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Collateral choice option valuation

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Abstract

A bank borrowing some money has to give some securities to the lender, which is called collateral. Different kinds of collateral can be posted, like cash in different currencies or a stock portfolio depending on the terms of the contract, which is called a Credit Support Annex (CSA). Those contracts specify eligible collateral, interest rate, frequency of collateral posting, minimum transfer amounts, etc. This guarantee reduces the counterparty risk associated with this type of transaction.

If a CSA allows for posting cash in different currencies as collateral, then the party posting collateral can, now and at each future point in time, choose which currency to post. This choice leads to optionality that needs to be accounted for when valuing even the most basic of derivatives such as forwards or swaps.

In this thesis, we deal with the valuation of embedded optionality in collateral contracts. We consider the case when collateral can be posted in two different currencies, which seems sufficient since collateral contracts are soon going to be simplified.

This study is based on the conditional independence approach proposed by Piterbarg [8]. This method is compared to both Monte-Carlo simulation and finite-difference method.

A practical application is finally presented with the example of a contract between Natixis and Barclays.

Keywords: collateral, optimal collateral posting, multi-currency collateral, collateral pricing, collateral discounting, conditional independence.

Acknowledgments

Contents

1   Introduction

In the wake of the financial crisis, practitioners witnessed divergence in rates and significantly widened swap rates, which impacts the discounting curves used in financial derivative valuation. The traditional assumption of a risk-free counterparty and rate was thus jeopardized. Piterbarg developed theoretical foundations for a model of an economy without a risk-free rate and with all assets traded on a collateralised basis [4], [8].

Credit Support Annexes (CSA) are part of the legal foundation for over-the-counter (OTC) derivatives trading, specifically focusing on counterparty credit risk mitigation. Among other things, these documents specify rules for posting collateral. If a CSA allows for posting cash in different currencies as collateral, then the party posting collateral has, now and at each future point in time, a choice of which currency to post. This choice leads to optionality that needs to be accounted for when valuing even the most basic of derivatives such as forwards or swaps.

We consider the important case when collateral can be delivered in exactly two different currencies. In this case, the adjustment to the discount factor applied to a cash flow paid at time $T \in \mathbb{R}_+$ reduces to calculating the expression of the form:

$$D(T) \triangleq \mathbb{E}\left[\exp\left(-\int_0^T q(s)^+\, ds\right)\right],$$

where q is a stochastic process representing the collateral basis, i.e. the difference between FX-adjusted collateral rates in the two currencies, and $x^+ \triangleq \max(x, 0)$.

The exact calculation of the expected value above in closed form appears to be impossible. However, a way to efficiently calculate $\{D(T_n)\}_{n=1}^N$ for a collection of times $\{T_n\}_{n=1}^N$, where $T_n = nT/N$ for $N \in \mathbb{N}^*$, is of critical importance as they are needed for discounting of all collateralised OTC derivatives.

In this thesis, we have been able to compute the adjusted discount factor used to price cash collateralised derivatives when collateral can be posted in two different currencies. This optionality valuation was performed by implementing Piterbarg's conditional independence method [8], which was compared to both Monte-Carlo simulation and finite-difference method (see Section 2).

Given the approximations considered in this method, the results are accurate and above all fast to obtain, which is the main advantage of this approach. This framework has been used in a practical application for a collateralised contract amendment between Natixis and Barclays, hence the need to price the compensation amount for removing the choice of currency (see Section 3).

2 Collateralised contracts

2.1 Context

A bank borrowing some money has to give some securities to the lender, which is called collateral. For instance, some cash or stocks can be post as collateral depending on the terms of the contract, which is called a Credit Support Annex (CSA). CSA specifies:

- eligible collateral (cash in a number of currencies, bonds);

- rates paid on collateral (party holding collateral typically pays a certain rate to the collateral owner);

- frequency of collateral posting (e.g. daily);

- thresholds, minimum transfer amounts, etc.

Let us consider the following example where party A buys a call option from party B: (see Figure 1)

1. A pays $V(0)$ dollars to B;

2. B posts $V(0)$ dollars as collateral to A and will post or claim back collateral during the life of the option since its value fluctuates;

3. A pays an agreed-upon overnight rate on the outstanding collateral to B;

4. B promises to pay the payoff of the option at expiry to A.

At any point in time t, the total collateral posted by B is $V(t)$, which is the value of the option on that day.

Figure 1: Collateralised contract

A huge increase of collateralised contracts has been noticed in recent years: from 30% in terms of trade volume for all OTC derivatives in 2003 to 70% in 2011 according to the ISDA Margin Survey.

More than 80% of collateral is cash and about half of the cash collateral is USD. In the case of posting cash as collateral, different currencies are usually allowed by the CSA. It thus leads to optionality.

We consider the special but important case of having the choice between posting two different currencies as discussed in Piterbarg [4]. An extension of this model to multi-currency collateral was not considered since legislation is soon going to simplify collateral posting. When dealing with a foreign counterparty, it is common to consider the domestic and the foreign currencies as eligible collateral.

2.2   Traders' approach

We first start by explaining how collateral is currently handled by traders.

With the derivative markets having changed dramatically since the 2008 financial crisis, regulatory reform and structural changes to the financial markets have resulted in the increased collateralisation of trades and a move to central clearing of vanilla trades. Financial practitioners are witnessing increased usage of collateral as a way to mitigate the risk of counterparty default. Many of these changes had a dramatic impact on how derivatives are fundamentally priced, with collateral choices impacting the discounting curves used in valuations. Adding to the post-crisis drama has been the divergence in rates and significantly widened basis swap spreads.

Traders acting on behalf of treasurers and other financial practitioners also face challenges surrounding the overwhelming complexity of the Credit Support Annex (CSA) in terms of embedded optionality. We explain below how to construct cheapest-to-deliver (CTD) curves and demonstrate how they enable practitioners to select appropriate collateral. A significant number of CSAs allow counterparties to choose collateral from a big list of eligible currencies and securities; furthermore, different currency collateral and types of collateral have different impacts on valuation. Given that almost every CSA agreement is unique, it is no wonder that a lack of transparency prevails and valuation discrepancies between counterparties abound, even for the simplest of trades. Many market participants have come to see that it is nearly impossible to compare prices between dealers.

A CTD curve is constructed via the following steps:

- **Step 1:** Construct all the appropriate curves necessary to build Overnight Indexed Swap (OIS), standard London Interbank Offered rate (Libor) curves and basis curves to translate local curves into the trade currency. See Figure 2 for an example with six currencies: US dollar (USD), euro (EUR), pounds sterling (GBP), Japanese yen (JPY), Swiss franc (CHF) and Canadian dollar (CAD).

Collateral/Currency	USD	EUR	CAD	GBP	JPY	CHF
Cash	USD FedFunds OIS Curve	EONIA OIS Curve	CORRA OIS Curve	SONIA OIS Curve	MUTAN OIS Curve	TOIS Curve
Corporate Bonds	USD LIBOR+ Spread	EURIBOR+ Spread	CDOR + Spread	GBP LIBOR + Spread	MUTAN OIS Curve	CHF LIBOR+ Spread

Figure 2: Curves to construct for six currencies

- **Step 2:** Translate curves in different currencies to the trade currency (see Figure 3).
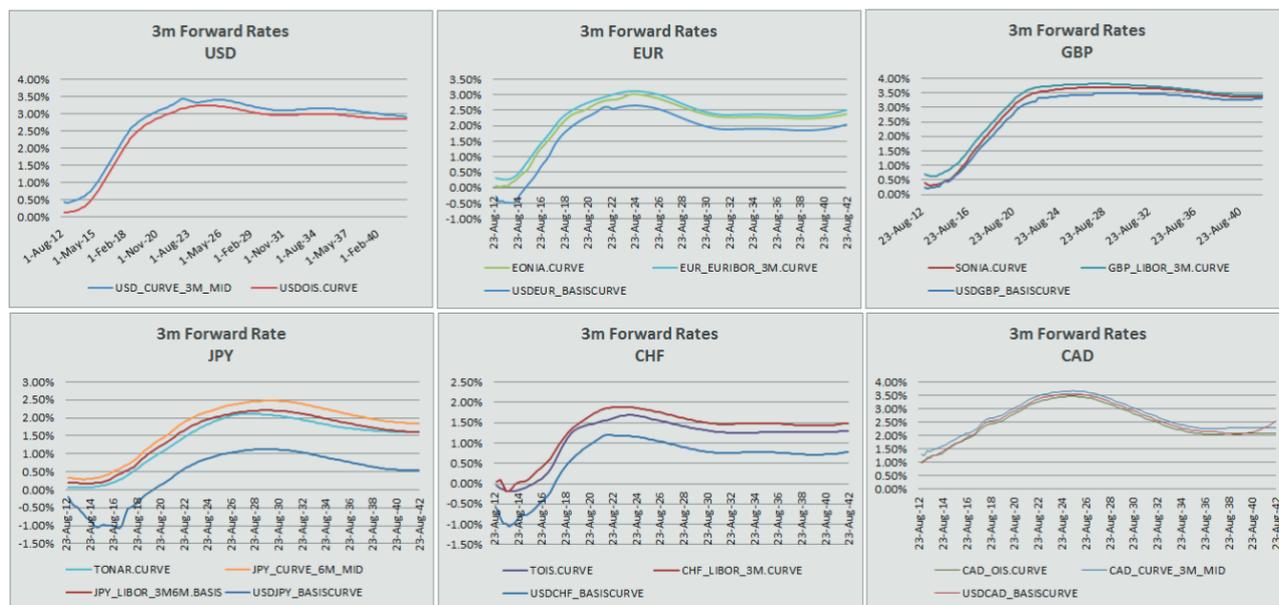
Figure 3: Curves construction for six currencies

- **Step 3:** Pick cheapest throughout the life of the trade i.e. the maximum instantaneous forward discount rate of collateral curves at each point of time, which corresponds to the lowest discount factor.

- **Step 4:** Construct the blended CTD curve obtained from the maxima forward rates (see Figure 4).
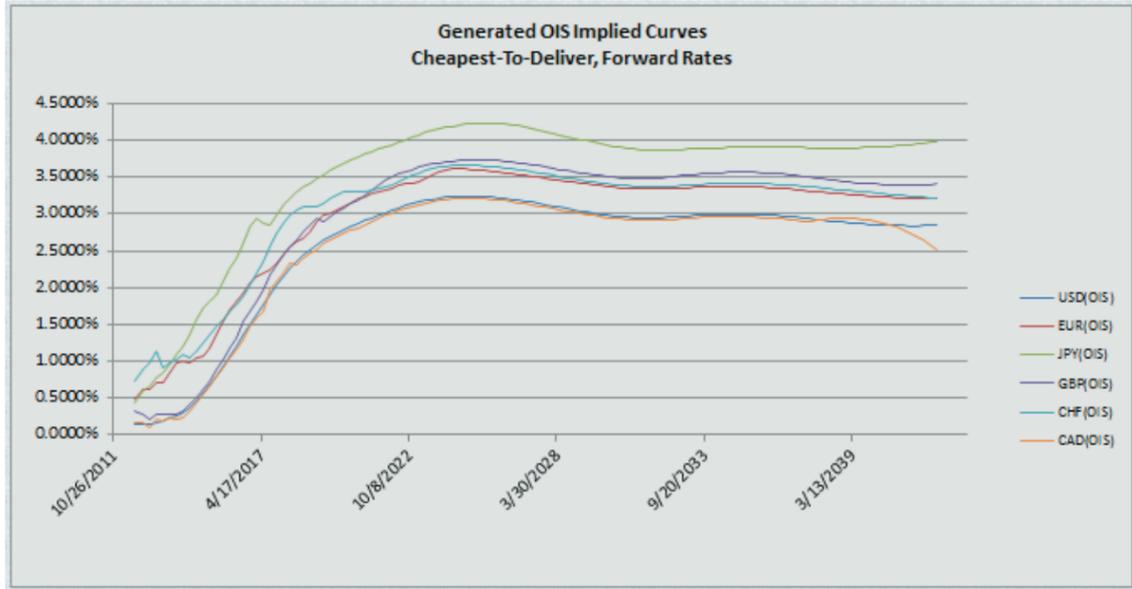
Figure 4: Cheapest-To-Deliver (CTD) curve

We can then discount future cash flows of the trades and derivatives with the CTD curve.

2.3   Model setup

We now consider a quantitative approach and model the problem as follows. We model q as a Gaussian process of the form:

$$q(t) \;=\; f(t) + x(t), \qquad t \in [0, T],$$

where f is a deterministic function and x is an Ornstein-Uhlenbeck (OU) process with dynamics:

$$
\begin{aligned}
dx(t) &= -\theta\, x(t)\, dt + \sigma\, dW(t), \\
x(0) &= 0,
\end{aligned}
$$

where θ and σ are respectively called the mean reversion and the volatility of the process q and W denotes a standard Brownian motion.

The stochastic differential equation (SDE) above can be solved explicitly:

$$
\begin{aligned}
x(t) &= h(t)\, y(t) \\
&= \sigma \int_0^t e^{-\theta(t-s)}\, dW(s),
\end{aligned}
$$

where

$$
\begin{aligned}
h(t) &\triangleq e^{-\theta t}, \\
y(t) &\triangleq \sigma \int_0^t e^{\theta s}\, dW(s).
\end{aligned}
$$

x can be represented as a stochastic integral of a deterministic function and is thus a Gaussian process:

$$
x(t) \sim \mathcal{N}\left(0, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\right).
$$

General properties of the OU process are stated and derived in Appendix A.1.

2.4   Numerical methods

Let us consider the three following numerical methods to compute the adjusted discount factor.

2.4.1   Monte-Carlo simulation

The Law of Large Numbers (LLN) states that the average of a sequence of independent and identically distributed (i.i.d.) random variables with equal expected value converges towards this common expected value.

Using this result, an expectation can be approximated by generating M sample paths $\{q_i\}_{i=1}^M$ and then taking the average as follows:

$$
\mathbb{E}\left[\exp\left(-\int_0^T q(t)^+\, dt\right)\right] \simeq S_M,
$$

where

$$
S_M \triangleq \frac{1}{M} \sum_{i=1}^M \exp\left(-\int_0^T q_i(t)^+\, dt\right).
$$

The algorithm used to simulate Gaussian random variables is detailed in Appendix A.2.

The rate of convergence of this method is given by the Central Limit Theorem (CLT), which states that:

$$
\frac{S_M - \mathbb{E}[S_M]}{\sqrt{\mathrm{Var}[S_M]}} \xrightarrow[M \to +\infty]{} \mathcal{N}(0,1),
$$

where $\mathrm{Var}[S_M]$ can be approximated by:

$$s_{M-1}^2 \triangleq \frac{1}{M-1} \sum_{i=1}^{M} (y_i - \overline{y})^2,$$

where

$$
\begin{aligned}
y_i &= \exp\left(-\int_0^T q_i(t)^+ \, dt\right), \\
\overline{y} &= \frac{1}{M} \sum_{i=1}^{M} y_i.
\end{aligned}
$$

2.4.2   Finite-difference method

The adjusted discount factor can be represented as follows:

$$D(T) = \int_{-\infty}^{+\infty} G(T, x) \, dx,$$

where

$$G(t, x) \triangleq \mathbb{E}\left[\exp\left(-\int_0^t q(s)^+ \, ds\right) \, \bigg| \, x_0 = x\right]$$

is the probability density function of the OU-process killed at rate $q(t)^+$.

The function G defined above satisfies the following forward Kolmogorov equation:

$$
\begin{aligned}
(\partial_t - L^*(t, x)) \, G(t, x) &= 0, \qquad t \in]0, T], \\
G(0, x) &= \delta(x),
\end{aligned}
$$

where

$$L^*(t, x) \, G(t, x) = -q(t)^+ \, G(t, x) + \theta \, \partial_x \left(x \, G(t, x)\right) + \frac{1}{2}\sigma^2 \, \partial_{xx} \, G(t, x),$$

where L^* is the dual of the infinitesimal generator of the OU-process killed at rate q^+ and δ is the Dirac function.

We define a spatial grid $\{x_n\}_{n=0}^{N_x+1}$ as follows:

$$x_n = x_0 + n\Delta x$$

and discretize the diffusion operator in matrix form:

$$A(t) = -q(t)^+ - \theta \, x \, \delta_x^* + \frac{1}{2}\sigma^2 \, \delta_{xx}.$$

We then define the following first and second-order difference operators:

$$\delta_x^* G(t, x_j) = \begin{cases} (G(t, x_{j+1}) - G(t, x_{j-1}))(2\Delta x)^{-1} & \text{if } |\theta(t) x_j| \Delta_x \le \sigma(t)^2 \\ (G(t, x_j) - G(t, x_{j-1}))(\Delta x)^{-1} & \text{if } \theta(t) x_j \Delta_x > \sigma(t)^2 \\ (G(t, x_{j+1}) - G(t, x_j))(\Delta x)^{-1} & \text{if } \theta(t) x_j \Delta_x < -\sigma(t)^2, \end{cases}$$

$$\delta_{xx} G(t, x_j) = (G(t, x_{j+1}) - 2G(t, x_j) + G(t, x_{j-1}))(\Delta x)^{-2}.$$

The corresponding spatial boundary conditions are the following:

$$G(t, x_0) = 0,$$
$$G(t, x_{N_x+1}) = 0.$$

The discrete version of the forward PDE reads:

$$(T_{k+1} - T_k)^{-1}[G(T_{k+1}) - G(T_k)] = A(T_{k+1})^T G(T_{k+1}),$$
$$G(0, x_n) = \begin{cases} (\Delta x)^{-1} & \text{if } x_n \neq 0 \\ 0 & \text{if } x_n = 0. \end{cases}$$

We then get the following iterative scheme:

$$G(T_{k+1}) = (I - (T_{k+1} - T_k)A(T_{k+1})^T)^{-1} G(T_k).$$

We obtain the $G(T_{k+1})$ by solving a matrix equation using Thomas' algorithm, which is presented in Appendix A.3.

We can finally compute the desired adjusted discount factor as follows:

$$D(T_k) = \Delta x \sum_{n=1}^{N_x} (G(T_k))_n.$$

2.4.3   Conditional independence approach

(a) General Idea:

We want to find a random variable $Z_0 \sim \mathcal{N}(0, 1)$ such that $\mathbb{E}[Q(t) Q(s)] = 0$, where

$$Q(t) = y(t) - \gamma(t) Z_0,$$
$$\gamma(t) = \mathbb{E}[y(t) Z_0].$$

We then get:

$$\mathbb{E}[Q(t) Q(s)] = \text{Var}[y(\min(t, s))] - \gamma(t) \gamma(s),$$

where

$$\text{Var}[y(\min(t, s))] = \sigma^2 \int_0^{\min(t,s)} e^{2\theta s} ds$$
$$= \frac{\sigma^2}{2\theta}(e^{2\theta \min(t,s)} - 1).$$

Let us assume that we have found γ such that $\mathbb{E}[Q(t)\,Q(s)] = 0$.

This leads to:

$$
\begin{aligned}
D(T_n) &= \mathbb{E}\left[\exp\left(-\int_0^{T_n} q(t)^+\,dt\right)\right] \\
&\simeq \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\sum_{i=1}^n q(T_i)^+\Delta T_i\right)\,\middle|\,Z_0\right]\right] \\
&\simeq \widehat{D}_{\mathrm{CI}}(T_n),
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{D}_{\mathrm{CI}}(T_n) &\triangleq \mathbb{E}\left[\widehat{D}_{\mathrm{CI}}(T_n, Z_0)\right], \\
\widehat{D}_{CI}(T_n, z) &\triangleq \prod_{i=1}^n B_i(z), \\
B_i(z) &= \mathbb{E}\left[\min\left(e^{-\mu_i-\sigma_i(\beta_i z+\widehat{\beta}_i Z_i)}, 1\right)\right],
\end{aligned}
$$

where $Z_i \sim \mathcal{N}(0,1)$ and the constants involved in the previous formula are defined by:

$$
\begin{aligned}
\mu_i &= \Delta T_i\, f(T_i), \\
\sigma_i &= \Delta T_i\, h(T_i)\,\sqrt{\mathrm{Var}[y(T_i)]}, \\
\beta_i &= \frac{\gamma(T_i)}{\sqrt{\mathrm{Var}[y(T_i)]}}, \\
\widehat{\beta}_i &= \sqrt{1-\beta_i^2}.
\end{aligned}
$$

$B_i(z)$ is a Black-like formula and can be computed as follows:

$$
B_i(z) = \int_{-\infty}^{+\infty} \min\left(e^{-\mu_i-\sigma_i(\beta_i z+\widehat{\beta}_i x)}, 1\right)\varphi(x)\,dx,
$$

which leads to the following formula:

$$
B_i(z) = \Phi(d_1) + e^{-\mu_i-\sigma_i\beta_i z+\sigma_i^2\widehat{\beta}_i^2/2}\,\Phi(d_2),
$$

where

$$
\begin{aligned}
d_1 &= -\frac{\mu_i}{\sigma_i\widehat{\beta}_i} - \frac{\beta_i}{\widehat{\beta}_i}z, \\
d_2 &= -d_1 - \sigma_i\widehat{\beta}_i
\end{aligned}
$$

and

$$\varphi : x \in \mathbb{R} \ \mapsto \ \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$\Phi : x \in \mathbb{R} \ \mapsto \ \int_{-\infty}^{x} \varphi(u)\, du.$$

The derivation of the Black-like formula is detailed in Appendix A.4.

The cumulative distribution function (CDF) of the standard normal distribution is computed using the approximation detailed in Appendix A.5.

(b) Small volatility expansion:

Now that we have motivated the conditional independence approach, let us find γ which is the only remaining unknown.

Let us define:

$$D(T, z) \ \triangleq \ \mathbb{E}\left[\exp\left(-\int_0^T q(t)^+\, dt\right) \bigg| Z_0 = z\right]$$

$$= \ \mathbb{E}\left[\exp\left(-\int_0^T (\ell(t, z) + h(t)\, Q(t))^+\, dt\right)\right]$$

and

$$D_{\mathrm{CI}}(T, z) \ \triangleq \ \lim_{N \to +\infty} \mathbb{E}\left[\exp\left(-\sum_{i=1}^{N} q(T_i)^+\right) \bigg| Z_0 = z\right]$$

$$= \ \lim_{N \to +\infty} \mathbb{E}\left[\exp\left(-\sum_{i=1}^{N}(\ell(T_i, z) + h(T_i)\, Q(T_i))^+\right) \bigg| Z_0 = z\right],$$

where

$$\ell(t, z) \ \triangleq \ q(t) + h(t)\, \gamma(t)\, z.$$

An expansion around a small ε yields:

$$D(T, z) \ \simeq \ e^{-\int_0^T \ell(t,z)^+\, dt}\left(1 + \int_0^T \int_0^{t'} h(t)\, \theta(t, z)\, h(t')\, \theta(t', z)\, (\mathrm{Var}[y(t)] - \gamma(t)\, \gamma(t'))\, dt\, dt'\right.$$

$$\left. - \frac{1}{2}\int_0^T \delta(\ell(t, z))\, h(t)^2\, (\mathrm{Var}[y(t)] - \gamma(t)^2)\, dt\right)$$

and

$$D_{\mathrm{CI}}(T, z) \ \simeq \ e^{-\int_0^T \ell(t,z)^+\, dt}\left(1 - \frac{1}{2}\int_0^T \delta(\ell(t, z))\, h(t)^2\, (\mathrm{Var}[y(t)] - \gamma(t)^2)\, dt\right).$$

The details are given in Appendix A.6.

Let us introduce the following notations:

$$R(T) \triangleq \mathbb{E}\left[e^{-\int_0^T \ell(t,Z_0)\,dt} \int_0^T \int_0^{t'} h(t)\,h(t')\,(\mathrm{Var}[y(t)] - \gamma(t)\,\gamma(t'))\,dt\,dt'\right],$$

$$R_+(T) \triangleq \mathbb{E}\left[e^{-\int_0^T \ell(t,Z_0)^+\,dt} \int_0^T \int_0^{t'} h(t)\,\theta(t,Z_0)\,h(t')\,\theta(t',Z_0)\,(\mathrm{Var}[y(t)] - \gamma(t)\,\gamma(t'))\,dt\,dt'\right].$$

(c) Variance fit:

Solving $R(T) = 0$ to the following equation:

$$\int_0^T \int_0^{t'} h(t)\,h(t')\,(\mathrm{Var}[y(t)] - \gamma(t)\,\gamma(t'))\,dt\,dt' \;=\; 0.$$

Separating the integral into two parts yields:

$$\left(\int_0^T h(t)\,\gamma(t)\,dt\right)^2 \;=\; 2\int_0^T \int_0^{t'} h(t')\,h(t)\,\mathrm{Var}[y(t)]\,dt'\,dt.$$

We recognize that the RHS is the variance of the integral of the OU-process:

$$\left(\int_0^T h(t)\,\gamma(t)\,dt\right)^2 \;=\; \mathrm{Var}\left[\int_0^T x(t)\,dt\right].$$

Differentiating w.r.t. T yields:

$$\gamma(T) \;=\; \frac{1}{h(T)}\frac{d}{dT}\left(\mathrm{Var}\left[\int_0^T x(t)\,dt\right]\right)^{1/2}.$$

The computation of the variance above is given in Appendix A.1.

(d) Optimal fit:

Solving $R_+(T) = 0$ yields to the following equation:

$$\mathbb{E}\left[e^{-\int_0^T \ell(t,Z_0)^+\,dt} \int_0^T \int_0^{t'} h(t)\,\theta(\ell(t,Z_0))\,h(t')\,\theta(\ell(t',Z_0))\,(\mathrm{Var}[y(t)] - \gamma(t)\,\gamma(t'))\,dt\,dt'\right] \;=\; 0.$$

Differentiating w.r.t. T yields:

$$\mathbb{E}\left[h(T)\,\theta(\ell(T,Z_0)) \int_0^T h(t)\,\theta(\ell(t,Z_0))\,(\mathrm{Var}[y(t)] - \gamma(t)\,\gamma(T))\,dt\right] \;=\; 0.$$

Inverting integral and expectation yields:

$$\int_0^T h(T)\, h(t)\, (\mathrm{Var}[y(t)] - \gamma(t)\, \gamma(T))\, \mathbb{E}\left[\theta(\ell(T, Z_0))\, \theta(\ell(t, Z_0))\right]\, dt \;=\; 0.$$

By definition of θ, ℓ and Z_0, we get:

$$\int_0^T \Phi\left(\min\left(\frac{f(T)}{\gamma(T)\, h(T)}, \frac{f(t)}{\gamma(t)\, h(t)}\right)\right) h(T)\, h(t)\, (\mathrm{Var}[y(t)] - \gamma(t)\, \gamma(T))\, dt \;=\; 0.$$

This non-linear equation can be solved numerically. We can also compute $\gamma(T_n)$ iteratively replacing γ by γ_0, solution to $R_+(\cdot) = 0$, for $t < T$:

$$\gamma(T) \;=\; \frac{\int_0^T \Phi\left(\min\left(\frac{f(T)}{\gamma_0(T)\, h(T)}, \frac{f(t)}{\gamma_0(t)\, h(t)}\right)\right) h(t)\, \mathrm{Var}[y(t)]\, dt}{\int_0^T \Phi\left(\min\left(\frac{f(T)}{\gamma_0(T)\, h(T)}, \frac{f(t)}{\gamma_0(t)\, h(t)}\right)\right) h(t)\, \gamma(t)\, dt}.$$

3   Implementation

3.1   Existing solution and constraints

There is currently no existing quantitative solution dealing with collateral optimal posting.

This issue is currently handled by the traders themselves and the idea of this internship is to quantify this issue and try to develop an optimal strategy.

The constraints are to get a fast and efficient method that gives the adjusted discount factor to take into account when pricing a collateralised derivative, so that there cannot be any arbitrage opportunity.

3.2   Architecture

The following object-oriented architecture was chosen: (see Figure 5)

- Class **Program**;

- Class **Gaussian** for the simulation of Gaussian random variables and the computations of their density / cumulative distribution functions;

- Class **Common** for common attributes and methods;

- Class **Numerical methods** and derived classes for the different numerical methods so that common properties can be factorized.
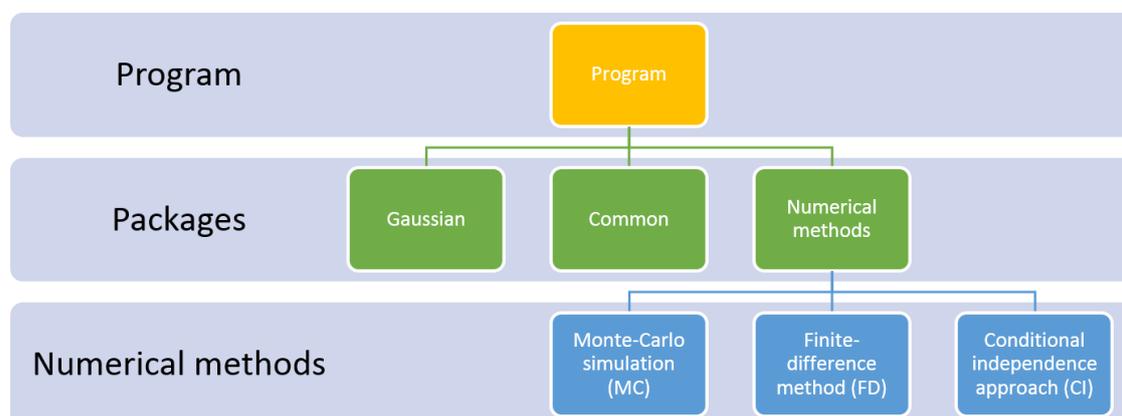
Figure 5: Program architecture

3.3   Algorithms

We considered the following different numerical methods implemented in C#:

1. Monte-Carlo simulation (MC);

2. finite-difference method (FD);

3. conditional independence approach (CI).

The key method of the article is the conditional independence one. The two others were also implemented as benchmarks to check our numerical results and to compare both the efficiency and the speed of the CI method.

3.4   Numerical results

3.4.1   Tests

The implemented numerical methods were tested using:

- unitary tests: numerical computation of an integral tested using integrands whose integral values are known, Black-like formula tested using Monte-Carlo simulation;

- formulas given in the article: the computational details were performed, which enabled me to detect a mistake in the Black-like formula since there was a maximum written instead of a minimum as well as a sign mistake. I contacted Piterbarg who was able to correct the electronic version available online.

3.4.2   Method comparison

The implemented methods were compared by first computing the adjusted discount rate, which is derived from the adjusted discount factor as follows:

$$r(T) \triangleq -\frac{\ln D(T)}{T}.$$

Using the following typical set of model parameters:

$$\theta(t) = 0.4, \qquad \sigma(t) = 0.01, \qquad f(t) = -0.015,$$

the following results are obtained considering a time horizon of 40 years:

T	$r_{\mathrm{MC}}(T)$	$r_{\mathrm{FD}}(T)$	$r_{\mathrm{CI}}(T)$
1	0.4	0.6	0.4
5	2.9	3.1	3.0
10	3.7	4.0	3.8
15	4.0	4.3	4.1
20	4.2	4.5	4.3
30	4.3	4.7	4.4
40	4.4	4.8	4.5

with T in years and r in basis points.

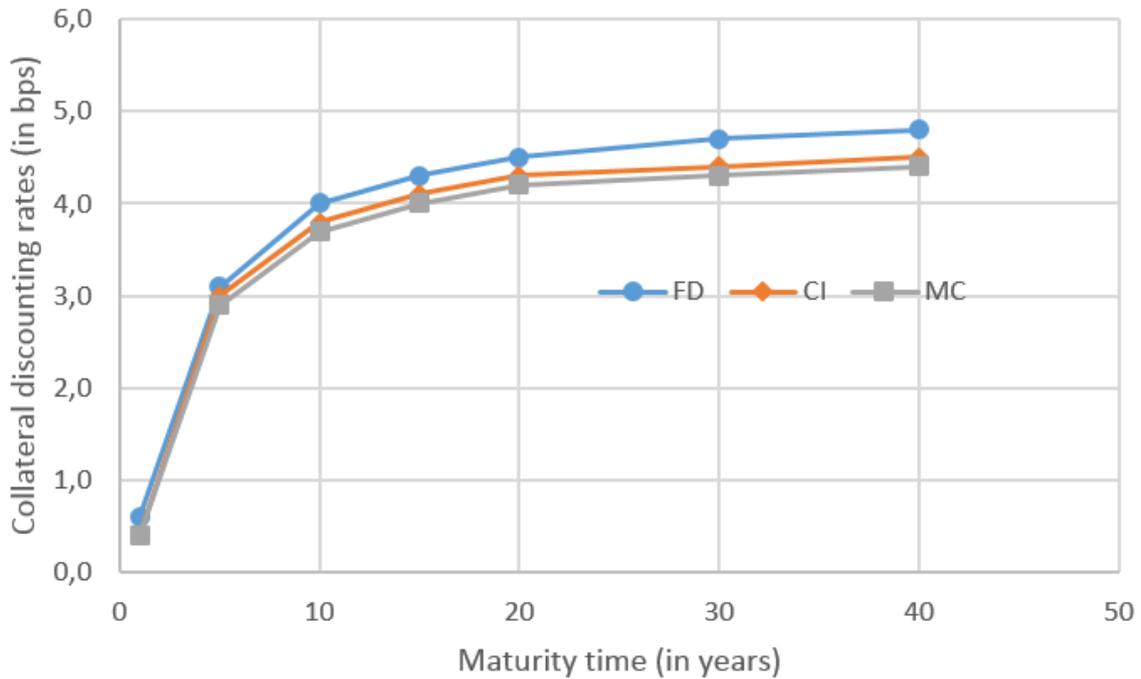Figure 6: Numerical methods comparison

The time-step size and the number of simulations used in the Monte-Carlo simulation are respectively $N = 100$ and $M = 1000$.

The time-step and space-step sized used in the finite-difference method are respectively $N = 150$ and $N_x = 3000$.

The time-step and space-step sized used in the conditional independence method are respectively $N = 25$ and $N_x = 20$.

The computational time (in ms) is given below for each numerical method:

MC	FD	CI
82	88	11

The CI method is thus very accurate and above all faster than the other usual ones.

3.4.3   Practical application

We now apply the previous conditional independence approach to a contract between Natixis and Barclays that is soon going to be amended.

Barclays used to post collateral in GBP or EUR to Natixis. But Natixis is soon going to accept only EUR as collateral. Natixis will thus have to compensate this lack of choice. The issue is to evaluate how much Natixis would have to pay Barclays as a compensation. Indeed, Barclays will ask for a certain amount of money and the idea is to be able to evaluate whether this amount is reasonable. Besides, we have to bear in mind that Piterbarg, who wrote the main article studied in this thesis, is the Head Quant of Barclays!

The first step is to find appropriate values for the model parameters. The model cannot be calibrated using market data since there are no derivatives from which those parameters could be implied. The way to go is thus historical estimation. Using the CSA curve, which is the spread between Overnight Interest Swap (OIS) rate and USD, and the BSEUR curve, which is the spread between EURIBOR and USD, we obtain the deterministic component $f(t)$ of the Gaussian process $q(t)$. See Figure 7.
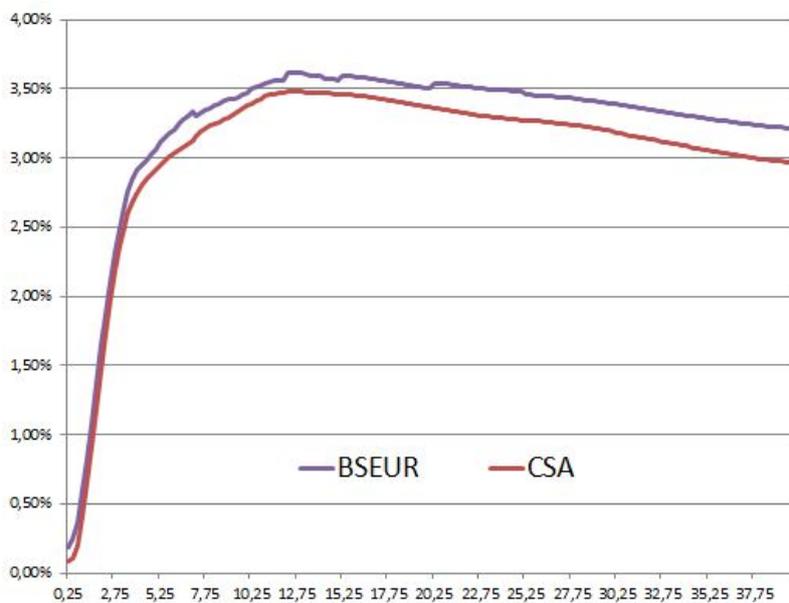
Figure 7: Spreads versus maturity

We use typical parameters for the mean reversion and the volatility of the Gaussian process:

$$\theta = 0.1, \qquad \sigma = 0.01.$$

Using those inputs, the program developed in C# can now be run to compute the adjusted discount rate. See Figure 8.
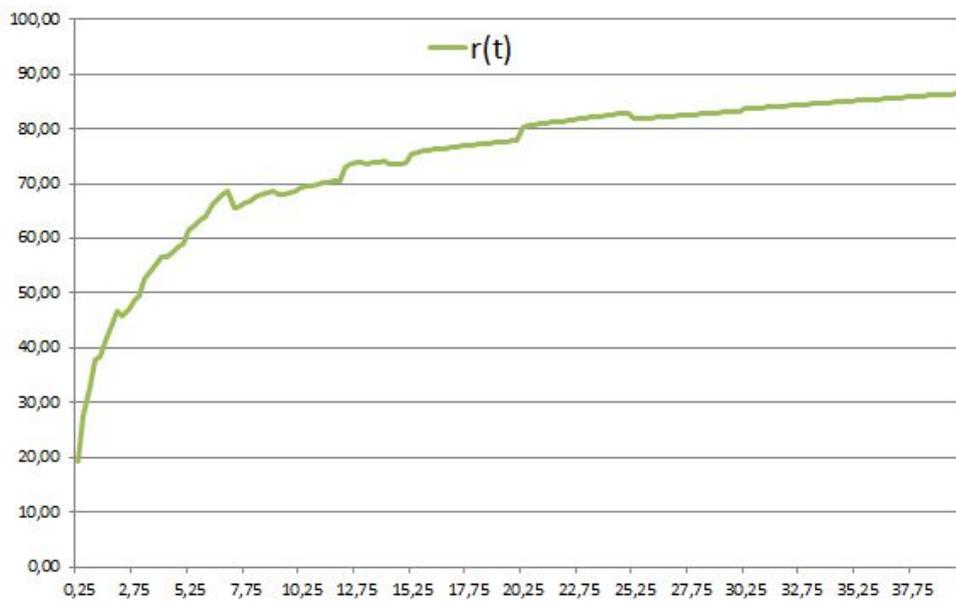
Figure 8: Adjustment rate versus maturity

4   Conclusion

In this thesis, we have been able to compute the adjusted discount factor used to price cash collateralised derivatives when collateral can be posted in two different currencies.

This optionality valuation was performed by implementing Piterbarg's conditional independence method, which was compared to both Monte-Carlo simulation and finite-difference method.

Given the approximations considered in this method, the results are accurate and above all fast to obtain, which is the main advantage of this approach.

This framework has been used in a practical application for a collateralised contract amendment between Natixis and Barclays, hence the need to price the compensation amount for removing the choice of currency.

An extension to multi-currency collateral has not been considered. Indeed, Credit Support Annexes that give rules for collateral posting are soon going to be simplified in order to make optimal collateral posting easier. The simple case of two different currencies is thus significant.

A   Appendices

A.1   Ornstein-Uhlenbeck process

An Ornstein-Uhlenbeck process is a stochastic process x with the following dynamics:

$$\begin{aligned} dx(t) &= -\theta\,(x(t) - \mu)\,dt + \sigma\,dW(t), \\ x(0) &= x_0 \in \mathbb{R}_+^*, \end{aligned}$$

where $(\theta, \mu, \sigma) \in (\mathbb{R}_+^*)^3$ and W is a standard Brownian motion.

The explicit solution of this stochastic differential equation (SDE) can be derived by applying Itô's lemma to the following process:

$$X(t) \triangleq e^{\theta t}\,x(t).$$

Itô's lemma yields:

$$\begin{aligned} dX(t) &= \theta\,e^{\theta t}\,x(t)\,dt + e^{\theta t}\,dx(t) \\ &= \mu\,\theta\,e^{\theta t}\,dt + \sigma\,e^{\theta t}\,dW(t). \end{aligned}$$

Integrating on $[0, t]$ yields:

$$X(t) = x_0 + \mu\left(e^{\theta t} - 1\right) + \sigma \int_0^t e^{\theta s}\,dW(s)$$

i.e.

$$\begin{aligned} x(t) &= e^{-\theta t}\,X(t) \\ &= x_0\,e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \sigma \int_0^t e^{-\theta\,(t-s)}\,dW(s). \end{aligned}$$

x can be represented as a stochastic integral of a deterministic function and is thus a Gaussian process.

- Its mean is given by:

$$\mathbb{E}[x(t)] = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$$

  since the expectation of a stochastic integral is 0.

- And its covariance is given by:

$$\begin{aligned} \mathrm{Cov}[x(t), x(s)] &= \sigma^2\,e^{-\theta\,(t+s)} \int_0^{\min(t,s)} e^{2\theta u}\,dW(u) \\ &= \frac{\sigma^2}{2\theta}\,e^{-\theta(t+s)}\left(e^{2\theta\min(t,s)} - 1\right) \end{aligned}$$

  using the Itô isometry

In particular:

$$\begin{aligned} \mathrm{Var}[x(t)] &= \mathrm{Cov}[x(t), x(t)] \\ &= \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}). \end{aligned}$$

In the case we consider through this paper, we have $x_0 = 0$ and $\mu = 0$.
The following process:

$$K(t) \triangleq \int_0^t x(s)\, ds$$

is also a Gaussian random variable.

Integrating the SDE given above on $[0, t]$ yields:

$$\int_0^t x(s)\, ds = \mu t + \frac{x(t) - x_0}{\theta} + \frac{\sigma}{\theta}\int_0^t dW(s).$$

Replacing x by its expression derived above yields:

$$\int_0^t x(s)\, ds = \mu t + (x_0 - \mu)\frac{1 - e^{-\theta t}}{\theta} + \frac{\sigma}{\theta}\int_0^t (1 - e^{-\theta(t-s)})\, dW(s).$$

The first moments of the Gaussian process K are easily obtained as follows:

$$\begin{aligned} \mathbb{E}[K(t)] &= \mu t + (x_0 - \mu)\frac{1 - e^{-\theta t}}{\theta}, \\ \mathrm{Var}[K(t)] &= \frac{\sigma^2}{\theta^2}\left(t - \frac{1 - e^{-\theta t}}{\theta}\right) - \frac{\sigma^2}{2\theta^3}\left(1 - e^{-\theta t}\right)^2. \end{aligned}$$

A.2   Simulation of Gaussian random variables

We can simulate a standard normal random variable Z using Box-Muller algorithm given below:

1. independently simulate $U_1 \sim \mathcal{U}([0, 1])$ and $U_2 \sim \mathcal{U}([0, 1])$;

2. set $Z \triangleq \sqrt{-2\ln U_1}\cos(2\pi U_2)$.

We can then obtain a normal random variable X with mean m and variance σ^2 by setting:

$$X \triangleq m + \sigma Z.$$

A.3   Thomas algorithm

Let us consider the following matrix equation involving a tridiagonal matrix:

$$
\begin{bmatrix}
f_1 & g_1 \\
e_2 & f_2 & g_2 \\
 & e_3 & f_3 & g_3 \\
 & & & \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & & \cdot & \cdot & \cdot \\
 & & & & & e_{n-1} & f_{n-1} & g_{n-1} \\
 & & & & & & e_n & f_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
r_1 \\ r_2 \\ r_3 \\ \cdot \\ \cdot \\ \cdot \\ r_{n-1} \\ r_n
\end{bmatrix},
$$

i.e. $Ax = r$.

The solution x can be obtained using Thomas' algorithm given below:

1. **Decomposition:**
   for $k = 2, 3, ..., n$
   $$e_k = e_k / f_{k-1}$$
   $$f_k = f_k - e_k \cdot g_{k-1}$$

2. **Forward substitution:**
   for $k = 2, 3, ..., n$
   $$r_k = r_k - e_k \cdot r_{k-1}$$

3. **Back substitution:**
   $$x_n = r_n / f_n$$
   for $k = n - 1, n - 2, ..., 1$
   $$x_k = (r_k - g_k \cdot x_{k+1}) / f_k$$

A.4   Black-like formula

Let us recall the Black-like formula to derive:

$$
B_i(z) \;=\; \int_{-\infty}^{+\infty} \min\left(e^{-\mu_i - \sigma_i(\beta_i z + \widehat{\beta}_i x)}, 1 \right) \varphi(x)\, dx.
$$

Step 1: Split the integral into two parts:

$$
\begin{aligned}
B_i(z) &= \int_{-\infty}^{d_1} \varphi(x)\,dx + \int_{d_1}^{+\infty} e^{-\mu_i - \sigma_i(\beta_i z + \widehat{\beta}_i x)}\,\varphi(x)\,dx \\
&= \Phi(d_1) - \int_{d_1}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\mu_i - \sigma_i(\beta_i z + \widehat{\beta}_i x) - x^2/2}\,dx \\
&= \Phi(d_1) - e^{-\mu_i - \sigma_i \beta_i z + \sigma_i^2 \widehat{\beta}_i^2/2} \int_{d_1}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(x + \sigma_i \widehat{\beta}_i\right)^2/2}\,dx,
\end{aligned}
$$

where

$$
d_1 = -\frac{\mu_i}{\sigma_i \widehat{\beta}_i} - \frac{\beta_i}{\widehat{\beta}_i} z.
$$

Step 2: Make the change of variable $u = x + \sigma_i \widehat{\beta}_i$:

$$
\begin{aligned}
B_i(z) &= \Phi(d_1) - e^{-\mu_i - \sigma_i \beta_i z + \sigma_i^2 \widehat{\beta}_i^2/2} \int_{d_1 + \sigma_i \widehat{\beta}_i}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du \\
&= \Phi(d_1) - e^{-\mu_i - \sigma_i \beta_i z + \sigma_i^2 \widehat{\beta}_i^2/2} \left[1 - \Phi\left(d_1 + \sigma_i \widehat{\beta}_i\right) \right] \\
&= \Phi(d_1) - e^{-\mu_i - \sigma_i \beta_i z + \sigma_i^2 \widehat{\beta}_i^2/2} \Phi\left(d_2\right),
\end{aligned}
$$

where

$$
d_2 = -d_1 - \sigma_i \widehat{\beta}_i.
$$

A.5   Approximation of the standard normal cumulative distribution function

The standard normal cumulative distribution function:

$$
\Phi : x \in \mathbb{R} \mapsto \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du
$$

can be approximated as follows:

$$
\Phi(x) \simeq 1 - \frac{1}{\sqrt{2\pi}} (b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) e^{-x^2/2},
$$

where

$$
\begin{aligned}
p &= 0.2316419, \\
b_1 &= 0.319381530, \\
b_2 &= -0.356563782, \\
b_3 &= 1.781477937, \\
b_4 &= -1.821255978, \\
b_5 &= 1.330274429, \\
t &= \frac{1}{1+px}
\end{aligned}
$$

for $x \in \mathbb{R}_+$ and $\Phi(x) = 1 - \Phi(-x)$ for $x \in \mathbb{R}_-^*$.

The approximation error is of the order 10^{-7}.

A.6 Small volatility expansion

A expansion around a small ε yields:

$$
\mathbb{E}\left[e^{-\int_0^T (\ell(t,z)+\varepsilon\, h(t)\, Q(t))^+\, dt}\right] \simeq e^{-\int_0^T \ell(t,z)^+\, dt}\left(1 - \varepsilon\, \alpha + \frac{1}{2}\,\varepsilon^2\,\beta - \frac{1}{2}\,\varepsilon^2\,\gamma\right),
$$

where

$$
\begin{aligned}
\alpha &\triangleq \mathbb{E}\left[\int_0^T \theta(\ell(t,z))\, h(t)\, Q(t)\, dt\right], \\
\beta &\triangleq \mathbb{E}\left[\left(\int_0^T \theta(\ell(t,z))\, h(t)\, Q(t)\, dt\right)^2\right], \\
\gamma &\triangleq \mathbb{E}\left[\int_0^T \delta(\ell(t,z))\, h(t)^2\, Q(t)^2\, dt\right].
\end{aligned}
$$

We then get:

$$
\begin{aligned}
\alpha &= 0, \\
\beta &= 2\int_0^T \int_0^{t'} \theta(\ell(t',z))\, h(t')\, \theta(\ell(t,z))\, h(t)\, \mathbb{E}\left[Q(t)Q(t')\right]\, dt\, dt', \\
\gamma &= \int_0^T \delta(\ell(t,z))\, h(t)^2\, \mathbb{E}\left[Q(t)^2\right]\, dt
\end{aligned}
$$

since $\mathbb{E}[Q(t)] = 0$.

Finally, we obtain:

$$
D(T,z) \simeq e^{-\int_0^T \ell(t,z)^+\, dt}\left(1 + \frac{1}{2}\,\beta - \frac{1}{2}\,\gamma\right).
$$

Doing the same for the conditionally independent discount factor yields:

$$\mathbb{E}\left[e^{-(\ell(T_i,z)+\varepsilon\,h(T_i)\,Q(T_i))^+\,\Delta T}\,\middle|\,Z_0=z\right] \;\simeq\; e^{-\ell(T_i,z)^+\,\Delta T}\left(1-\varepsilon\,\alpha_i+\frac{1}{2}\,\varepsilon^2\,\beta_i-\frac{1}{2}\,\varepsilon^2\,\gamma_i\right),$$

where

$$\alpha_i \;\triangleq\; \mathbb{E}\left[\theta(\ell(T_i,z))\,h(T_i)\,Q(T_i)\,\Delta T\,\middle|\,Z_0=z\right],$$

$$\beta_i \;\triangleq\; \mathbb{E}\left[(\theta(\ell(T_i,z))\,h(T_i)\,Q(T_i)\,\Delta T)^2\,\middle|\,Z_0=z\right],$$

$$\gamma_i \;\triangleq\; \mathbb{E}\left[\delta(\ell(T_i,z))\,h(T_i)^2\,Q(T_i)^2\,\Delta T\,\middle|\,Z_0=z\right].$$

We then get:

$$D_{\mathrm{CI}}(T,z) \;\simeq\; e^{-\int_0^T \ell(t,z)^+\,dt}\left(1-\frac{1}{2}\gamma\right)$$

when we let ΔT go to 0 and only keep terms of order $\mathcal{O}(T_i)$.

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