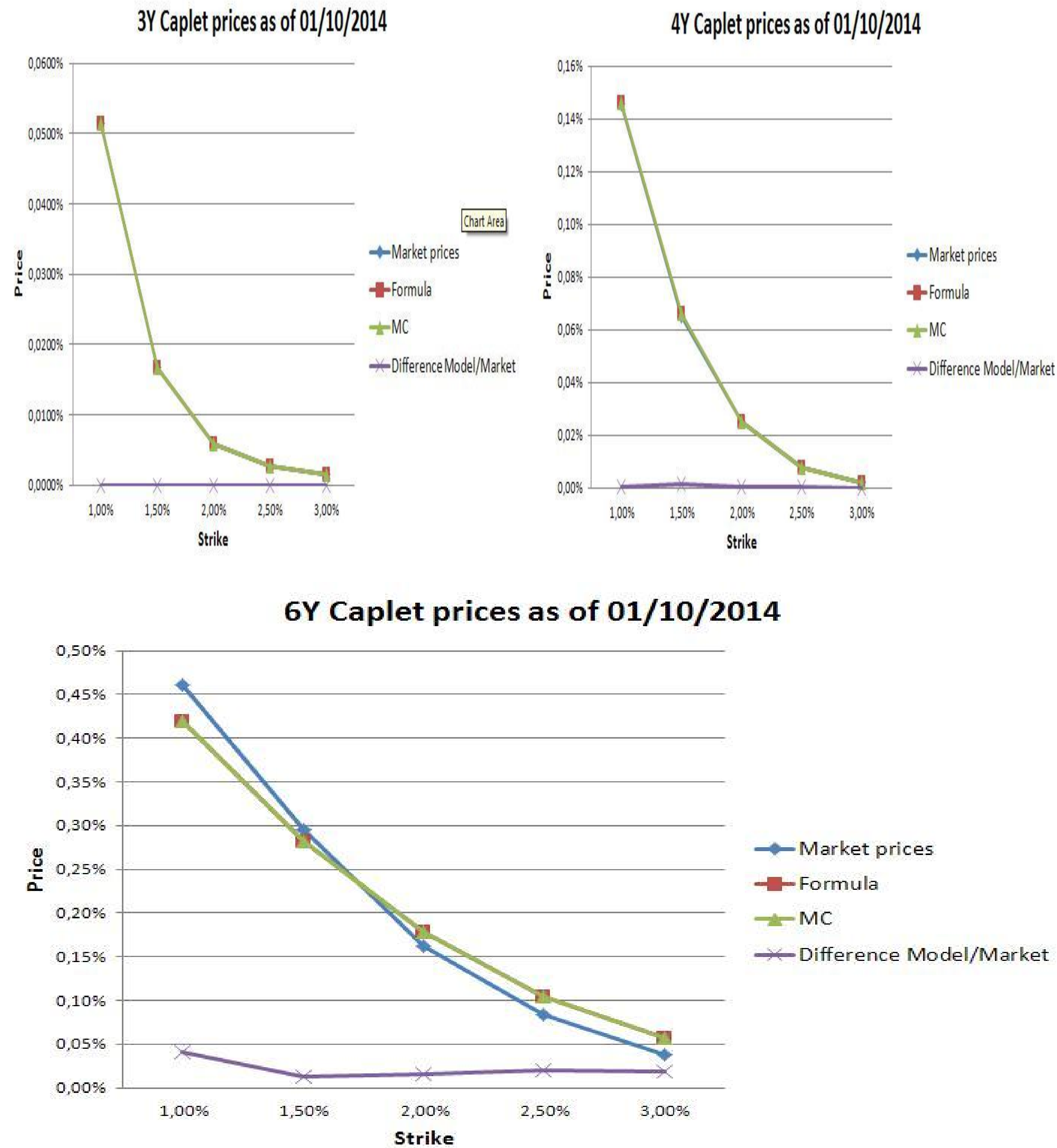


Figure 7: Market and model (formula / MC) caplet prices comparison as of 01/10/2014, notional = 1.

Figure 8 shows the amplitudes of the calibration errors for the terms 3Y, 4Y and 6Y. According to the results obtained, a calibration of the model is possible for middle terms such as 3Y and 4Y, where the calibrated parameters allow to reproduce the market prices almost perfectly for a 3Y and 4Y (the results for a 3Y maturity are better than for 4Y though). But for higher maturities such as 6Y, the calibration becomes unaccurate, as supported by the resulting model prices not matching market prices anymore.
As a matter of fact, the calibration errors given by:

$$\frac{|\Phi_C^{model}(t,K,T) - \Phi_C^{mkt}(t,K,T)|}{\Phi_C^{mkt}(t,K,T)},$$

for $K \in \{1\%, 1.5\%, 2\%, 2.5\%, 3\%\}$ and maturity $T \in \{3Y, 4Y, 6Y\}$, are almost negligible for the 3Y and 4Y maturities but increase for higher maturitites (6Y).

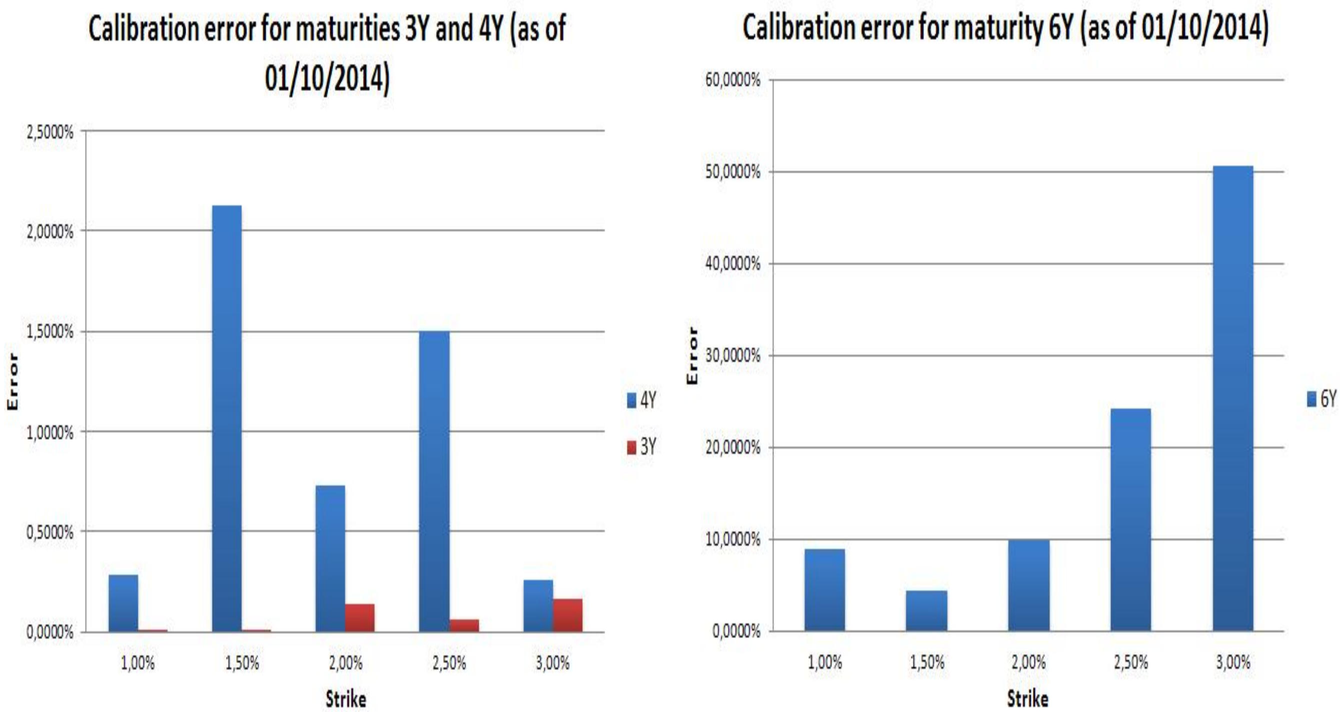


Figure 8: Calibration errors as of 01/10/2014.

Remark: If we consider the case where the spread is deterministic, we will only have one parameter to calibrate, σ_k^{6M}.
We run a calibration for this degenerated case for maturities 3Y, 4Y and 6Y as before and obtain the results reported in figure 9.

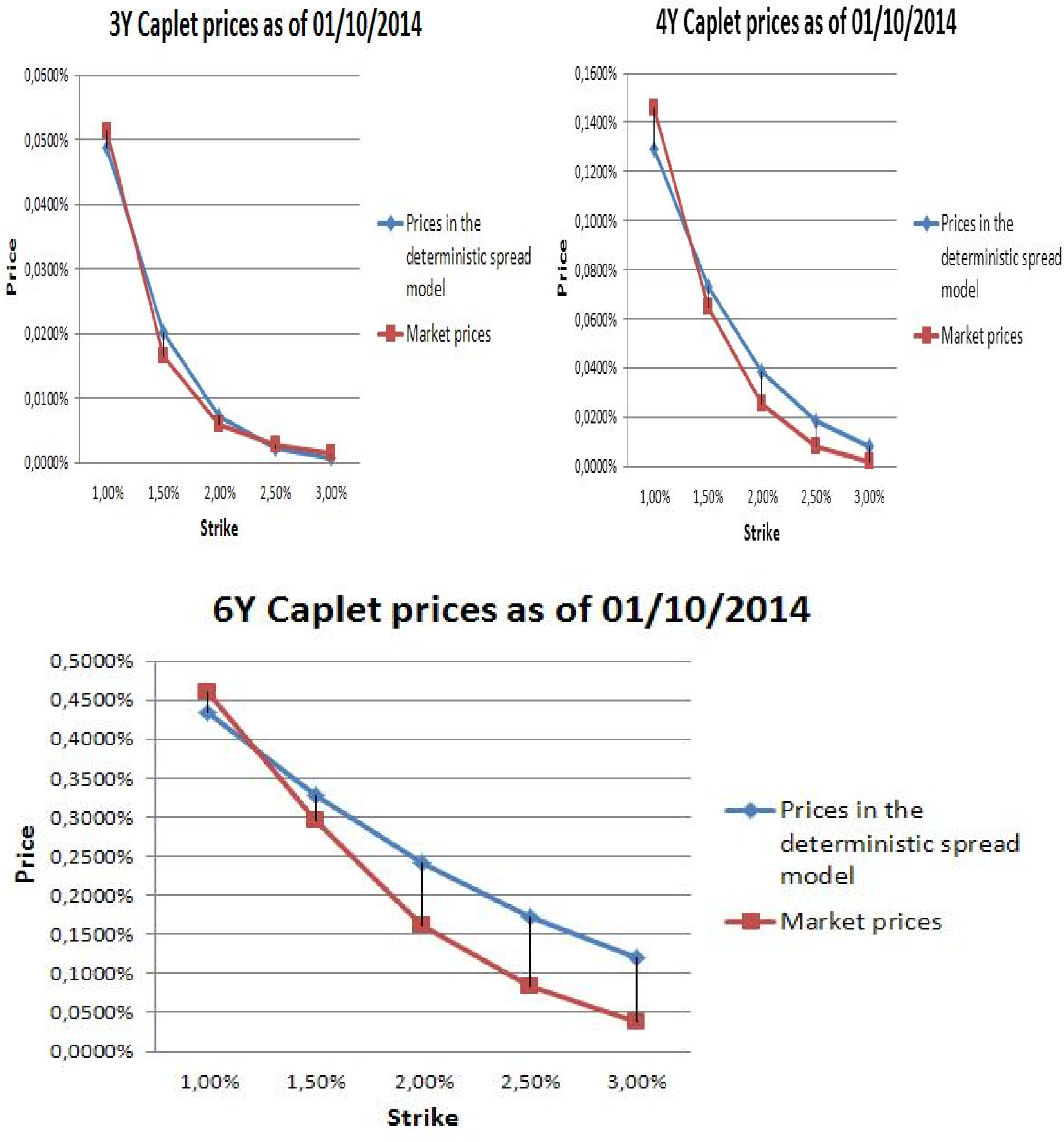


Figure 9: Market and model caplet prices comparison as of 01/10/2014 when using a model with a deterministic basis spread: $\sigma_k^{6M} = 0.335$, $\sigma_k^{6M} = 0.379$ and $\sigma_k^{6M} = 0.485$ for 3Y, 4Y and 6Y maturities respectively, notional = 1.

As we can observe from the figures, the calibration is not as accurate as when using a stochastic spread for maturity 3Y and quickly becomes unaccurate for higher maturities as the prices obtained via such a model cannot match the market prices for any choice of the parameter σ_k^{6M}.
From this comparison, we can understand the importance of including a stochastic feature and its impact on caplet prices (among other interest rate derivatives). Thus, models that use a deterministic spread don't allow a good description of the market, which can be viewed through our example, where the impact of the choice of the spread becomes obvious, now that the basis spread is no longer negligible as in the pre-crisis period.

The results emphasize the efficiency of the Libor Market Model when we calibrate using caplets of average maturities such as 3Y and 4Y. Meanwhile, the deterministic spread model is clearly not efficient even for average maturities.
One difficulty here is that the level of the forward Euribor rates is relatively weak, so we have less data to calibrate with as we cannot consider market data for strikes more than 3% for example.
The fact that for the date chosen (01/10/2014) the rates obtained are low, when performing a calibration using market caplets of strikes that are much higher than the FRA rates, the results can be disappointing. In our case, the lack of data for lower strikes forced us to consider strikes up to 3% (but not more), even if the level of forward rates as of 01/10/2014 is less than 1.5%.

We have tested our model on the same data set as used in Mercurio's article, which corresponded to February 8th, 2010, where the market was less stressed as the rates weren't as low as in our example. We managed to obtain good fits for even higher strikes (more than 3%), corroborating the results obtained in the article. We wanted, through this last test, emphasize that the model calibration is very sensitive to the level of rates in the market.

To improve the calibration more, one can also add weights in order to give more importance to one region of strikes where the FRA rates fall and less importance to the tails (very low or very high strikes).

In our approach, we have been able to approach the market behaviour by using a classical stochastic model for the basis spread, for average maturity caplets.
Our model can be more adapted to other sets of data corresponding to a time when the market is not stressed. In fact, If we take a look at the rates as of 01/10/2014 in the annex, we see that some rates are negative, which corresponds to a stressed situation.

We insist on the fact that the LMM model is a tractable model, allowing to add more hypotheses and/or relax some others. It allows to derive closed formulae consequently. The hardest part is to choose the most suitable dynamics for the quantitites we want to model as this choice depends on the market changes. For example, in case of low rates, a log-normal model wouldn't be a clever choice. In our example, the market situation as of 01/10/2014 (low rates) didn't allow for a very accurate fitting, which is the main dfficulty in this field.

Moreover, the choice of the instruments used in the calibration is very important as well. Unfortunately, there still doesn't exist enough liquid instruments tied to the basis spread. Such instruments would have been more appropriate to use for the calibration.

4 Annex

4.1 Demonstrations

- We demonstrate the equality in equation (9):
$$\frac{\partial}{\partial K}\left\{\int_K^{+\infty}[y-K]f_{F_k^m(T_{k-1}^m)}(y)dy\right\} = -\int_K^{+\infty} f_{F_k^m(T_{k-1}^m)}(y)dy.$$
We have
$$\begin{aligned}\frac{\partial}{\partial K}\left\{\int_K^{+\infty}[y-K]f_{F_k^m(T_{k-1}^m)}(y)dy\right\} &= \frac{\partial}{\partial K}\left\{\int_0^{+\infty}[y-K]f_{F_k^m(T_{k-1}^m)}(y)\mathbb{1}_{\{y\geq K\}}dy\right\}\\ &= \int_0^{+\infty} -\mathbb{1}_{\{y\geq K\}}f_{F_k^m(T_{k-1}^m)}(y)dy + \int_0^{+\infty}(y-K)\delta_K(y)f_{F_k^m(T_{k-1}^m)}(y)dy,\end{aligned}$$
where δ_K is the Dirac function such that $\delta_K(y)=+\infty$ if $y=K$ and $\delta_K(y)=0$ otherwise. So
$$\int_0^{+\infty}(y-K)\delta_K(y)f_{F_k^m(T_{k-1}^m)}(y)dy = 0.$$
Which leads to the result.

- Equation (20):
$$\begin{aligned}
\frac{\partial N(d_2^G)}{\partial K} &= N'(d_2^G)\frac{\partial d_2^G}{\partial K}\\
&= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(d_2^G)^2}{2}\right\}\frac{\partial d_2^G}{\partial K}\\
&= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(d_1^G)^2 - 2d_1^G\sigma_k^m\sqrt{T_{k-1}^m - t} + (\sigma_k^m)^2(T_{k-1}^m - t)}{2}\right\}\frac{\partial d_2^G}{\partial K}\\
&= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(d_1^G)^2}{2}\right\}\exp\left\{\frac{-(\sigma_k^m)^2(T_{k-1}^m - t)}{2} + d_1^G\sigma_k^m\sqrt{T_{k-1}^m - t}\right\}\frac{\partial d_2^G}{\partial K}\\
&= N'(d_1^G)\exp\left\{\ln\left(\frac{F_k^m(t)+\frac{1}{\gamma_k^m}}{K+\frac{1}{\gamma_k^m}}\right)\right\}\frac{\partial d_1^G}{\partial K}\\
\frac{\partial N(d_2^G)}{\partial K} &= \left(\frac{F_k^m(t)+\frac{1}{\gamma_k^m}}{K+\frac{1}{\gamma_k^m}}\right)\frac{\partial N(d_1^G)}{\partial K}.
\end{aligned}$$

4.2 Market Data

The market data used for yield curve construction/bootstrapping are the EUR data as of 01/10/2014 is given in the tables below:

Pillar	Type	Quotes
ON	deposit	0,19700%
1W	swap OIS	-0,04900%
1M	swap OIS	-0,04200%
2M	swap OIS	-0,04000%
3M	swap OIS	-0,04000%
4M	swap OIS	-0,04600%
5M	swap OIS	-0,04500%
6M	swap OIS	-0,04400%
7M	swap OIS	-0,04700%
8M	swap OIS	-0,05100%
9M	swap OIS	-0,05500%
10M	swap OIS	-0,05700%
11M	swap OIS	-0,06000%
12M	swap OIS	-0,06185%
2Y	swap OIS	-0,06193%
3Y	swap OIS	-0,03349%
4Y	swap OIS	0,02536%
5Y	swap OIS	0,11869%
6Y	swap OIS	0,24356%
7Y	swap OIS	0,38456%
8Y	swap OIS	0,52860%
9Y	swap OIS	0,66653%
10Y	swap OIS	0,79554%
12Y	swap OIS	1,01661%
15Y	swap OIS	1,26524%
20Y	swap OIS	1,51071%
25Y	swap OIS	1,62227%
30Y	swap OIS	1,67617%
40Y	swap OIS	1,75114%
50Y	swap OIS	1,78049 %

Table 7: Market quotes for the instruments used in bootstrapping the **EUR OIS discount curve** as of 01/10/2014.

Pillar	**Type**	**Quotes**
ON	deposit	0,18100%
1W	deposit	0,18470%
1M	deposit	0,19458%
2M	deposit	0,20692%
3M	deposit	0,19745%
4M	deposit	0,18905%
5M	deposit	0,18706%
6M	deposit	0,18100%
7M	deposit	0,17889%
8M	deposit	0,17782%
9M	deposit	0,17577%
10M	deposit	0,17241%
11M	deposit	0,17062%
12M	swap 6M	0,16996%
2Y	swap 6M	0,18629%
3Y	swap 6M	0,23700%
4Y	swap 6M	0,31500%
5Y	swap 6M	0,42200%
6Y	swap 6M	0,55000%
7Y	swap 6M	0,692%
8Y	swap 6M	0,835 %
9Y	swap 6M	0,97100 %
10Y	swap 6M	1,09500 %
12Y	swap 6M	1,30500 %
15Y	swap 6M	1,53100 %
20Y	swap 6M	1,74300 %
25Y	swap 6M	1,83200 %
30Y	swap 6M	1,86800 %
40Y	swap 6M	1,91600 %
50Y	swap 6M	1,92500 %

Table 8: Market quotes for the instruments used in bootstrapping the **EURIBOR 6M forward curve** as of 01/10/2014.

The following table provides the market cap implied volatility data as of 01/10/2014 used in the calibration process, precisely in the bootstrapping of caplet market prices.

Maturity	**1.00%**	**1.50%**	**2.00%**	**2.50%**	**3.00%**
1Y	133,00	129,00	127,00	127,00	127,00
18M	133,00	128,00	126,00	125,00	124,00
2Y	127,00	122,00	120,00	119,00	118,00
3Y	80,60	81,20	82,00	82,80	83,60
4Y	75,60	72,50	70,90	70,10	69,50
5Y	71,20	65,90	63,10	61,40	60,30
6Y	66,80	60,40	56,90	54,70	53,10
7Y	62,80	56,00	52,20	49,70	47,90
8Y	59,50	52,60	48,60	46,00	44,10
9Y	56,80	49,80	45,70	43,10	41,20
10Y	54,50	47,60	43,60	40,90	39,10
12Y	51,40	44,70	40,60	37,90	36,10
15Y	48,10	41,60	37,80	35,10	33,30
20Y	45,10	38,90	35,30	32,60	30,80
25Y	43,70	37,70	34,20	31,60	29,90
30Y	43,00	37,00	33,60	31,00	29,40

Table 9: Market cap implied volatilities in (%) as of 01/10/2014.

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