Pricing Interest Rate Derivatives in the Multi-Curve Framework

Stochastic Basis Spread

Zakaria El Menouni
910504-8772

KTH - Royal Institute of Technology
SF299X - Degree Project in Mathematical Statistics

March 30th, 2015
Summary

1 Introduction
   The Interest Rate Market Transformation
   Consequences

2 The Multi-Curve Framework
   Pricing Interest Rate Derivatives: What has changed?
   Pricing Plain Vanilla Instruments: Deposits and Swaps

3 Case of a Stochastic Spread
   Modeling the Basis Spread
   Caplet Pricing
   Introducing Shifted Log-Normal and SABR Dynamics
   Model Calibration

4 Results

5 Discussion & Improvement Suggestions

6 Conclusion
Summary

1. Introduction
   The Interest Rate Market Transformation
   Consequences

2. The Multi-Curve Framework

3. Case of a Stochastic Spread

4. Results

5. Discussion & Improvement Suggestions

6. Conclusion
Pricing Interest Rate Derivatives in the Multi-Curve Framework

Introduction

The Interest Rate Market Transformation

- Divergence between interest rates $\Rightarrow$ Appearance of a spread.
- Explosion of the spread in 2008 (August) and in 2011.
- Libor not considered risk-free anymore.
- Risky transactions.
• Interest rate divergence ⇒ Market subdivision.

• Expansion of collateralized transactions (Over Night rate as the risk-free rate).

• Replacement of the now risky Libor by the approximately risk-free Over night rate.

• Invalidity of the single-curve framework.

• Use of the Multi-curve framework: One discounting curve and different forward curves.
Summary

1 Introduction

2 The Multi-Curve Framework
   Pricing Interest Rate Derivatives: What has changed?
   Pricing Plain Vanilla Instruments: Deposits and Swaps

3 Case of a Stochastic Spread

4 Results

5 Discussion & Improvement Suggestions

6 Conclusion
Pricing procedure in the Multi-Curve framework

1. Construct the discount curve,
2. Construct the forward curves homogeneously with the tenors,
3. Use each forwarding curve to compute the FRA rates with tenor \( m \) for each coupon \( i \in \{1, \ldots, n\} \):

\[
FRA_m(t; T_{i-1}, T_i) = \frac{1}{\gamma_m(T_{i-1}, T_i)} \left( \frac{P_m(t, T_{i-1})}{P_m(t, T_i)} - 1 \right), \quad t \in [T_{i-1}, T_i)
\]

4. Compute the cash flows \( c_i \):

\[
c_i(t, T_i) = \mathbb{E}_t^{Q_{T_i}} [\Phi_i],
\]

5. Use the discount curve to compute the discount factors \( P_{d}(t, T_i) \),
6. Compute the \( t \)-price \( \phi(t) \) of the derivative of interest such that:

\[
\phi(t) = \sum_{i=1}^{m} P_d(t, T_i) \mathbb{E}_t^{Q_{T_i}} [\Phi_i].
\]
**Definition: Deposit**

A **deposit** is a zero coupon contract where a counterparty A lends a nominal N at $T_0$ to another counterparty, and at maturity $T$, receives the notional amount back as well as the interest accrued over the period $[T_0, T]$ at a simply compounded rate $R_m(T_0, T)$ fixed at a date $T_F \leq T_0$ and of tenor m.

**Payoff:**

$$\Phi_{Deposit}(T) = N(1 + R_m(T_0, T)\gamma(T_0, T)),$$

**Price at time** $t \in [T_F, T]:$

$$\Phi_{Deposit}(t) = P_m(t, T)E_{t}^{Q_m} [\Phi_{Deposit}(T)] = NP_m(t, T)(1 + R_m(T_0, T)\gamma(T_0, T)).$$
**Definition: Swap**

A **swap** is a $T$-maturity contract between two counterparties where they exchange a fixed rate for a floating rate, typically the Libor rate. Let $\Omega_T = \{T_0, T_1, ..., T_n\}$ be the cash flow schedule of the floating leg and $\Omega_S = \{S_0, S_1, ..., S_{n'}\}$ that of the fixed leg (with rate $K$). $n$ and $n'$ are the numbers of floating and fixed payments of the swap, respectively.

### Fixed Leg ($\Omega_S$)

- $t$
- $S_0$
- $S_1$
- $S_2$
- $S_3$
- $\ldots$
- $S_{n'}$

### Floating Leg ($\Omega_T$)

- $t$
- $T_0 = S_0$
- $T_1$
- $T_2$
- $T_3$
- $\ldots$
- $T_n$
Fixed Leg Cash Flows:
At the end of a period \([S_{j-1}, S_j]\), the fixed leg pays off:

\[ NK \gamma_K(S_{j-1}, S_j). \]

Float Leg Cash Flows:
At the end of a period \([T_{i-1}, T_i]\), the float leg pays off:

\[ N \gamma_{\text{float}}(T_{i-1}, T_i)L_m(T_{i-1}, T_i). \]

Price at time \( t \in [T_0, T] \):

\[
\Phi_{IRS}(t) = N \omega \left[ \sum_{i=1}^{n} P_d(t, T_i) \gamma_{\text{float}}(T_{i-1}, T_i) \mathbb{E}_{Q}^{T_i}_{d} [L_m(T_{i-1}, T_i)] \right] - \sum_{j=1}^{n'} K P_d(t, S_j) \gamma_K(S_{j-1}, S_j),
\]
Example of a yield curve:

OIS discount curve as of 01/10/2014

Interest rates as of 01/10/2014
Pricing Interest Rate Derivatives in the Multi-Curve Framework

The Multi-Curve Framework

Pricing Plain Vanilla Instruments: Deposits and Swaps

EUR003M Index (Euribor 3 Month ACT/360)
EUSWEC Curncy (EUR SWAP (EONIA) 3 MO)
EUR003M Index - EUSWEC Curncy

Graph showing the price movements of EUR003M Index and EUSWEC Currency from 2005 to 2015.
Summary

1 Introduction

2 The Multi-Curve Framework

3 Case of a Stochastic Spread
   Modeling the Basis Spread
   Caplet Pricing
   Introducing Shifted Log-Normal and SABR Dynamics
   Model Calibration

4 Results

5 Discussion & Improvement Suggestions

6 Conclusion
Notations:
Let us consider a tenor $m$ for the Ibor and the corresponding term structure $\{T^m_0, T^m_1, ..., T^m_N\}$, such that $\forall k \in \{1, ..., N\}, T^m_k - T^m_{k-1} = m$.

$$F^m_k(t) = F_d(t, T^m_{k-1}, T^m_k) = \frac{1}{\gamma^m(T^m_{k-1}, T^m_k)} \left( \frac{P_d(t, T^m_{k-1})}{P_d(t, T^m_k)} - 1 \right),$$

$$L^m_k(t) = FRA(t, T^m_{k-1}, T^m_k) = \mathbb{E}^{Q^m_t}_{t} [L_m(T^m_{k-1}, T^m_k)].$$

The spread is given by:

$$\forall t \geq T^m_0, S^m_k(t) = L^m_k(t) - F^m_k(t).$$

Aim: Adapt the Libor Market Model (LMM) to the Multi-Curve setting: We choose to model $S^m_k$ and $F^m_k$ and deduce $L^m_k$ needed for the pricing.

Assumptions:

$$\forall t \geq T^m_0, S^m_k(t) \geq 0, F^m_k(t)$$ can be negative and $F^m_k$ and $S^m_k$ are independent.
Definition: Caplet

A caplet of maturity $T_{k-1}^m$ is a call option indexed on a Libor rate of tenor $m$, $L_m(T_{k-1}^m, T_k^m)$, struck at $K$ (with a strike $K$). $L_m(T_{k-1}^m, T_k^m)$ is fixed (determined) at $T_{k-1}^m$ and payed at $T_k^m$. The payoff of the caplet at maturity $T_{k-1}^m$ is:

$$\gamma_m(T_{k-1}^m, T_k^m) \left[ L_m(T_{k-1}^m, T_k^m) - K \right]^+.$$

Index IBOR of tenor $m$

\[ L_m(T_{k-1}^m, T_k^m) \]

Caplet maturing at $T_{k-1}^m$ and paying off at $T_k^m$

\[ \gamma_m(T_{k-1}^m, T_k^m)[L_m(T_{k-1}^m, T_k^m) - K]^+ \]
Notation: $Q^{T^m_k}$ the measure when the numeraire is $T \to P_d(., T)$.

Recall the price of an interest rate derivative (here a caplet) in the Multi-Curve framework:

<table>
<thead>
<tr>
<th>Caplet price at time $t \leq T^m_{k-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_C(t, K; T^m_{k-1}, T^m_k) = \gamma_m(T^m_{k-1}, T^m_k) P_d(t, T^m_k) \mathbb{E}<em>t^{Q^{T^m_k}} [(L_m(T^m</em>{k-1}, T^m_k) - K)^+]$.</td>
</tr>
</tbody>
</table>

After some algebra, we obtain:

$$
\Phi_C(t, K; T^m_{k-1}, T^m_k) = \int_{-\infty}^{K} \Phi^{S}_C(t, K - y; T_{k-1}, T_k) f^{m}_{F}(T^m_{k-1})(y) dy
- S^m_k(t) \frac{\partial}{\partial K} \Phi^{F}_C(t, K; T_{k-1}, T_k) + \Phi^{F}_C(t, K; T_{k-1}, T_k).
$$

Where

$$
\Phi^{S}_C(t, K, T^m_{k-1}, T^m_k) = \gamma_m(T^m_{k-1}, T^m_k) P_d(t, T^m_k) \mathbb{E}_t^{Q^{T^m_k}} [(S^m_k(T^m_{k-1}) - K)^+],
$$

$$
\Phi^{F}_C(t, K, T^m_{k-1}, T^m_k) = \gamma_m(T^m_{k-1}, T^m_k) P_d(t, T^m_k) \mathbb{E}_t^{Q^{T^m_k}} [(F^m_k(T^m_{k-1}) - K)^+].
$$
We choose the following dynamics:

- **OIS forward rates: Shifted Log-Normal**

\[
    dF_k^m(t) = \sigma_k^m \left( F_k^m(t) + \frac{1}{\gamma_k^m} \right) dZ_k^m(t),
\]

(1)

- **The basis spread: SABR**

\[
\begin{align*}
    dS_k^m(t) &= [S_k^m(t)]^\beta_k V_k(t) dX_k(t), \\
    dV_k(t) &= \varepsilon_k V_k(t) dY_k(t), \quad V_k(0) = \alpha_k,
\end{align*}
\]

where \( \{X_k\}_{t \geq 0} \) and \( \{Y_k\}_{t \geq 0} \) are correlated standard Wiener processes under the \( Q_{d_k}^{T_m} \) measure, i.e. \( dX_k(t) dY_k(t) = \rho_k dt \) with \( \rho_k \in [-1, 1) \) and \( \alpha_k > 0, \varepsilon_k > 0, \beta_k \in (0, 1] \) are constants.
Pricing Interest Rate Derivatives in the Multi-Curve Framework

Case of a Stochastic Spread

Introducing Shifted Log-Normal and SABR Dynamics

After some algebraical manipulations, we replace the quantities appearing in the caplet price formula with their expressions and obtain:

\[
g(y, t, F^m_k(t)) = \left[ \ln \left( \frac{y}{F^m_k(t) + \frac{1}{\gamma^m_k}} \right) + \frac{1}{2} \left( \sigma^m_k \right)^2 (T^m_{k-1} - t) \right]^2,
\]

\[
h(y, t, F^m_k(t), S^m_k(t)) = \frac{\Phi^{SABR}_C(t, K + \frac{1}{\gamma^m_k} - y; T^m_{k-1}, T^m_k)}{\sigma^m_k y \sqrt{2\pi(T^m_{k-1} - t)}} \exp \left\{ -\frac{g(y, t, F^m_k(t))}{2(\sigma^m_k)^2 (T^m_{k-1} - t)} \right\}
\]

\[
\Phi_C(t, K; T^m_{k-1}, T^m_k) \approx \int_{\frac{1}{\gamma^m_k}}^{K + \frac{1}{\gamma^m_k}} h(y, t, F^m_k(t), S^m_k(t)) dy
\]

\[
+ \gamma^m_k P_d(t, T^m_k) \left[ \left( F^m_k(t) + \frac{1}{\gamma^m_k} \right) N(d^G_1) + \left( S^m_k(t) - K - \frac{1}{\gamma^m_k} \right) N(d^G_2) \right].
\]
\[ d_1^G = \frac{\ln \left( \frac{F^m_k(t) + \frac{1}{\gamma_k^m}}{K + \frac{1}{\gamma_k^m}} \right) + \frac{1}{2} (\sigma_k^m)^2 (T^m_{k-1} - t)}{\sigma_k^m \sqrt{T^m_{k-1} - t}}, \]

\[ d_2^G = d_1^G - \sigma_k^m \sqrt{T^m_{k-1} - t}. \]

Formula Assessment: A Monte Carlo Method with an antithetic variance reduction method.
Aim: Estimate the parameters $\alpha_k$, $\beta_k$, $\rho_k$, $\varepsilon_k$ and $\sigma^m_k$ for each $k$ using market data: Implied volatilities of EUR caps (underlying tenor 6M) as of 01/10/2014 ( Strikes 1%, 1.5%, 2%, 2.5%, 3%).

1. Compute caplet market prices from the market caps’ implied volatilities,
2. Fix $\beta_k = 0.5$ for example.
3. Perform a least squares minimization:

\[
(\hat{\alpha}_k, \hat{\rho}_k, \hat{\varepsilon}_k, \hat{\sigma}^m_k) = \arg\min_{\alpha_k, \rho_k, \varepsilon_k, \sigma^m_k} \sum_i \left[ \Phi_{\text{Model\ caplet}}(t, K_i, T) - \Phi_{\text{mkt\ caplet}}(t, K_i, T) \right]^2.
\]
Summary

1 Introduction

2 The Multi-Curve Framework

3 Case of a Stochastic Spread

4 Results

5 Discussion & Improvement Suggestions

6 Conclusion
Calibration errors as of 01/10/2014:

**Calibration error for maturities 3Y and 4Y (as of 01/10/2014)**

- **Strike**: 1.00%, 1.50%, 2.00%, 2.50%, 3.00%
- **Error**: 0,0000% to 2,5000%

- **3Y**: Red bars
- **4Y**: Blue bars

**Calibration error for maturity 6Y (as of 01/10/2014)**

- **Strike**: 1.00%, 1.50%, 2.00%, 2.50%, 3.00%
- **Error**: 0,0000% to 60,0000%

- **6Y**: Blue bars
3Y Caplet prices as of 01/10/2014

4Y Caplet prices as of 01/10/2014

- Market prices
- Formula
- MC
- Difference Model/Market
6Y Caplet prices as of 01/10/2014

- Market prices
- Formula
- MC
- Difference Model/Market
Results

Model Vs. Market Smile (3Y maturity)

Model Vs. Market Smile (4Y maturity)

Model Vs. Market Smile (6Y maturity)
Deterministic Spread:

3Y Caplet prices as of 01/10/2014

4Y Caplet prices as of 01/10/2014
6Y Caplet prices as of 01/10/2014

- Prices in the deterministic spread model
- Market prices
Summary

1. Introduction

2. The Multi-Curve Framework

3. Case of a Stochastic Spread

4. Results

5. Discussion & Improvement Suggestions

6. Conclusion
Results of the calibration as of 01/10/2014:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Caplet 3Y</th>
<th>Caplet 4Y</th>
<th>Caplet 6Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expiration date</td>
<td>02/10/2017</td>
<td>01/10/2018</td>
<td>01/10/2020</td>
</tr>
<tr>
<td>$F_{k}^{6M}(t)$</td>
<td>0.169%</td>
<td>0.438%</td>
<td>1.157%</td>
</tr>
<tr>
<td>$L_{k}^{6M}(t)$</td>
<td>0.501%</td>
<td>0.784%</td>
<td>1.465%</td>
</tr>
<tr>
<td>$S_{k}^{6M}(t)$</td>
<td>0.333%</td>
<td>0.346%</td>
<td>0.308%</td>
</tr>
<tr>
<td>$\alpha_{k}$</td>
<td>2.27%</td>
<td>2.05%</td>
<td>1.98%</td>
</tr>
<tr>
<td>$\beta_{k}$</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$\rho_{k}$</td>
<td>0.99</td>
<td>-0.99</td>
<td>0.99</td>
</tr>
<tr>
<td>$\varepsilon_{k}$</td>
<td>40.00%</td>
<td>0.35%</td>
<td>1.00%</td>
</tr>
<tr>
<td>$\sigma_{k}^{6M}$</td>
<td>0.1871%</td>
<td>0.236%</td>
<td>0.295%</td>
</tr>
</tbody>
</table>
Remarks:

- Week sensitivity to the correlation $\rho_k \Rightarrow$ Set $\rho_k = 0$,
- The correlation between Forward OIS rates and the stochastic volatility is set to zero $\Rightarrow$ same volatility dynamics under different measures,
- The rates are low so the calibration becomes hard because of data lacking for low strikes.

Imagination suggestions:

- Perform a weighted calibration (emphasis on the region where the FRA rate falls),
- Calibrate using other data sets (choose another reference date).
Summary

1. Introduction
2. The Multi-Curve Framework
3. Case of a Stochastic Spread
4. Results
5. Discussion & Improvement Suggestions
6. Conclusion
Advantages:

- Tractability,
- Possibility of closed formula derivation and thus easy implementation and calibration,
- The model could be calibrated better in an unstressed market (non negative rates and/or higher rates) (F. Mercurio’s example).

Limitations:

- In our example (i.e. low rates), the model only works for low and average maturities,
- All market data (for all quoted strikes) cannot be fitted using this model because of the level of rates.