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SF299X Master Thesis in Mathematical Statistics

Optimization of hydro power on the Nordic electricity exchange using financial derivatives

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Abstract

Since the deregulation of the Nordic electricity market in 1996, electricity has become one of the most traded commodities in the Nordic region. The electricity price is characterized by large fluctuations as the supply and demand of electricity are seasonally dependent. The main interest of the hydro power producers is to assure that they can sell their hydro power at an attractive rate over time. This means that there is a demand for hedging against these fluctuations which in turn creates trading opportunities for third party actors that offer solutions between consumers and producers. Telge Krafthandel is one of these actors interested in predicting the future supply of hydro power, and consequently the resulting price of electricity. Several existing models employ the assumption of perfect foresight regarding the weather in the future. In this thesis, the authors develop new models for hydro power optimization that take hydrological uncertainty into account by implementing a variation of multi-stage optimization in order to maximize the income of the hydro power producers. The optimization is performed with respect to prices of financial derivatives on electricity. This gives insights into the expected supply of hydro power in the future which in turn can be used as an indicator of the price of electricity. The thesis also discusses, among other things, different methods for modeling stochastic inflow to the reservoirs and scenario construction. This practice will result in different methods that are suitable for various key players in the industry.

Keywords: Optimization, hydro power, linear programming, stochastic programming, scenario construction, stochastic inflow modeling, financial derivatives on electricity.

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Nomenclature

Input variables	
c_i	Discounted price of forward contract on electricity in time step i (EUR/GWh).
Inflow_i	Scenario forecast of future inflow to the reservoirs during time step i (GWh).
n	Number of time steps in the optimization problem (the time period under study).
Reservoirs_0	Water level in the reservoirs before the optimization problem begins (GWh).
$\text{Reservoirs}_{\text{Target}}$	Future target water level in the reservoirs after the optimization problem ends (GWh).
x_0	Observed hydro power production during the time step before the optimization problem begins (GWh).
Output variables	
x_i	Hydro power production in time step i (GWh).
Constraints	
HP_i	Hydro power production in time step i (GWh).
Reservoirs_i	Water level in the reservoirs in time step i (GWh).
x_{Total}	Maximum available hydro power production during time step 1 to time step n (GWh).
Symbols	
Θ_i	Stochastic inflow to the reservoirs during time step i (GWh).

Table 1: A selection of variables used in the report.

1 Introduction

A general introduction to the electricity market and the broad characteristics surrounding this thesis is provided in Section 1.1. Section 1.2 presents the objective of this thesis. In Section 1.3, the limitations are introduced. The structure of the report is outlined in Section 1.4.

1.1 The electricity market

The Swedish electricity market was deregulated in 1996 which opened up for competition between producers and the introduction of an electricity exchange. The demand for electricity is seasonally dependent both on a daily, weekly and yearly basis (see Figure 1 for weekly forward prices), resulting in large fluctuations in electricity prices. The main interest of a hydro power producers is therefore to assure that they can sell their hydro power at an attractive rate over time.

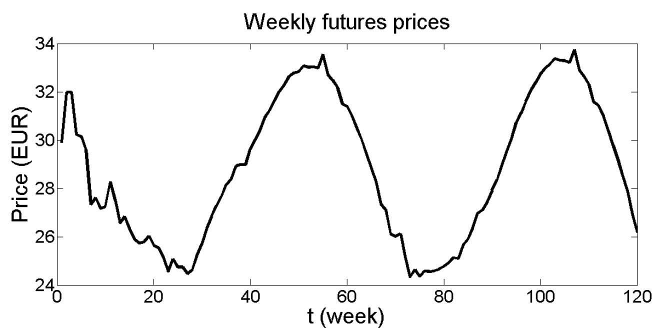

Figure 1: Prices of forward contracts each week (EUR).

Analysts argue that before the deregulation, the focus of the hydro power producers was more oriented towards public welfare than corporate interests. In present time, hydro power producers have shifted towards becoming revenue maximizing utilities. Currently, hydro power production covers more than 50% of the total electricity consumption in the Nordic countries. Due to the large volume in combination with the high flexibility in production and low production costs, hydro power is an important factor impacting the price of electricity. This relatively large share of the total electricity consumption originating from hydro power is unique compared to the rest of the world, and this can partially be explained by the different geographical characteristics.

The Nordic region has a common electricity market called Nord Pool Spot. This market enables the buying and selling of power for physical delivery during all hours under the following day. Using bids from the buyers and sellers, the price for each hour during the following day is calculated. The spot price for each hour is set where the demand curves intersect with the supply curves.

Furthermore, there is a financial market for electricity on the exchange Nasdaq OMX. This market enables buying and selling of forwards and futures contracts on electricity. These contracts are financially settled, which means that the price difference between the decided price and the spot price will be settled between the buyer and the seller of the contract.

Large electricity consumers, such as industries and real estate companies, usually want their expenses to be foreseeable. This means that there is a demand for hedging against price fluctuations which in turn creates trading opportunities for third party actors that offer solutions between consumers and producers. Telge Krafthandel is one of these actors interested in predicting the future supply of hydro power, and consequently the resulting price of electricity.

Electricity can be regarded as a perishable commodity since it needs to be consumed instantly, without the possibility to stockpile for later use. The utilization factor of the storing capabilities for electricity are so small in modern times that it is generally considered as a non-storable asset on a large scale.

A common dynamic in the market can be observed when there are large volumes of hydro power being produced combined with a large amount of water in the reservoirs as a result of favorable inflow to the reservoirs. In these circumstances, the price of electricity tends to decrease due to an increased supply but relatively stable demand.

Several existing optimization models employ the assumption of perfect foresight regarding the weather in the future. These models do not explicitly take advantage of new information that continuously becomes available over time. In reality, perfect foresight when it comes to weather in the future is an unrealistic assumption. Changes in weather forecasts regarding future inflow to the reservoirs is an example of new information that typically could impact the strategy decisions of hydro power producers.

1.2 Thesis objectives

The main objective of this thesis is to develop an optimization model for the purposes of Telge Krafthandel. The model should take hydrological uncertainty into account by implementing a variation of multi-stage optimization in order to maximize the income of the hydro power producers. This will give insights into the expected supply of hydro power in the future which in turn can be used as an indicator of future electricity prices.

Furthermore, this thesis will consider different methods to model stochastic inflow and scenario construction. This practice will result in different methods that are suitable for various key players in the industry.

Specifically, the objective is to find a model that in a realistic manner captures the uncertainty a hydro scheduler is faced with when planning the production. Moreover, the model should capture how the decision-maker would use new information as it occurs. Hence, the following research question will be answered:

RQ: *How should a model be developed for different actors in the area of hydro power optimization to take advantage of new information regarding uncertain inflow to the reservoirs?*

1.3 Limitations

This thesis does not deal with models that capture the details of short-term planning. This is mainly because of the nature of the thesis-specific problem and limitations in data. A more sophisticated type of optimization model requires detailed data that is not easily accessible, such as details for each individual hydro power plant, water reservoirs, etc. The objective of this thesis is to develop an optimization model that uses public data that are directly observable from the market. Furthermore, the objective is to perform optimization on a large system (the Nordic region), as opposed to optimizing individual hydro power plants.

1.4 Structure of the thesis

In Section 2, the nature of the problem is explained and the optimization problem is mathematically formulated. Section 3 presents relevant theory on the most frequently used optimization models for solving long-term hydro scheduling problems. Section 4 covers the methods used to model the distribution of inflow, construct scenarios, solve the optimization problem using both linear and stochastic programming, and finally how to measure the performance of the different models under study. In Section 5, the results are presented for all optimization models (different combinations of modeling inflow, constructing scenarios and modeling time steps) considered in this thesis. The results are discussed in Section 6 and the conclusions are presented in Section 7. Finally, some suggestions for further research are given in Section 8.

2 Problem formulation

In order to answer the research question, the desired optimization problem needs to be defined. The optimal solution will be defined as the production that generates the greatest income for the hydro power producer. Understandably, there is a trade-off between producing in the short-term perspective, when the prices are more predictable, versus saving water resources for production in the long-term perspective, when the prices are uncertain. The electricity prices in the future are represented by the prices of forward contracts traded on the electricity exchange. Moreover, a number of constraints regarding the production and the management of the reservoirs further complicates the problem. The objective function of this problem is defined in Section 2.1 and the constraints are formulated in Section 2.3. Additionally, Section 2.2 describes the input parameters to the problem and how the optimal solution will look like.

2.1 Objective function

To model the optimal supply of hydro power, the objective function this problem seeks to maximize is the income of the producers over time. The income of the producers is modeled as the discounted price of forward contracts on electricity multiplied by the planned production during the corresponding time periods:

$$\text{Income} = c_1 x_1 + \ldots + c_n x_n = \mathbf{c}^T \mathbf{x}$$

where

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and

$c_i =$ Discounted price of forward contract on electricity (EUR/GWh) in time step i, $i = 1, \ldots, n$.

$x_i =$ Hydro power production (GWh) in time step i, $i = 1, \ldots, n$.

Hence, the optimization problem can be formulated as:

$$\begin{bmatrix} \underset{\mathbf{x}}{\text{maximize}} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{bmatrix}$$

where the matrix $\mathbf{A}$ and the vector $\mathbf{b}$, which are corresponding to the constraints of the problem, are derived in Section 4.2 Constraints.

2.2 Input/output

The data input to the model is the number of time steps for the optimization problem (n, the time period under study), the current water level in the reservoirs (Reservoirs_0, the starting position), the production during the previous time step (x_0), the best guess forecast of future inflow to the reservoirs (**Inflow**), the future target water level in the reservoirs after the optimization problem ends ($\text{Reservoirs}_{\text{Target}}$), and the prices of forward contracts on electricity ($\mathbf{c}$). Furthermore, the model needs input data for the different constraints (see Section 2.3 Constraints).

$$\textbf{Inflow} = \begin{pmatrix} \text{Inflow}_1 \\ \vdots \\ \text{Inflow}_n \end{pmatrix}, \; \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where

Inflow_i = Scenario forecast of future inflow to the reservoirs (GWh) in time step i, $i = 1, \ldots, n$.

c_i = Discounted price of forward contract on electricity (EUR/GWh) in time step $i, \; i = 1, \ldots, n$.

In the case when the inflow of water to the reservoirs is modeled as stochastic, the inflow is represented by a stochastic process X_t. The stochastic inflow during a specific time step is denoted by the random variable Θ. A stochastic process can be represented as a sum of random variables:

$$X_t = \sum_{i=1}^{t} \Theta_i, \; t = 1, \ldots, n.$$

where

$t \in \{1, \ldots, n\}$ is the set of discrete time steps.

Θ_i = Stochastic inflow of water (GWh) during time step $i, \; i = 1, \ldots, n$.

The purpose of the future target water level in the reservoirs is to make the solution compatible in the long-term perspective. Without the inclusion of a future target level, the optimal solution would always end up at the minimum allowed reservoirs level. Thus, it would be considered optimal to consume all the permitted water resources. However, a reasonable assumption is that hydro power production in the future will continue in a similar fashion as in modern times. In an ideal world, the optimization problem would be solved for $n \to \infty$, but practical reasons combined with an increased uncertainty of the far future render this approach unfeasible.

The optimization will be modeled for the Nordic region. The output of the model is a vector of production volumes ($\mathbf{x}$) for each time step that is optimal for the time period under study:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where

x_i = Hydro power production (GWh) in time step i, $i = 1, \ldots, n$.

Moreover, the water level in the reservoirs (GWh) for a specific time step is determined by the ingoing reservoir level, the hydro power production and the new inflow of water. The best guess meteorological forecast of inflow for the upcoming 120 weeks can be seen in Figure 2. The water level in the reservoirs can be formulated as:

$$\begin{aligned} \text{Reservoirs}_1 &= \text{Reservoirs}_0 + \text{Inflow}_1 - x_1 \\ &\vdots \\ \text{Reservoirs}_n &= \text{Reservoirs}_{n-1} + \text{Inflow}_n - x_n \end{aligned}$$

Thus, a generalized expression for time step i can be formulated as:

$$\text{Reservoirs}_i = \text{Reservoirs}_{i-1} + \text{Inflow}_i - x_i,\ i = 1, \ldots, n. \tag{1}$$

where

Reservoirs_i = Water level in the reservoirs (GWh) in time step i, $i = 1, \ldots, n$.
Inflow_i = Inflow of water to the reservoirs (GWh) in time step i, $i = 1, \ldots, n$.
x_i = Hydro power production (GWh) in time step i, $i = 1, \ldots, n$.

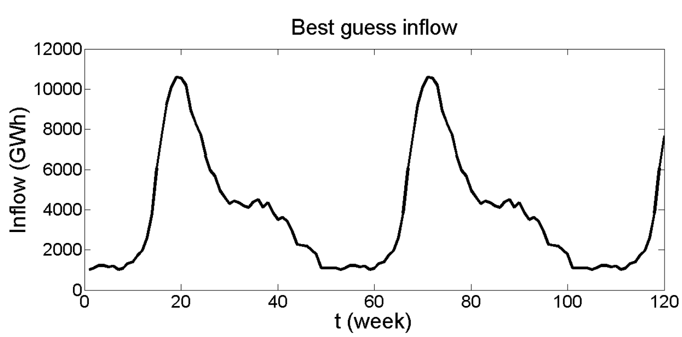

Figure 2: Best guess meteorological forecast of the weekly inflow to the reservoirs.

2.3 Constraints

The constraints consist of different parts which captures different characteristics of the hydro power plants. These characteristics are total production, production for each time step (minimum and maximum level of production and a minimum and maximum difference in production between two consecutive time steps) and the water level in the reservoirs each time step. For convenience, these constraints are called (in order of appearance) total production, min/max production, min/max Δ production and min/max reservoirs.

These constraints reflect historical extreme observations (all time high/low for the respective time steps of the year), representing an upper and a lower bound. Thus, the purpose of the constraints is to generate a solution that is realistically feasible given the previous practices in the hydro power production industry.

It may be theoretically possible to breach these limits in practice, but in that case the solution is in the context of unknown territory since similar conditions have not been observed before. This means that, if the historical constraints are not satisfied, the solution might be unrealistic since it's not in harmony with historical experiences.

If the model were to use technical constraints instead of historical constraints, a situation of potential power outage could occur when the solution approaches the theoretical minimum and the inflow is low. This is a scenario that the producers seek to avoid since they are typically unwilling to risk to expose themselves to such an extreme situation. Therefore, the historical extreme values are chosen as the constraints in this thesis.

For example, when minimizing or maximizing over the set Reservoirs_1, the members of the set are the recorded water levels in the reservoirs during time step 1 for all the previous years in the data set (year 1996 to 2014). The reason why these years are selected is because off the deregulation in 1996, which increased the availability of public data. Furthermore, the data prior to 1996 is not relevant in this thesis since it does not represent current market dynamics. A general observation from the energy industry is that larger volumes of hydro power have been produced after the deregulation, presumably due to the increased economic incentives.

2.3.1 Total production

The constraint on the total production is reflecting the total available hydro power production for the whole time period under study, according to the scenario inflow to the reservoirs.

Total hydro power production (*total production*):

$$\sum_{i=1}^{n} x_i = x_1 + \ldots + x_n \leq x_{\text{Total}}$$

where

x_{Total} = Maximum available hydro power production (GWh) during time step 1 to week n.

$x_{\text{Total}} \geq 0$.

x_i = Hydro power production (GWh) in time step i, $i = 1, \ldots, n$.

$x_i \geq 0, \ i = 1, \ldots, n$.

2.3.2 Min/max production

The minimum and maximum production for each time step (*min/max production*, see Figure 3):

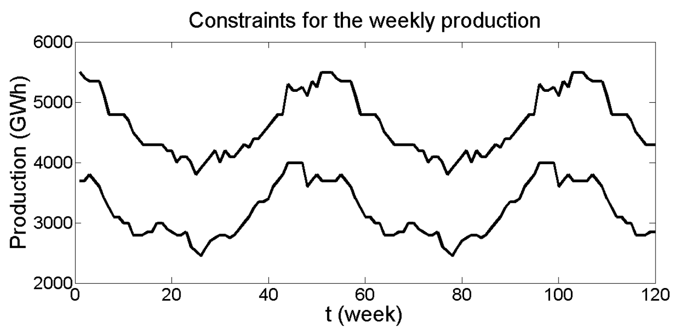

Figure 3: Constraints for the minimum and maximum production.

Hydro power production each time step (*min/max production*):

$$min(\text{HP}_1) \leq x_1 \leq max(\text{HP}_1)$$
$$\vdots$$
$$min(\text{HP}_n) \leq x_n \leq max(\text{HP}_n)$$

Thus, a generalized expression for time step i can be formulated as:

$$min(\text{HP}_i) \leq x_i \leq max(\text{HP}_i), \ i = 1, \ldots, n.$$

where

$min(\text{HP}_i)$ = Minimum hydro power production (GWh) during time step i, $i = 1, \ldots, n$.

$max(\text{HP}_i)$ = Maximum hydro power production (GWh) during time step i, $i = 1, \ldots, n$.

$\text{HP}_i \geq 0, \ i = 1, \ldots, n$.

2.3.3 Min/max Δ production

The difference in hydro power production between two consecutive time steps (*min/max Δ production*, see Figure 4):

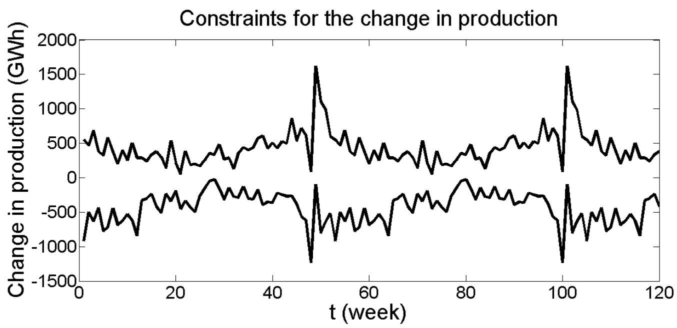

Figure 4: Constraints for the minimum and maximum change in production between two consecutive time steps. The change in production must be located within these boundaries.

$$\begin{aligned}
&x_0 - x_1 \leq max(|\text{HP}_1^{\downarrow}|)\\
&x_1 - x_0 \leq max(\text{HP}_1^{\uparrow})\\
&\vdots\\
&x_{n-1} - x_n \leq max(|\text{HP}_n^{\downarrow}|)\\
&x_n - x_{n-1} \leq max(\text{HP}_n^{\uparrow})
\end{aligned}$$

The constraint for the difference in hydro power production between time step i and $i-1$ can be expressed as the following closed interval:

$$min(\text{HP}_i^{\downarrow}) \leq x_i - x_{i-1} \leq max(\text{HP}_i^{\uparrow}),\ i = 1, \ldots, n.$$

$$\begin{aligned}
x_{i-1} - x_i &\leq max(|\text{HP}_i^{\downarrow}|),\ i = 1, \ldots, n.\\
x_i - x_{i-1} &\leq max(\text{HP}_i^{\uparrow}),\ i = 1, \ldots, n.
\end{aligned}$$

where

$min(\mathrm{HP}_i^{\downarrow})$ = Maximum downward change in hydro power production (GWh) between time step $i-1$ and i, $i = 1, \dots, n$.

$\mathrm{HP}_i^{\downarrow} \leq 0$, $i = 1, \dots, n$.

$max(\mathrm{HP}_i^{\uparrow})$ = Maximum upward change in hydro power production (GWh) between time step $i-1$ and i, $i = 1, \dots, n$.

$\mathrm{HP}_i^{\uparrow} \geq 0$, $i = 1, \dots, n$.

x_0 = Observed hydro power production (GWh) during the time step before the optimization problem begins.

$x_0 \geq 0$.

2.3.4 Min/max reservoirs

Furthermore, the system has constraints on the water level in the reservoirs for each time step (*min/max reservoirs*, see Figure 5).

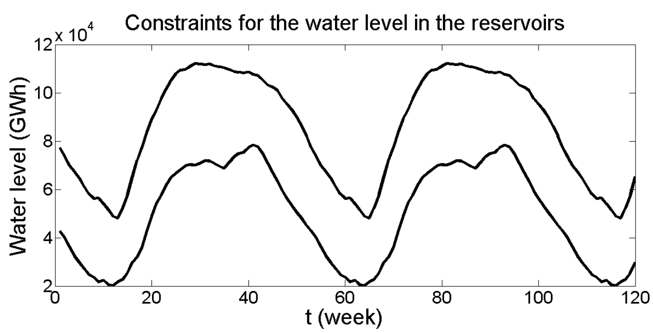

Figure 5: Constraints for the minimum and maximum water level in the reservoirs.

$$
\begin{aligned}
&min(\text{Reservoirs}_1) \leq y_1 - x_1 && \leq max(\text{Reservoirs}_1) \\
&min(\text{Reservoirs}_2) \leq y_2 - x_1 - x_2 && \leq max(\text{Reservoirs}_2) \\
&\vdots \\
&min(\text{Reservoirs}_{n-1}) \leq y_{n-1} - \sum_{i=1}^{n-1} x_i && \leq max(\text{Reservoirs}_{n-1}) \\
&y_n - \sum_{i=1}^{n} x_i \geq \text{Reservoirs}_{\text{Target}}
\end{aligned}
$$

Thus, a generalized expression for time step i ($1 \leq i \leq n-1$, there is a special case for $i = n$ due to the future target level as can be seen above) can be formulated as:

$$min(\text{Reservoirs}_i) \leq y_i - \sum_{j=1}^{i} x_j \quad \leq max(\text{Reservoirs}_i),\ i = 1, \ldots, n-1.$$

where

$y_i = \text{Reservoirs}_0 + \sum_{j=1}^{i} \text{Inflow}_j,\ i = 1, \ldots, n.$

Reservoirs_0 = Observed water level in the reservoirs (GWh) in the time step before the optimization problem begins.

$\text{Reservoirs}_{\text{Target}}$ = Future target water level in the reservoirs (GWh) in the time step the optimization problem ends (week n).

$min(\text{Reservoirs}_i)$ = Minimum water level in the reservoirs (GWh) in time step $i,\ i = 1, \ldots, n.$

$max(\text{Reservoirs}_i)$ = Maximum water level in the reservoirs (GWh) in time step $i,\ i = 1, \ldots, n.$

$\text{Reservoirs}_i \geq 0,\ i = 0, \ldots, n.$

$\text{Inflow}_i \geq 0,\ i = 1, \ldots, n.$

3 Literature review

This section presents relevant theory on some of the most frequently used optimization models that are considered as candidates for models in this thesis. The arguments that support the selection of certain models from this section are communicated in Section 4.

There are commonly two types of scheduling problems considered for hydro power production in larger systems (optimization for singular plants will not be considered due to the magnitude of the problem in this thesis). These can be categorized into short-term and long-term planning. The overall goal for a hydro power producer is to gain as much income as possible from the current water in the reservoirs and the expected inflow in the future. To achieve this, it is important to combine the two types of optimization problems in order to take advantage of the daily fluctuations in prices for the short-term production at the same time as water should be saved for later periods with potentially higher prices.

For the short-term planning, key conditions that must be considered are start-up costs for opening or closing specific water gates, how the production of an upstream plant affects downstream plants and the amount of water that can be used during the specific period (Yildiran et al, 2015). In the literature, there exists different types of solution methods for these problems with specific advantages and disadvantages.

Often times, the objective of the optimization problem is to either maximize the profit (by producing when prices are expected to be high) or to maximize the amount of produced electricity (by minimizing the spillage in the system). In the Nordic region, where spillage is rare, profit maximization is more common. In this setting, the aim is to try to save the optimal amount of water for periods of higher expected prices. Spillage minimization objective functions are common in for instance Brazil, when considering cascades of hydro power plants. In this situation, there is a larger amount of water in circulation and downstream plants are affected by upstream plants. Thus, the downstream plants might not be able to use all the available water if the upstream plant is running maximum production during an extended period of time (Yildiran et al, 2015).

In order to optimize the total income from the production, a proxy of the future electricity prices are needed. As these are unknown in advance, Fleten & Wallace (1998) suggest that the prices of financial forward prices traded on the electricity exchange can be used as an approximation. Fleten et al (2009) also state that 9 of the 14 largest hydro power producers in Norway use forward prices in order to plan their own production. This thesis will only focus on long-term scheduling with the objective to optimize the production on a weekly, or monthly basis. As long-term scheduling usually uses time steps of one week or longer, a common simplification of the problem is to ignore the short-term fluctuations in the spot price and production, and instead optimize subject to the weekly forward prices and constraints rescaled to weekly values.

Solving the long-term hydro optimization problem can be done in several ways. Linear programming (LP) is the most basic form of optimization and requires that both the objective function and the constraints are linear functions. As can be expected, the simplicity is associated with certain limitations. This makes LP unable to capture all the constraints needed for short-term optimization. For long-term planning, key constraints are the long-term water level in the reservoirs and constraints on the minimum and maximum production in the system. Since these constraints can be formulated as linear functions, LP is useful in the long-term perspective.

In several systems, depending on the design of the reservoirs, the amount of production is not only dependent on the discharge of water, but also on the difference in height between the upstream reservoir and the downstream water level (called the head effect). Explained by the laws of physics, the transformation from potential energy to kinetic energy creates a nonlinear objective function. To solve these types of problems, non-linear-programing (NLP) is needed (Feltmark & Lindberg, 1997; Catalao et al, 2011).

In reality, it is not possible to solve hydro scheduling problems by optimizing deterministic objective functions. Hydro scheduling is associated with uncertainty both when it comes to available water volumes, as the inflow is unknown, and when it comes to prices, as we do not know future prices. Modelling uncertainty is a challenging task, and stochastic optimization problems can not be solved by only solving one deterministic LP or NLP.

Most practitioners use stochastic programming (SP) where the objective function is expressed as an expectation rather than a deterministic function. To solve these types of problems, some information regarding the distribution of the stochastic variable is needed. The general idea of SP is to create a large amount of scenarios from the estimated distribution and use LP or NLP to find an optimal solution for each of the scenarios.

The construction of a comprehensive and representative scenario set for the involved stochastic quantities is a challenging task. Depending on the underlying stochastic variable, there are several different approaches that can be used to generate a meaningful set of scenarios. For example, the inflow scenarios can be based on a combination of the current meteorological forecasts and historical inflows, assuming that the inflow obeys a seasonal pattern (Albers, 2011).

The weighted average solution is referred to as the deterministic equivalent and is the most basic form of solving a SP. Note however that the deterministic equivalent solution is usually not the same as the optimal solution for the expected value of the scenarios (Birge & Louveaux, 1997) .

A drawback of solving the deterministic equivalent is that it scenarios are regarded as if they were known with perfect information. This means that the outcomes of the random variables (the scenarios) for the examined periods are known in advance, which is never the case in reality. Therefore, a common practice is to solve the problem using a recourse decision and solving the recourse problem, usually called multi-stage optimization (Birge & Louveaux, 1997). A multi-period setting requires several decisions to be made throughout a series of uncertain occurrences. Since the outcomes are unknown to the decision maker, they can be described with the help of stochastic quantities. At each time step, the optimal decision is sought with respect to observed past outcomes and in anticipation of unknown future realizations. The expectations for the future are captured by a finite set of possible future scenarios $\mathbb{S} \in \{S_1, \dots, S_{n_S}\}$ (Albers, 2011). These scenarios correspond to different realizations of the underlying probability distribution for the stochastic inflow Θ.

4 Method

As previously stated, there exists several optimization methods, each with different ways of dealing with uncertainty and scenario creation. As the objective of this thesis is to find the most suitable optimization model that captures uncertainty and realistic decision making based on available information, different models need to be considered. In this section, the optimization models included in this thesis are described as well as how their performance will be measured. Specifically, Section 4.1 starts by describing how the problem can be solved using linear programming and stochastic programming. Section 4.2 describes the constraints in detail and how they are formed.

The majority of this thesis is focusing on stochastic programming. Hence, the upcoming sections are only relevant for SP. Section 4.3 describes the two time step models multi-stage (MS) and deterministic equivalent (DE). Section 4.4 describes three different methods of modelling the distribution of inflow: Normalization, Time Series and Bootstrap. Section 4.5 describes three methods of constructing scenarios: independent stochastic sampling (IS), trinomial deterministic tree (TD) and trinomial stochastic tree (TS). Combining these methods in all possible ways result in a total of 18 optimization models. Finally, Section 4.6 describes the parameters that are evaluated when measuring the performance of the different models.

4.1 Optimization model

Stochastic programs have the reputation of being computationally difficult to solve. This can lead to practitioners faced with real-world problems being naturally inclined to solve simpler versions of the problems. Frequently used simpler versions are, for example, to solve the deterministic program obtained by replacing all random variables by their expected values (the mean value problem) or to solve several deterministic problems, each corresponding to one particular scenario, and then to combine these different solutions by some heuristic rule (Birge & Louveaux, 1997).

Hence, from a mathematical standpoint, the stochastic optimization problem (SP) can be formulated and solved as a deterministic linear program (LP). In order to incorporate uncertainties, probabilities can be assigned to different scenarios to create the corresponding deterministic equivalent problem (Albers, 2011).

4.1.1 Linear programming

Since the optimization problem can be expressed as a linear objective function subject to linear inequality constraints, the problem can be solved using linear programming (LP). Within the framework of linear programming, there exists several well-known methods such as the simplex algorithm and the interior-point method.

The LP problem can be formulated as (Sasane & Svanberg, 2010):

$$\text{(LP)} \begin{bmatrix} \underset{\mathbf{x}}{\text{maximize}} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{bmatrix}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where

c_i = Discounted price of forward contract on electricity (EUR/GWh) in time step i, $i = 1, \dots, n$.

x_i = Hydro power production (GWh) in time step i, $i = 1, \dots, n$.

The optimization problem is solved using Matlab's built-in function *linprog*, available via Optimization Toolbox (MathWorks, 2015). Using this implementation, the problem is solved as a minimization problem. To transform a maximization problem into a minimization problem, the objective function is multiplied by the factor (-1). Hence, the following optimization problem is solved:

$$\text{(LP)} \begin{bmatrix} \underset{\mathbf{x}}{\text{minimize}} & -\mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{bmatrix}$$

4.1.2 Stochastic programming

Production planning, energy planning and water resource modeling are areas that have been the subject of stochastic programming (SP) models for many years. Stochastic programming can model uncertain future situations so that informed policy decisions may be made (Birge & Louveaux, 1997).

A key component when modeling long-term water optimization is the future inflow of water to the reservoirs. Especially when applying a future target level of the reservoirs, the future inflow represent the total available production volume over the examined time period. The inflow can be forecasted for short time periods, however the accuracy of the forecast decrease rapidly for large time horizons. A common approach to capture this uncertainty is to express the inflow to the reservoirs as a stochastic variable and solve the optimization problem using stochastic programming.

In this thesis, the stochastic feature is the inflow of water to the reservoirs. The production of hydro power is a function of, among other things, inflow of water to the reservoirs. This means that the optimization problem needs to be reformulated to suit the requirements of the problem setting in this thesis.

Hence, the SP problem can be formulated as:

$$\text{(SP)}\begin{bmatrix} \underset{\mathbf{x}(\Theta)}{\text{maximize}} & \mathbb{E}[\mathbf{c}^T\mathbf{x}(\Theta)] \\ \text{subject to} & \mathbf{A}\mathbf{x}(\Theta) \leq \mathbf{b} \end{bmatrix}$$

4.2 Constraints

The inequality constraints, represented by the matrix **A** and the vector **b**, consists of different parts which captures different characteristics of the hydro power plants, explained in Section 2.3. In order to solve the problem, all inequality constraints need to be formulated on matrix form as "less than or equal" ($\leq$) constraints. This section presents the derivations of the constraint matrices:

$$\mathbf{A} = \underbrace{\begin{pmatrix} \mathbf{A}_{\text{total production}} \\ \mathbf{A}_{\text{min production}} \\ \mathbf{A}_{\text{max production}} \\ \mathbf{A}_{\text{min/max }\Delta\text{ production}} \\ \mathbf{A}_{\text{min/max }\Delta\text{ production }x_0} \\ \mathbf{A}_{\text{min reservoirs}} \\ \mathbf{A}_{\text{max reservoirs}} \end{pmatrix}}_{(6n\times n)} \begin{matrix} \}(1\times n) \\ \}(n\times n) \\ \}(n\times n) \\ \}\big(2(n-1)\times n\big) \\ \}(2\times n) \\ \}(n\times n) \\ \}\big((n-1)\times n\big) \end{matrix}, \quad \mathbf{b} = \underbrace{\begin{pmatrix} \mathbf{b}_{\text{total production}} \\ \mathbf{b}_{\text{min production}} \\ \mathbf{b}_{\text{max production}} \\ \mathbf{b}_{\text{min/max }\Delta\text{ production}} \\ \mathbf{b}_{\text{min/max }\Delta\text{ production }x_0} \\ \mathbf{b}_{\text{min reservoirs}} \\ \mathbf{b}_{\text{max reservoirs}} \end{pmatrix}}_{(6n\times 1)} \begin{matrix} \}(1\times 1) \\ \}(n\times 1) \\ \}(n\times 1) \\ \}\big(2(n-1)\times 1\big) \\ \}(2\times 1) \\ \}(n\times 1) \\ \}\big((n-1)\times 1\big) \end{matrix}$$

4.2.1 Total production

On matrix form, the constraint on the total hydro power production (*total production*) can be expressed as:

$$\underbrace{\begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}}_{(1\times n)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{(n\times 1)} \leq x_{\text{Total}}$$

Thus, we get that:

$$\mathbf{A}_{\text{total production}} = \underbrace{\begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}}_{(1 \times n)}, \ \mathbf{b}_{\text{total production}} = x_{\text{Total}}$$

4.2.2 Min/max production

On matrix form, the constraints on the minimum and maximum production each time step (*min/max production*) can be expressed as:

$$\underbrace{\begin{pmatrix} min(\text{HP}_1) \\ \vdots \\ min(\text{HP}_n) \end{pmatrix}}_{(n \times 1)} \leq \underbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}}_{(n \times n)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{(n \times 1)} \leq \underbrace{\begin{pmatrix} max(\text{HP}_1) \\ \vdots \\ max(\text{HP}_n) \end{pmatrix}}_{(n \times 1)}$$

Thus, we obtain:

$$\mathbf{A}_{\text{min production}} = \underbrace{\begin{pmatrix} -1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}}_{(n \times n)}, \ \mathbf{b}_{\text{min production}} = \underbrace{\begin{pmatrix} -min(\text{HP}_1) \\ \vdots \\ -min(\text{HP}_n) \end{pmatrix}}_{(n \times 1)}$$

$$\mathbf{A}_{\text{max production}} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}}_{(n \times n)}, \ \mathbf{b}_{\text{max production}} = \underbrace{\begin{pmatrix} max(\text{HP}_1) \\ \vdots \\ max(\text{HP}_n) \end{pmatrix}}_{(n \times 1)}$$

where

$\mathbf{A}_{\text{max production}}$ = The identity matrix of size n ($\boldsymbol{I}_n$).
$\mathbf{A}_{\text{min production}} = -\boldsymbol{I}_n$

4.2.3 Min/max Δ production

On matrix form, the constraints on the difference in hydro power production between two consecutive time steps (*min/max Δ production*) can be expressed as:

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}}_{\left(\,2(n-1)\times n\,\right)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{(n\times 1)} \leq \underbrace{\begin{pmatrix} max(|\mathrm{HP}_2^{\downarrow}|) \\ max(\mathrm{HP}_2^{\uparrow}) \\ \vdots \\ max(|\mathrm{HP}_n^{\downarrow}|) \\ max(\mathrm{HP}_n^{\uparrow}) \end{pmatrix}}_{\left(\,2(n-1)\times 1\,\right)}$$

The special case for the constraint on x_1, compared to x_0, can be expressed as:

$$\underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \end{pmatrix}}_{(2\times n)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{(n\times 1)} \leq \underbrace{\begin{pmatrix} max(\mathrm{HP}_1^{\uparrow}) + x_0 \\ max(|\mathrm{HP}_1^{\downarrow}|) - x_0 \end{pmatrix}}_{(2\times 1)}$$

Hence, the following constraint matrices are acquired:

$$\mathbf{A}_{\text{min/max }\Delta\text{ production}} = \underbrace{\begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}}_{\left(\,2(n-1)\times n\,\right)}, \ \mathbf{b}_{\text{min/max }\Delta\text{ production}} = \underbrace{\begin{pmatrix} max(|\mathrm{HP}_2^{\downarrow}|) \\ max(\mathrm{HP}_2^{\uparrow}) \\ \vdots \\ max(|\mathrm{HP}_n^{\downarrow}|) \\ max(\mathrm{HP}_n^{\uparrow}) \end{pmatrix}}_{\left(\,2(n-1)\times 1\,\right)}$$

$$\mathbf{A}_{\text{min/max }\Delta\text{ production }x_0} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \end{pmatrix}}_{(2\times n)},\ \mathbf{b}_{\text{min/max }\Delta\text{ production }x_0} = \underbrace{\begin{pmatrix} max(\text{HP}_1^{\uparrow}) + x_0 \\ max(|\text{HP}_1^{\downarrow}|) - x_0 \end{pmatrix}}_{(2\times 1)}$$

4.2.4 Min/max reservoirs

In order to express the constraints on the water level in the reservoirs (*min/max reservoirs*) on matrix form, first note that:

$$\begin{aligned}
&min(\text{Reservoirs}_1) - y_1 \leq -x_1 && \leq max(\text{Reservoirs}_1) - y_1 \\
&min(\text{Reservoirs}_2) - y_2 \leq -x_1 - x_2 && \leq max(\text{Reservoirs}_2) - y_2 \\
&\vdots \\
&min(\text{Reservoirs}_{n-1}) - y_{n-1} \leq -\sum_{i=1}^{n-1} x_i && \leq max(\text{Reservoirs}_n) - y_{n-1} \\
&-\sum_{i=1}^{n} x_i \geq \text{Reservoirs}_{\text{Target}} - y_n
\end{aligned}$$

On matrix form, the first part of the inequalities ($1 \leq i \leq n-1$) can be expressed as:

$$\underbrace{\begin{pmatrix} min(\text{Reservoirs}_1) - y_1 \\ \vdots \\ min(\text{Reservoirs}_{n-1}) - y_{n-1} \end{pmatrix}}_{\left(\ (n-1)\times 1\ \right)} \leq \underbrace{\begin{pmatrix} -1 & 0 & \cdots \\ \vdots & \ddots & \ddots \\ -1 & \cdots & -1 \end{pmatrix}}_{\left(\ (n-1)\times(n-1)\ \right)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}}_{\left(\ (n-1)\times 1\ \right)} \leq \underbrace{\begin{pmatrix} max(\text{Reservoirs}_1) - y_1 \\ \vdots \\ max(\text{Reservoirs}_{n-1}) - y_{n-1} \end{pmatrix}}_{\left(\ (n-1)\times 1\ \right)}$$

Furthermore, the last part of the inequalities ($i = n$) can be expressed as:

$$\underbrace{\begin{pmatrix} -1 & \cdots & -1 \end{pmatrix}}_{(1\times n)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{(n\times 1)} \geq \text{Reservoirs}_{\text{Target}} - y_n$$

Thus, the set of inequalities can be expressed as:

$$\underbrace{\begin{pmatrix} -1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & \cdots & -1 \end{pmatrix}}_{(n \times n)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{(n \times 1)} \geq \underbrace{\begin{pmatrix} min(\text{Reservoirs}_1) - y_1 \\ \vdots \\ min(\text{Reservoirs}_{n-1}) - y_{n-1} \\ \text{Reservoirs}_{\text{Target}} - y_n \end{pmatrix}}_{(n \times 1)}$$

$$\underbrace{\begin{pmatrix} -1 & 0 & \cdots \\ \vdots & \ddots & \ddots \\ -1 & \cdots & -1 \end{pmatrix}}_{\left((n-1) \times (n-1) \right)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}}_{\left((n-1) \times 1 \right)} \leq \underbrace{\begin{pmatrix} max(\text{Reservoirs}_1) - y_1 \\ \vdots \\ max(\text{Reservoirs}_{n-1}) - y_{n-1} \end{pmatrix}}_{\left((n-1) \times 1 \right)}$$

Multiplied by the factor (-1) to transform the "greater than or equal" inequality into a "less than or equal" constraint:

$$\underbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix}}_{(n \times n)} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{(n \times 1)} \leq \underbrace{\begin{pmatrix} y_1 - min(\text{Reservoirs}_1) \\ \vdots \\ y_{n-1} - min(\text{Reservoirs}_{n-1}) \\ y_n - \text{Reservoirs}_{\text{Target}} \end{pmatrix}}_{(n \times 1)}$$

Hence, the following matrices are obtained:

$$\mathbf{A}_{\text{min reservoirs}} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix}}_{(n \times n)}, \ \mathbf{b}_{\text{min reservoirs}} = \underbrace{\begin{pmatrix} y_1 - min(\text{Reservoirs}_1) \\ \vdots \\ y_{n-1} - min(\text{Reservoirs}_{n-1}) \\ y_n - \text{Reservoirs}_{\text{Target}} \end{pmatrix}}_{(n \times 1)}$$

$$\mathbf{A}_{\text{max reservoirs}} = \underbrace{\begin{pmatrix} -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -1 & \cdots & -1 & 0 \end{pmatrix}}_{\left(\; (n-1) \times n \; \right)}, \; \mathbf{b}_{\text{max reservoirs}} = \underbrace{\begin{pmatrix} max(\text{Reservoirs}_1) - y_1 \\ \vdots \\ max(\text{Reservoirs}_{n-1}) - y_{n-1} \end{pmatrix}}_{\left(\; (n-1) \times 1 \; \right)}$$

where

$\mathbf{A}_{\text{min reservoirs}} =$ The lower triangular matrix of size $n,$ where all the elements are 1.
$\mathbf{A}_{\text{max reservoirs}} =$ The lower triangular matrix of size $n-1,$ where all the elements are $-1,$ with a column of zeros concatenated from the right.

4.3 Model time steps

The optimization model will use different settings for the time steps depending on different objectives.

4.3.1 Multi-stage

The starting position is known, which enables a prediction of the available water resources in the reservoirs using a meteorological forecast of inflow in the future (known as the "best guess" forecast). The prices of forward contracts on electricity are also known, which means that the production can be scheduled for time periods when the price is supposed to be high, thereby maximizing the profit for the producers.

In reality, the inflow after time step 1 will most likely be different than what was anticipated according to the best guess forecast. In that case, the realized water level in the reservoirs after time step 1 is different than what was previously accounted for. In the next time step, the model shall perform a new optimization from that point in the future (time step 1), with the new starting level in combination with the same forecast of the remaining time period that was used from the beginning. This will yield a new result for the production in time step 2, compared to the earlier result obtained when the optimization first started (iteration 1). This series of occurrences is illustrated in Figure 6.

The model constantly takes one additional time step into the future and perform the optimization from a new starting location, and thus acquires a new optimal production curve of hydro power in the future. In each time step, the deterministic equivalent problem is solved. This process is repeated until the model reaches time n (the total number of time steps under study), resulting in a number of optimizations along the way. The model will use a number of different scenarios for the inflow to the reservoirs, each scenario representing a possible outcome of real weather. The final result from the optimization consists of a vector of the optimal production of hydro power each time step, taking into account a specific hydrological scenario.

4.3.2 Deterministic equivalent

In the case of solving the deterministic equivalent, the model performs the optimization as if perfect foresight of the future was given. Hence, the optimization problem is solved only once at time $t = 0$ for the complete time period under study.

However, the method of considering the deterministic equivalent does not necessarily mean that the practitioner has to employ the assumption of perfect foresight. The method can be used by considering a number of different scenarios of the future, and solving them one at a time with "perfect foresight". The final solution can then be represented by a weighting of

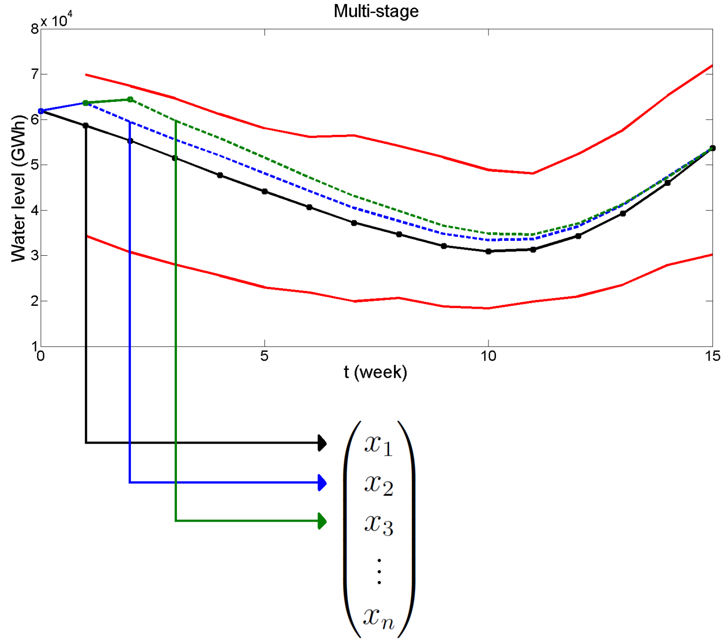

Figure 6: The water level in the reservoirs for the optimal solution of one scenario. The multi-stage solution is constructed from the first element of the deterministic equivalent solutions (black, blue and green). The red lines are corresponding to the constraints.

these different scenarios.

4.4 Inflow distribution

The creation of stochastic scenarios requires information of the underlying stochastic variable, in this case the inflow to the reservoirs. This information is obtained from historical observations. To capture the different ways of creating stochastic inflows, this thesis will examine three different stochastic representations of inflow. The aim is not to find the best way to model the distribution of inflow, but rather to test different methods that are deemed plausible. Section 4.4.1 describes a Normalization approach, where the intention is to fit a parametric Normal distribution for each time step. Section 4.4.2 describes time series techniques to estimate a trend, a seasonal component and a noise variable to describe the data. Finally, Section 4.4.3 describes a Bootstrap approach, where scenarios are created by randomly sampling inflows from the historical observations.

4.4.1 Normalization

The Normalization (N) approach relies on the assumption that inflow to the reservoirs are outcomes from a Normal distribution. This assumption will be discussed more in Section 6.

By nature, weather is not constant over time, but rather seasonally dependent. This implies that it is unreasonable to assume that the inflows over time are outcomes of the same Normal distribution. A more reasonable assumption is that historical outcomes from different times of the year (same calendar week or four week periods will be used) have the same distribution. Hence, we assume that the inflow at time step i (X_i):

$$X_i \sim N(\mu_i, \sigma_i)$$

From this assumption, normalizing the data by the time period specific mean and standard deviation, it follows that:

$$Z_i = \frac{X_i - \mu_i}{\sigma_i} \sim N(0, 1)$$

If μ_i and σ_i are estimated by the empirical mean and standard deviation ($\hat{\mu}_i$ and $\hat{\sigma}_i$) for the corresponding time periods, the fit of the Normal distribution can be tested by analyzing the empirical residual $\hat{Z}_t$. A histogram of normalized data $\hat{Z}_t$, with time step four weeks and a total of 247 data points, is shown in Figure 7.

To create scenarios in this method, a sample from the standard Normal distribution is simulated, and then scaled by the specific mean and standard deviation for the corresponding time period.

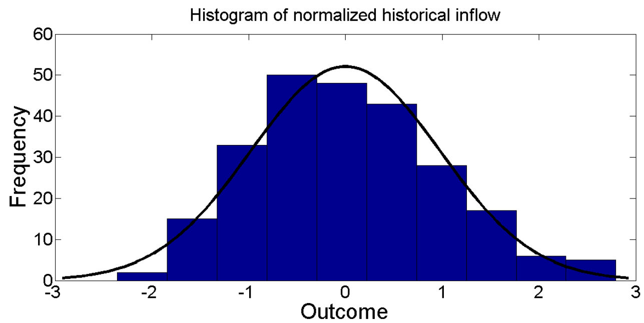

Figure 7: Histogram of normalized historical inflow for the time step of 4 weeks (247 data points). The black line is the probability density function for the $N(0, 1)$ distribution.

4.4.2 Time series analysis

In this method, techniques from time series (TS) analysis are used in order to model inflow. Data for weekly inflows to the reservoirs for the Nordic hydro system is publicly available for the time period after the deregulation in 1996. For simplicity, all calendar years are assumed to have 52 weeks, and the data points for the occasional 53rd calendar week are removed from the sample. This gives a historical sample of 988 observed weeks (19 years).

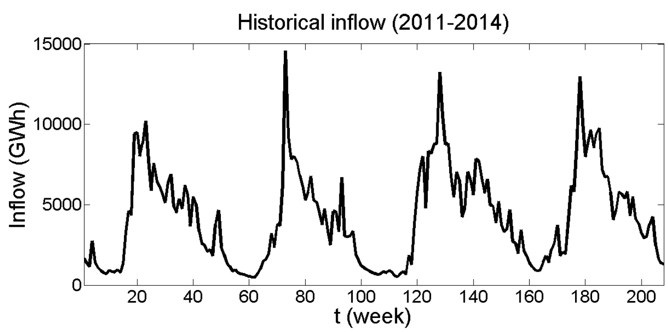

Figure 8: Realization of weekly inflow to the Nordic reservoirs for the years $2011 - 2014$.

A realization of the last 4 years of inflow is shown in Figure 8. As can be observed, the weekly inflows are seasonally dependent. Hence, the stochastic model needs to capture this property.

In time series analysis, a stochastic process can be expressed as:

$$X_t = m_t + s_t + Y_t$$

In this expression, X_t is the realization of a stochastic process, m_t is the trend component, s_t is the seasonal component and Y_t is the stationary random noise component. The objective is to estimate and extract the deterministic trend and seasonal components from the data so that the residual, $Y_t = X_t - m_t - s_t$, becomes a stationary time series. If this is carried out successfully, the fitted model can be used to simulate an arbitrary number of independent sequences of weekly inflows (Brookwell & Davis, 2001).

Estimating the trend and seasonal components

There are different methods to estimate both the trend and seasonal components, e.g. moving average filter, polynomial fitting, exponential smoothing, etc. In this thesis, Brookwell & Davis (2001) so called S1 method for estimation of trend and seasonal component is used.

The method is performed in three steps:

i) The trend is estimated by fitting a moving average filter with lag d, where d is the period of the seasonal variation, in order to eliminate the seasonal component and dampen the noise. If d is odd, $d = 2q + 1$ ($q \in \mathbb{R}$):

$$\hat{m}_t = d^{-1} \sum_{j=-q}^{q} X_{t-j}$$

If n is the number of observations and d is even ($d = 2q$):

$$\hat{m}_t = \frac{0.5x_{t-q} + x_{t-q+1} + \cdots + x_{t+q-1} + 0.5x_{t+q}}{d}, \; q < t \leq n - q.$$

It is critical to find a value of d that represents the period over the entire time series. To model the weekly inflow, a reasonable choice of d is 52 weeks, i.e. the number of calendar weeks in an ordinary year (some years have 53 weeks). However, to check this, the autocorrelation function (ACF) of X_t can be analyzed. For observations $x_1, \ldots, x_n$ of a time series, the sample autocorrelation function of lag h is:

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}, \; -n < h < n$$

where $\hat{\gamma}(h)$ is the sample autocovariance function of lag h, defined as:

$$\hat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}), \; -n < h < n$$

where $\bar{x}$ is the sample mean. The sample ACF of X_t is shown in Figure 9. As can be observed, the ACF has a period of 52 which supports the choice of $d = 52$. The upper left

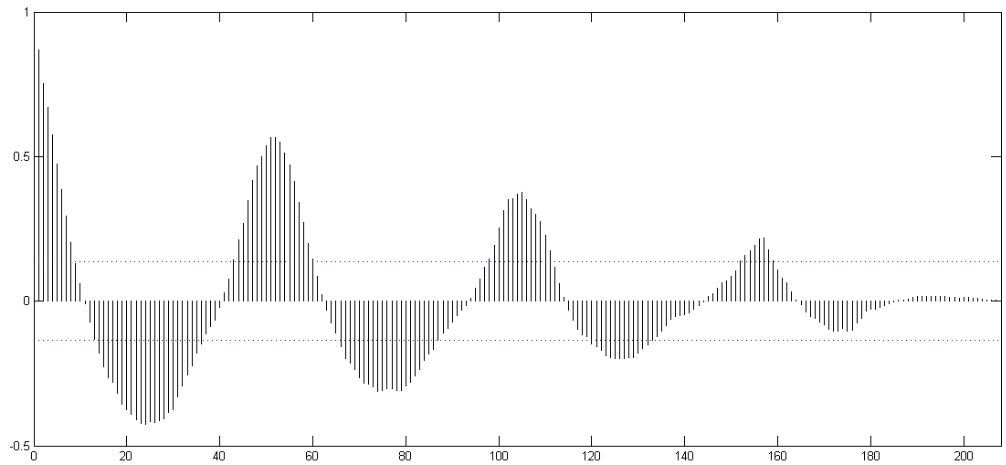

Figure 9: Autocorrelation function of the weekly inflows for the period 2011-2014. **Y-axis**: Correlation. **X-axis**: Lag.

plot in Figure 10 shows the estimation of m_t using $d = 52$.

ii) Estimate the seasonal component by taking the period wise average of the deviations from $\hat{m}_t$.

$$w_k = \text{the average of the deviations } \{(x_{k+jd} - \hat{m}_{k+jd}),\ q < k + jd \leq n - q\}.$$

$$\hat{s}_k = w_k - d^{-1} \sum_{i=1}^{d} w_i,\ k = 1, \dots, d.$$
$$\hat{s}_k = \hat{s}_{k-d},\ k > d.$$

The deseasonalized data is then represented by $d_t = x_t - \hat{s}_t,\ t = 1, \dots, n$.

The estimated seasonal component s_k can be observed in the upper right plot in Figure 10.

iii) Re-estimate the trend from the deseasonalized data by fitting a polynomial. In this case, a linear trend is estimated by fitting a first degree polynomial. The re-estimation of $\hat{m}_t$ can be observed in the lower left plot of Figure 10.

Finally, the noise sequence, which can be observed in the lower right plot of Figure 10, is given by:

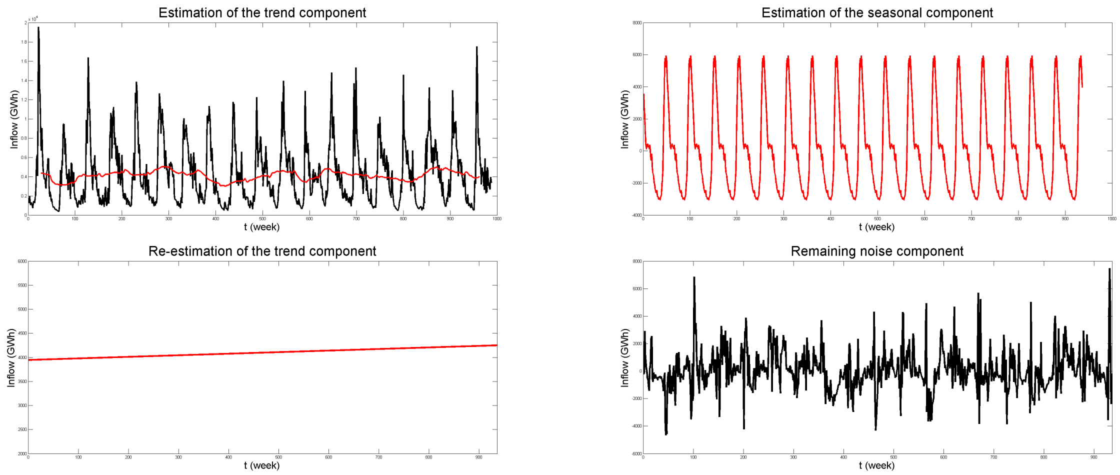

Figure 10: **Upper left**: Empirical estimation of the trend component (m_t).
Upper right: Empirical estimation of the seasonal component (s_t).
Lower left: Re-estimation of the trend component (m_t).
Lower right: Remaining noise component (Y_t).

$$\hat{Y}_t = X_t - \hat{m}_t - \hat{s}_t$$

Testing and finding the distribution of the noise sequence

In order to use the fitted model for prediction or sampling, m_t and s_t need to be deterministic functions of t. I.e., $s_t = s_{t+d}$, $\forall t$. Additionally, the remaining noise term Y_t needs to be stationary.

By estimating the distribution of the noise term, different scenarios of the stochastic process can be simulated. As the simplest form of a stationary process is a sequence of i.i.d. random variables, the first hypothesis to test is if $\hat{Y}_t$ can represent observed values from such a sequence. If not, $\hat{Y}_t$ is probably described by some more advanced stationary process.

IID test

In this thesis, three different tests for i.i.d. are used (Brockwell & Davis, 2010).

i) Analyzing the sample autocorrelation function

For large n, the sample autocorrelation for a sequence of i.i.d. random variables $Y_1, \ldots, Y_n$ with finite variance is approximately distributed as $N(0, 1/n)$. This means that 95% of the sample autocorrelations should lie within the confidence bounds $\pm 1.96 \frac{1}{\sqrt{n}}$.

If more than 5% of the sample correlations are outside this confidence interval, then the hypothesis that Y_t is i.i.d. can be rejected at the confidence level 95%.

ii) The rank test

The rank test is effective to test if there is a remaining trend in the data. Define P as the number of pairs (i, j) such that $y_j > y_i$, $\forall j > i$, $i = 1, \ldots, n-1$. There is a total of $\left(\begin{array}{c} n \\ 2 \end{array} \right) = \frac{1}{2}(n-1)$ pairs. If $Y_1, \ldots, Y_n$ are observations of i.i.d. noise, the probability that $y_j > y_i = 0.5$. This means that $\mu_P = E[P] = \frac{1}{4}(n-1)$. A large positive value of $P - \mu_p$ indicates an increasing trend in the data, and a large negative value indicates a decreasing trend in the data. The variance of P can be expressed as $\sigma_P^2 = \frac{n(n-1)(2n+5)}{72}$. Hence, $P \sim N(\mu_P, \sigma_P^2)$ (Brookwell Davis, 2001).

The hypothesis that Y_t is i.i.d. can be rejected if $\frac{|P-\mu_P|}{\sigma_P} > \Phi_{1-\frac{\alpha}{2}}$, where α is the significance level and Φ is the standard Normal cumulative distribution function.

iii) The different sign test

In this test, the number S of times that $Y_i > Y_{i-1}$, $i = 1, \ldots, n$, is counted. If Y_t is i.i.d., then $P(Y_i > Y_{i-1}) = 0.5$, $E[S] = \mu_S = 0.5 \cdot (n-1)$ and the variance $\sigma_S^2 = \frac{(n+1)}{12}$.

The hypothesis that Y_t is i.i.d. can be rejected if $\frac{|S-\mu_S|}{\sigma_S} > \Phi_{1-\frac{\alpha}{2}}$, where α is the significance level.

If the residual does not pass the tests, it means that there is dependence among the residuals and that Y_t probably is described by a more complex stationary time series. However, noise dependence can be an advantage for prediction as past observations of the noise variable can be used to predict future values.

If the residual Y_t pass all the tests, the hypothesis that Y_t consists of observations from i.i.d. random variables can not be rejected. For the data set used in this thesis, Y_t pass all of the three tests and thus the hypothesis that Y_t is i.i.d. noise can not be rejected. This result is convenient for sampling purposes as the next step is to fit a parametric distribution to Y_t.

Different parametric distributions were considered in order to model the noise component Y_t. After consideration of different alternatives, the Normal distribution was deemed the most suitable choice. The parametric Normal distribution is found by using Matlab's built-in function *normfit*.

4.4.3 Bootstrap

The Bootstrap (B) method draws random samples with replacement from the empirical distribution. The empirical distribution is considered separately for each time step under study. This means that the data set used in this thesis, including 19 years of historical inflows, yields 19 different possible outcomes for each time step. The basic idea of the Bootstrap method is to gain more information about an unknown phenomenon by resampling from the limited data that is available.

Let θ_i be the outcome of Θ_i, where Θ_i is the stochastic inflow to the reservoirs during time step i. Then, the different outcomes in the bootstrap sample for time step i is:

$$\{\theta_{i,1}, \ldots, \theta_{i,19}\}.$$

where

$\theta_{i,1}$ is the observation for time step i from 1996.

$\vdots$

$\theta_{i,19}$ is the observation for time step i from 2014.

How the bootstrap approach is used for the different scenario construction methods is explained in more detail within their corresponding sections within Section 4.5. However, the general idea is to create scenarios by drawing with replacement from the historical observations in order to create a larger amount of scenarios than the history originally can provide.

4.5 Scenario construction

This section presents the scenario construction methods considered in this thesis. Section 4.5.1 explains the independent stochastic method. In Section 4.5.2, the trinomial tree model is described, including the two different implementations called trinomial deterministic and trinomial stochastic.

4.5.1 Independent stochastic

The independent stochastic (IS) method creates a fixed amount n_S of scenarios by independently sample n inflows from one of the three distribution models (see Section 4.4) for each time step under study. The sampled inflows are then merged to create n_S scenarios.

In the Normalization model, inflows for time step i are simulated by first sample from the $N(0,1)$ distribution, and then the outcomes are scaled with the time specific mean and standard deviation:

$$\Theta_i(\text{N,IS}) = Z_i \cdot \sigma_i + \mu_i,\ Z_i \sim N(0,1).$$

In the Time Series method, inflows for time step i are simulated by first sampling from the noise component Y_t, and then the trend (m_t) and seasonal components (s_t) are added:

$$\Theta_i(\text{TS,IS}) = Y_i + m_i + s_i,\ Y_i \sim Y_t.$$

In the Bootstrap approach, historical observations are drawn with replacement by using the uniform distribution so that the outcomes are selected with equal probability:

$$\Theta_i(\text{B,IS}) = \theta_{i,j},\ j \sim U(1,19).$$

where

$U(a,b)$ denotes the discrete Uniform distribution on the closed interval $[a,b]$.

4.5.2 Scenario tree model

The stochastic feature can be modeled by scenario trees. Scenario trees are used to illustrate the evolution of different realizations of a stochastic process. Common practices are to use binomial or trinomial trees. In the binomial tree model, the inflow to the reservoirs can be modeled as higher (upwards in the tree) and lower (downwards in the tree) for each time step. In comparison, the trinomial tree is a more detailed model than the binomial tree since it enables a larger variety of outcomes, namely three instead of two in each time step. A subsequent drawback of this feature is that the scenario tree grows larger in a faster manner, which increases the computation time. This thesis focuses on the trinomial tree model since it is more detailedl.

In the trinomial tree model, there are three possible inflow realizations for each time step, corresponding to low, medium and high. Each realization has its own particular probability distribution, representing different inflow expectations for each time step. In this model, the total number of scenarios is $n_S = 3^n$, where n is the number of time steps. Figure 11 illustrates an example of a trinomial tree with deterministic steps, where the different steps are equal for all time steps.

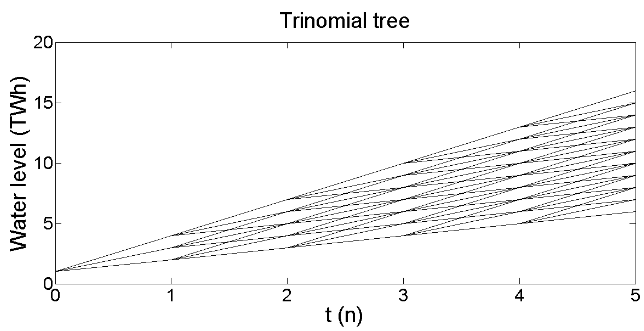

Figure 11: An illustrative example of a trinomial tree with deterministic steps. In this example, the number of time steps is $n = 5$, the number of scenarios is $n_S = 3^n = 243$ and the steps $(\text{low}, \text{medium}, \text{high}) = (1, 2, 3)$.

In this model, a critical issue is to determine the size of the different steps (low, medium and high) for moving in the tree. Two different implementations of the trinomial tree model are used. In the first implementation (*trinomial deterministic*), the steps for moving in the tree are defined as different quantiles of the period's distribution. In the second implementation (*trinomial stochastic*), the steps for moving in the tree are simulated samples from different probability distributions which are constructed from the corresponding period's distribution.

4.5.2.1 Trinomial deterministic The trinomial deterministic (TD) method is reminiscent of a deterministic model, where the different steps are predefined. The steps corresponding to an upwards respectively a downwards move in the tree are defined as a quantile of the period's inflow distribution, namely the $1-\alpha$ and the α quantile (the medium step is defined as the 50% quantile). Figure 12 illustrates the situation for one time step, where the different steps are marked by the dashed lines.

In the Normalization model, the following steps in the deterministic tree are used:

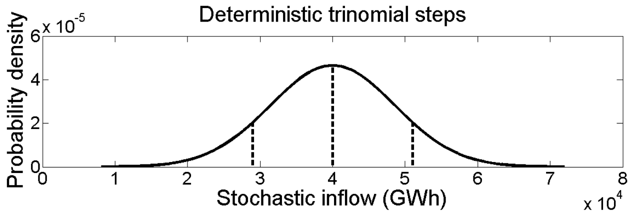

Figure 12: The different steps in the deterministic trinomial tree model. In this example, $\alpha = 10\%$.

$$\begin{aligned}
\text{Low}_i(\text{N,TD}) &= \Phi_{\alpha}^{-1} \cdot \sigma_i + \mu_i \\
\text{Medium}_i(\text{N,TD}) &= \Phi_{0.5}^{-1} \cdot \sigma_i + \mu_i \\
\text{High}_i(\text{N,TD}) &= \Phi_{1-\alpha}^{-1} \cdot \sigma_i + \mu_i
\end{aligned}$$

where

$\Phi_{\alpha}^{-1} =$ The quantile function of the $N(0,1)$ distribution evaluated at α.

In the Time Series method, the following steps in the deterministic tree are used:

$$\begin{aligned}
\text{Low}_i(\text{TS,TD}) &= Q(\alpha) + m_i + s_i \\
\text{Medium}_i(\text{TS,TD}) &= Q(0.5) + m_i + s_i \\
\text{High}_i(\text{TS,TD}) &= Q(1-\alpha) + m_i + s_i
\end{aligned}$$

where

Q(α) = The quantile function of the noise distribution Y_t evaluated at α.

In the Bootstrap model, the samples are drawn by selecting quantiles from the empirical distribution. In practice, this is simply a specific ordered element in the sorted sample (in ascending order) of historical observations. E.g., if considering the 10% quantile, the empirical 10% quantile is the 2nd element in the sorted sample of 19 observations. Hence, the following steps are obtained for time step i (for $\alpha = 10\%$):

$$\begin{aligned}
\text{Low}_i(\text{B,TD}) &= \theta_{i,2} \\
\text{Medium}_i(\text{B,TD}) &= \theta_{i,10} \\
\text{High}_i(\text{B,TD}) &= \theta_{i,18}
\end{aligned}$$

where $\{\theta_{i,1}, \ldots, \theta_{i,19}\}$ is the ordered sample of historical observations, in ascending order, corresponding to time step i.

4.5.2.2 Trinomial stochastic

The trinomial stochastic (TS) method divides the probability distribution for each time step into three different probability distributions which correspond to low, medium and high. The distribution's mean values μ are defined as the quantile values used in the trinomial deterministic method and the standard deviations σ are scaled in order for the distributions collective spread of possible inflow to approximately represent the same inflow spread as the period's distribution. Figure 13 provides an illustration of this method. The result from this method is a scenario tree constructed by sampled values from these distributions.

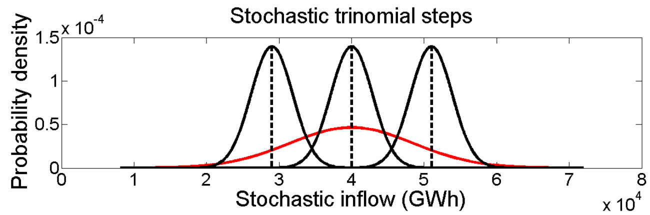

Figure 13: The distributions for the different steps in the stochastic trinomial tree model.

In the Normalization method, the following steps in the stochastic tree are used:

$$\text{Low}_i(\text{N,TS}) = Z_i \cdot \sigma_{\text{Low},i} + \mu_{\text{Low},i}$$
$$\text{Medium}_i(\text{N,TS}) = Z_i \cdot \sigma_{\text{Medium},i} + \mu_{\text{Medium},i}$$
$$\text{High}_i(\text{N,TS}) = Z_i \cdot \sigma_{\text{High},i} + \mu_{\text{High},i}$$

where

$\mu_{\text{Low},i} = \Phi_{\alpha}^{-1}$
$\mu_{\text{Medium},i} = \Phi_{0.5}^{-1}$
$\mu_{\text{High},i} = \Phi_{1-\alpha}^{-1}$
$\sigma_{\text{Low},i}, \sigma_{\text{Medium},i}$ and $\sigma_{\text{High},i}$ are scaled as in Figure 13.

In the Time Series method, the following steps in the stochastic tree are used:

$$\text{Low}_i(\text{TS,TS}) = Z_i \cdot \sigma_{\text{Low},i} + \mu_{\text{Low},i} + m_i + s_i$$
$$\text{Medium}_i(\text{TS,TS}) = Z_i \cdot \sigma_{\text{Medium},i} + \mu_{\text{Medium},i} + m_i + s_i$$
$$\text{High}_i(\text{TS,TS}) = Z_i \cdot \sigma_{\text{High},i} + \mu_{\text{High},i} + m_i + s_i$$

where

$\mu_{\text{Low},i} = Q(\alpha)$
$\mu_{\text{Medium},i} = Q(0.5)$
$\mu_{\text{High},i} = Q(1-\alpha)$
$\sigma_{\text{Low},i}, \sigma_{\text{Medium},i}$ and $\sigma_{\text{High},i}$ are scaled as in Figure 13.

In the Bootstrap model, the sorted sample of historical observations is divided into three subsets (corresponding to low, medium and high). These intervals are overlapping to maintain the feature from the overlapping Normal distributions of the other methods. E.g., this means that there is a small probability to take a smaller step in the tree for high compared to medium. A similar argument can be made for the other adjacent steps.

The setting for time step i can be illustrated as:

$$\underbrace{\{\theta_{i,1}, \ldots, \theta_{i,7}\}}_{\text{Low}}, \underbrace{\{\theta_{i,7}, \ldots, \theta_{i,13}\}}_{\text{Medium}}, \underbrace{\{\theta_{i,13}, \ldots, \theta_{i,19}\}}_{\text{High}}$$

where $\{\theta_{i,1}, \ldots, \theta_{i,19}\}$ is the ordered historical observations corresponding to time step i, in ascending order. Thus, the following steps are used:

$$\begin{aligned}
&\text{Low}_i(\text{B,TS}) = \theta_{i,j(\text{Low})}\\
&\text{Medium}_i(\text{B,TS}) = \theta_{i,j(\text{Medium})}\\
&\text{High}_i(\text{B,TS}) = \theta_{i,j(\text{High})}
\end{aligned}$$

where

$j(\text{Low}) \sim U(1,7)$
$j(\text{Medium}) \sim U(7,13)$
$j(\text{High}) \sim U(13,19)$

4.6 Measuring performance

In stochastic programming, uncertainty always has a negative impact on the predetermined optimal solution. This creates a challenge to compare different SP models as the captured level of uncertainty may differ. Factors that will affect the uncertainty is whether scenarios are viewed as deterministic or unknown, as well as the difference in possible inflow realizations between the scenarios. In order to capture these factors, the following parameters will be used to evaluate the performance: optimal stochastic solution (OSS), expected value of perfect information (EVPI), value of stochastic solution (VSS), standard deviation of scenario inflows ($\sigma(\mathbb{S})$), number of scenarios (n_S) and runtime.

Optimal stochastic solution (OSS)
The optimal stochastic solution is the average optimal solution for the different scenarios under study. Hence, it is the optimal solution in the long run when a specific set of scenarios is taken into account (Birge & Louveaux).

Expected value of perfect information (EVPI)
The expected value of perfect information is a measure of the cost associated with uncertainty in a system. The value represents the theoretical amount a decision maker would be willing to pay in return for complete information of the future (Birge & Louveaux, 1997, p.137).

For stochastic programming, this value is calculated as the difference between the optimal solution given perfect information and the weighted average value for the different scenarios under study. I.e., the difference between the deterministic equivalent solution and the multi-stage solution. This can be written as:

$$\begin{aligned} &\mathrm{EVPI}_{\mathrm{MS}} = \mathrm{OSS}_{\mathrm{DE}} - \mathrm{OSS}_{\mathrm{MS}} \\ &\mathrm{EVPI}_{\mathrm{DE}} = 0 \end{aligned}$$

An easy example (Hubbard, 2007) to understand the concept of EVPI is to consider an investment decision, where an investor can choose to invest into one out of three assets (A_1, A_2, A_3). It is known in advance that the market will go up 50% of the times, stay even 30% of the times and go down 20% of the times. Furthermore, the payouts of the assets in the different market outcomes are known:

$$A_1 = \begin{cases} \$1500, & \text{market up,} \\ \$300, & \text{market even,} \\ -\$800, & \text{market down,} \end{cases} \quad A_2 = \begin{cases} \$900, & \text{market up,} \\ \$600, & \text{market even,} \\ \$200, & \text{market down,} \end{cases} \quad A_3 = \$500, \text{ always.}$$

The optimal investment with uncertain future is the investment with highest expected value, also called the expected monetary value (EMV).

$$
\begin{aligned}
&\mathbb{E}[A_1] = 0.5 \cdot \$1500 + 0.3 \cdot \$300 + 0.2 \cdot (-\$800) = \$680 \\
&\mathbb{E}[A_2] = 0.5 \cdot \$900 + 0.3 \cdot \$600 + 0.2 \cdot \$200 = \$590 \\
&\mathbb{E}[A_3] = \$500
\end{aligned}
$$

Hence, the optimal investment is to invest in asset A_1, with EMV = \$680.

However, if an investor would know the market direction in advance, it is possible to always choose the best asset to invest in. Therefore, the expected value given perfect information is:

$$\mathrm{EV|PI} = 0.5 \cdot \$1500 + 0.3 \cdot \$600 + 0.2 \cdot \$500 = \$1030$$

The expected value of perfect information is then the difference between EV|PI and the optimal solution with uncertainty:

$$\mathrm{EVPI} = \$1030 - \$680 = \$350$$

This means that an investor would be willing to pay a maximum of \$350 in return for perfect information.

Value of the stochastic solution (VSS)
The value of the stochastic solution (VSS) is, similarly to EVPI, a measure of the cost of uncertainty in a system. Instead of comparing the optimal stochastic solution with the optimal deterministic solution as for EVPI, VSS is defined as the difference between the optimal stochastic solution and the mean value solution. The mean value solution (MVS) is the optimal deterministic solution for the mean of all stochastic scenarios. Thus, in the mean value problem, the outcome of the stochastic inflow is equal to the expected value of all scenarios for each time step. Hence, MVS is the optimal solution when ignoring uncertainty in the system and VSS measures the cost of ignorance. EVPI measures the value of knowing the future with certainty while VSS assesses the value of knowing and using distributions of future outcomes (Birge & Louveaux, 1997). Mathematically, this can be formulated as:

$$
\begin{aligned}
\mathrm{VSS}_{\mathrm{MS}} &= \mathrm{OSS}_{\mathrm{MS}} - \mathrm{MVS} \\
\mathrm{VSS}_{\mathrm{DE}} &= \mathrm{OSS}_{\mathrm{DE}} - \mathrm{MVS}
\end{aligned}
$$

Standard deviation of scenario inflows, $\sigma(\mathbb{S})$
$\sigma(\mathbb{S})$ measures the standard deviation of the total amount of inflow for the scenarios.

Runtime
Runtime measures the time for the optimization to finish.

The results will be presented in a table for the different optimization methods according to the layout in Table 2. One table of results will be presented for each of the different distribution methods.

	OSS	EVPI	VSS	$\sigma(\mathbb{S})$	n_S	Runtime
$\mathrm{IS}_{\mathrm{MS}}$						
$\mathrm{IS}_{\mathrm{DE}}$						
$\mathrm{TD}_{\mathrm{MS}}$						
$\mathrm{TD}_{\mathrm{DE}}$						
$\mathrm{TS}_{\mathrm{MS}}$						
$\mathrm{TS}_{\mathrm{DE}}$						

Table 2: Layout for the table of results.

$\mathrm{IS}_{\mathrm{MS}}$: Independent stochastic, multi-stage.
$\mathrm{IS}_{\mathrm{DE}}$: Independent stochastic, deterministic equivalent.
$\mathrm{TD}_{\mathrm{MS}}$: Trinomial deterministic, multi-stage.
$\mathrm{TD}_{\mathrm{DE}}$: Trinomial deterministic, deterministic equivalent.
$\mathrm{TS}_{\mathrm{MS}}$: Trinomial stochastic, multi-stage.
$\mathrm{TS}_{\mathrm{DE}}$: Trinomial stochastic, deterministic equivalent.

These methods are corresponding to different combinations of the alternatives for the model time step, scenario construction and methods for creating the inflow distribution.

Time step methods

- Multi-stage (MS): Future outcomes of scenarios are unknown to the decision maker. A new optimization is performed in each time step after new information is known, in anticipation of unknown future inflow. The total number of optimizations performed in this method is equal to the number of time steps under study.

- Deterministic equivalent (DE): Scenarios are regarded as certain and known outcomes of future inflow. Only one optimization is performed at $t = 0$ for the whole time period.

Scenario methods

- Independent stochastic (IS): The outcome in each time step is sampled independently.

- Trinomial deterministic (TD): The steps for moving in the tree are defined as different quantiles of the period's inflow distribution.

- Trinomial stochastic (TS): The steps for moving in the tree are simulated samples from different probability distributions which are constructed from the corresponding period's inflow distribution.

Distribution methods

- Parametric sampling:
  - Normalization: Assuming the historical inflow for each time step is following a Normal distribution, a sample can be generated by sampling from the $N(0,1)$ distribution and then scale the outcomes by the corresponding Normalization parameters $\hat{\mu}_i$ and $\hat{\sigma}_i$.
  - Time series analysis: A distribution is derived using methods from time series analysis by identifying the trend component, the seasonal component and the random noise component.
- Non-parametric sampling:
  - Bootstrap: A fixed number of scenarios are sampled from the historical observations for each time step. To create n scenarios, a total of n historical inflows are drawn with replacement from the empirical distribution for each time step.

5 Results

This section presents the results for the optimization models considered in this thesis. Section 5.1 showcases the results for linear programming, including the two different model time steps: deterministic equivalent and multi-stage. Section 5.2 presents the results for stochastic programming.

5.1 Linear programming

Two different types of problems are solved by using linear programming. The first type, the deterministic equivalent problem, is presented in Section 5.1.1. The second type, the multi-stage problem, is presented in Section 5.1.2.

5.1.1 Deterministic equivalent

The model will result in a vector of optimal hydro power production for each time step. The upper left plot in Figure 14 shows the optimal production for a scenario with perfect foresight, i.e. with the assumption that the inflow to the reservoirs will be exactly as expected.

As expected, the optimal production is highly correlated with the prices of forward contracts (seen in the lower left plot in Figure 14). Since the objective function is to maximize the revenue, the model wants to produce as much as possible when prices of forward contracts are high and produce as little as possible when prices are low. Perfect foresight makes it possible to schedule the production and the reservoirs so that neither the delta production constraint (see the upper right plot in Figure 14) nor the reservoir constraint (see the lower right plot in Figure 14) are violated.

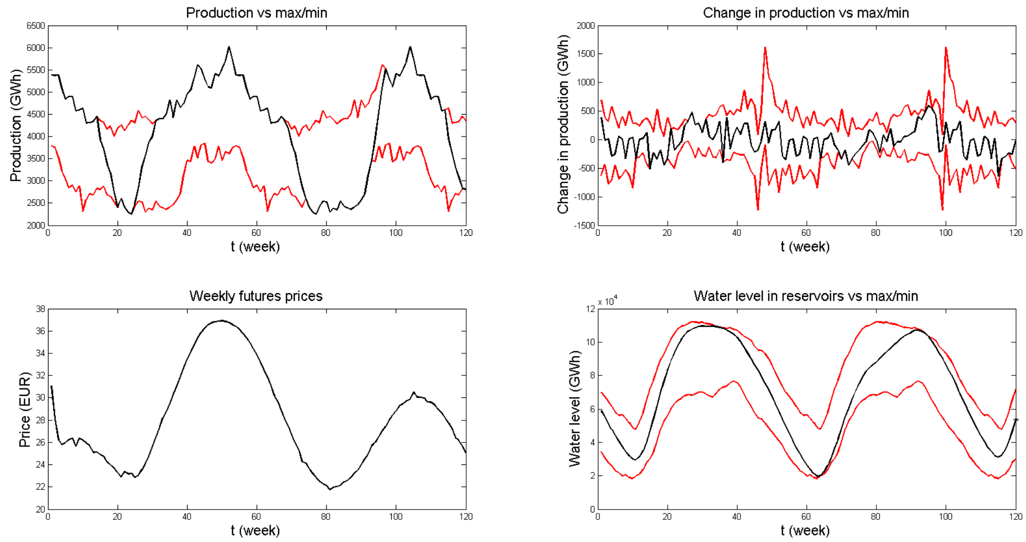

Figure 14: Optimal solution and feasibility check. The black lines correspond to the solution with perfect foresight, and the red lines show the constraints.

5.1.2 Multi-stage

The upper left plot in Figure 15 shows the optimal production in each time step. The lower left plot in Figure 15 shows the inflow of the scenario as well as the best guess forecast. As can be observed, the scenario is initially drier than the best guess forecast. As a consequence of this, the reservoir level for the scenario approaches the lower bound (see the lower right plot in Figure 15). This forces the model to run minimum production even when the forward prices are high (the same forward prices are used as in the lower left plot in Figure 14).

The upper right plot in Figure 15 shows the constraint for the change in production. The reason why this constraint can be violated is because the model is set to automatically generate maximum/minimum production if the constraint for the water level in the reservoirs is violated from above/below. This is a feature that ensures that the constraint for the reservoirs is satisfied to the extent possible.

In extreme scenarios it is impossible to respect the constraint for the water level in the reservoirs even if employing maximum or minimum production to offset for these circumstances. E.g., if the water level in the reservoirs is close to the lower bound and the scenario inflow is less than the best guess forecast, then the constraint will be violated even if the model runs minimum production. A similar argument can be made in the case when the upper bound is violated.

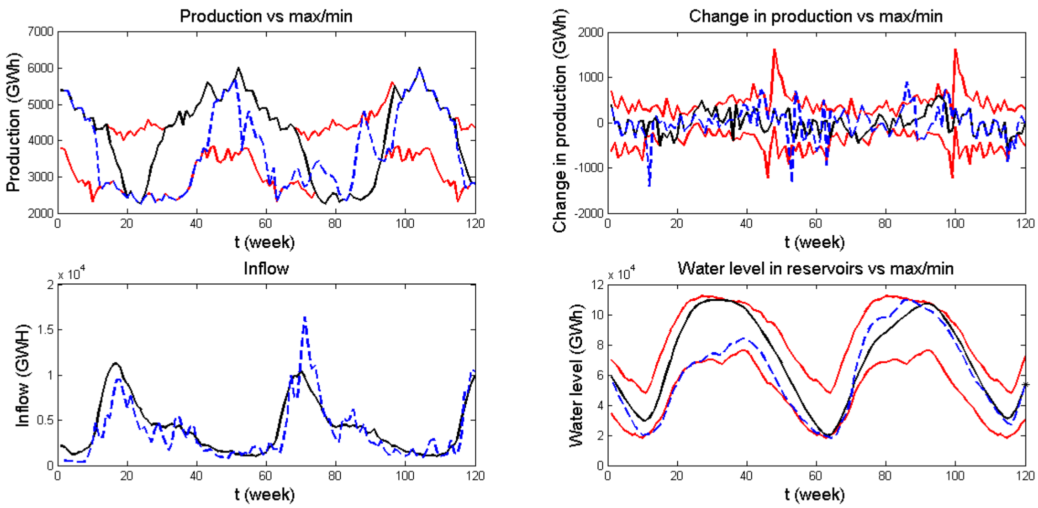

Figure 15: Weekly production and feasibility check for a scenario. The black lines are corresponding to the solution with perfect foresight, the blue lines correspond to the scenario solution and the red lines show the constraints.

5.2 Stochastic programming

In this section, the findings are presented in three tables, one for each of the inflow distribution methods Normalization, Time Series and Bootstrap. Some of the key findings and interesting observations are then presented to support the discussion in Section 6.

Normalization

	OSS (EUR)	EVPI (EUR)	VSS (EUR)	$\sigma(\mathbb{S})$ (GWh)	n_S (#)	Runtime (min)
IS_{MS}	$4.24 \cdot 10^9$	$6.35 \cdot 10^7$	$-8.99 \cdot 10^7$	$1.03 \cdot 10^4$	10 000	138.18
IS_{DE}	$4.31 \cdot 10^9$	0	$-2.65 \cdot 10^7$	$1.03 \cdot 10^4$	10 000	7.15
TD_{MS}	$4.14 \cdot 10^9$	$8.7 \cdot 10^7$	$-6.16 \cdot 10^7$	$1.59 \cdot 10^4$	59 049	67.99
TD_{DE}	$4.23 \cdot 10^9$	0	$2.54 \cdot 10^7$	$1.59 \cdot 10^4$	59 049	10.89
TS_{MS}	$4.14 \cdot 10^9$	$8.88 \cdot 10^7$	$-6.39 \cdot 10^7$	$1.67 \cdot 10^4$	59 049	64.38
TS_{DE}	$4.22 \cdot 10^9$	0	$2.49 \cdot 10^7$	$1.67 \cdot 10^4$	59 049	11.07

Table 3: Table of results for the Normalization inflow distribution.

For the optimal stochastic solution (OSS), the deterministic equivalent solution with independent stochastic scenarios (IS_{DE}) for the Bootstrap distribution generated the highest value ($4.31 \cdot 10^9$ EUR). The multi-stage solution with trinomial stochastic scenarios (TS_{MS}) for the Time Series distribution generated the lowest OSS ($4.12 \cdot 10^9$ EUR). For all optimization models, the deterministic equivalent outperforms the corresponding multi-stage solution. For all

Time Series

	OSS (EUR)	EVPI (EUR)	VSS (EUR)	$\sigma(\mathbb{S})$ (GWh)	n_S (#)	Runtime (min)
IS_{MS}	$4.19 \cdot 10^9$	$3.88 \cdot 10^7$	$-5.11 \cdot 10^7$	$8.44 \cdot 10^3$	10 000	131.48
IS_{DE}	$4.24 \cdot 10^9$	0	$-1.23 \cdot 10^7$	$8.44 \cdot 10^3$	10 000	6.54
TD_{MS}	$4.12 \cdot 10^9$	$7.33 \cdot 10^7$	$2.07 \cdot 10^7$	$1.38 \cdot 10^4$	59 049	70.34
TD_{DE}	$4.19 \cdot 10^9$	0	$9.39 \cdot 10^7$	$1.38 \cdot 10^4$	59 049	12.09
TS_{MS}	$4.12 \cdot 10^9$	$8.16 \cdot 10^7$	$1.24 \cdot 10^7$	$1.44 \cdot 10^4$	59 049	71.65
TS_{DE}	$4.20 \cdot 10^9$	0	$9.39 \cdot 10^7$	$1.44 \cdot 10^4$	59 049	12.26

Table 4: Table of results for the Time Series inflow distribution.

Bootstrap

	OSS (EUR)	EVPI (EUR)	VSS (EUR)	$\sigma(\mathbb{S})$ (GWh)	n_S (#)	Runtime (min)
IS_{MS}	$4.24 \cdot 10^9$	$6.63 \cdot 10^7$	$-9.36 \cdot 10^7$	$1.01 \cdot 10^4$	10 000	135.14
IS_{DE}	$4.31 \cdot 10^9$	0	$-2.72 \cdot 10^7$	$1.01 \cdot 10^4$	10 000	6.9
TD_{MS}	$4.16 \cdot 10^9$	$8.35 \cdot 10^7$	$-6.46 \cdot 10^7$	$1.61 \cdot 10^4$	59 049	80.51
TD_{DE}	$4.24 \cdot 10^9$	0	$1.89 \cdot 10^7$	$1.61 \cdot 10^4$	59 049	13.68
TS_{MS}	$4.14 \cdot 10^9$	$8.26 \cdot 10^7$	$-5.62 \cdot 10^7$	$1.43 \cdot 10^4$	59 049	77.23
TS_{DE}	$4.22 \cdot 10^9$	0	$2.64 \cdot 10^7$	$1.43 \cdot 10^4$	59 049	13.33

Table 5: Table of results for the Bootstrap inflow distribution.

distributions, the IS_{DE} has the highest OSS.

The expected value of perfect information (EVPI) is calculated as the difference between the OSS for the deterministic equivalent and the OSS for the scenario type under study. As a consequence of this, EVPI = 0 for all the deterministic equivalents. The highest EVPI ($8.88 \cdot 10^7$ EUR) was generated by the multi-stage solution with trinomial stochastic scenarios (TS_{MS}) for the Normalization distribution. The lowest EVPI ($3.88 \cdot 10^7$ EUR), excluding the deterministic equivalents, was generated by the multi-stage solution with independent stochastic scenarios (IS_{MS}) for the Time Series distribution.

For the value of the stochastic solution (VSS), the multi-stage solution with independent stochastic scenarios (IS_{MS}) for the Normalization distribution generated the highest value ($8.99 \cdot 10^7$ EUR). The deterministic equivalent solution with trinomial deterministic scenarios (TD_{DE}) for the Time Series distribution generated the lowest VSS ($-9.39 \cdot 10^7$ EUR). For the Normalization and the Bootsrap distributions, the DE solutions have negative VSS and the MS solutions have positive VSS. For the Time Series distribution, all trinomial tree scenarios generate negative VSS. The interpretation of negative VSS will be discussed thoroughly in

Section 6.

For all inflow distributions, the standard deviation of the total amount of inflow is lowest for the independent stochastic scenarios, with the Time Series model having the smallest standard deviation of $8.44 \cdot 10^3$ GWh. The highest standard deviation of total inflow is generated by the trinomial stochastic scenarios for the Normalization distribution.

The number of scenarios for the trinomial tree is determined by the number of time steps and increases with a factor 3 for every added time step. Ten time steps leads to $3^{10} =$ 59 049 scenarios. For the independent stochastic scenarios, 10 000 scenarios for each inflow distribution are examined.

As expected, the runtimes are significantly higher for the multi-stage optimizations in comparison with the deterministic equivalent optimizations. The DE runtimes are ranging from $6.5 - 13.5$ minutes and the MS runtimes are ranging from $64.4 - 138$ minutes. Note that the runtimes between IS and the trinomial tree models are not directly comparable as the number of scenarios and the length of the time steps are different.

6 Discussion

This section begins with a discussion about the three different methods to model the inflow distribution used in this thesis. The assumptions these models rely on are also discussed. Section 6.2 includes discussions about the three different scenario construction methods used. In Section 6.3, some of the key findings and interesting observations from the result section are discussed. Finally, possible implications of using different models are discussed with regards to different actors in Section 6.4.

6.1 Inflow distribution

6.1.1 Normalization

The Normalization method, as defined in this thesis, relies on the assumption that historical observations corresponding to the same time period can be described as outcomes from a Normal distribution. However, as our data set includes only 19 observations for each time step, it is rather hard to draw any conclusions by looking at individual time steps. A nice feature of the Normalization method, as defined in this thesis, is that the normalized data is expected to be outcomes of $N(0,1)$ variables for all time steps. To examine whether the assumption of normality is reasonable, the distribution of the normalized data can be compared to the standard Normal distribution. As can be seen in Figure 7, the histogram of the aggregated empirical residual $\hat{Y}_t$ for the examined time period indicates that this quantity is similar to the $N(0,1)$ distribution. This result indicates that Normalization is a reasonable method for this data set.

An advantage of this inflow distribution model in comparison to the Time Series method, is that the Normalization method allows for different standard deviations in every time step, whereas the Time Series model requires the standard deviation to be constant. Observance of historical data indicates that different time periods have different expected inflow and different variability in inflow. For example, the time of year that coincides with the annual snow melting typically has the highest volatility regarding the inflow of water to the reservoirs.

6.1.2 Time Series

A feature of the Time Series method is that the same noise component is estimated for the entire time period. A drawback of this estimation is that the standard deviation is constant over time, and hence uncorrelated with the amount of expected inflow. This characteristic is captured in the Normalization model, and might thus be more realistic as the historical variation of inflow has been much larger for the wet time periods during snow melting in comparison with the dry summer months.

However, i.i.d. tests of the residual Y_t cannot reject the hypothesis that Y_t is i.i.d. noise which indicates a good fit of the Time Series model as it has successfully removed the trend and the seasonal components.

6.1.3 Bootstrap

An advantage of the Bootstrap method is that it is non-parametric. Thus, it does not rely on any assumptions of the underlying distribution. It also solely uses historically observed values which ensures the absence of unrealistic values. This is in contrast to the other inflow distribution methods, Normalization and Time Series, which use a parametric distribution that has a small probability of extreme outcomes.

The Bootstrap model relies on the assumption that the outcomes of each time step are i.i.d., which might not always be the case for real weather. In reality, it is probable that there is some correlation between adjacent time periods. Thus, it is the independence part of the i.i.d. abbreviation that can arguably be problematic. The identically distributed postulation for each time step is considered a more reasonable assumption.

A risk with using the Bootstrap method for a small historical sample, as in this setting of only 19 observations per time step, is that the empirical distribution is not always a good representation of the true underlying distribution. Especially, when all observations are viewed as equally likely, extreme values among the historical observations will receive unrealistically high probabilities.

6.1.4 Comparison between the models

As has been mentioned above, the biggest differences between the models are that the Normalization and the Time Series models rely on parametric distributions and that the Time Series model has constant volatility for all time steps.

Figure 16 illustrates 10 000 sampled scenarios using the three inflow distribution methods, compared with the historical sample of 19 years of inflow. As can be observed, the Normalization scenarios (upper left plot) have their lowest variance during the first weeks of the year and the highest variance during the inflow peak when the snow is melting. It can also be seen that the effect of the constant volatility of the Time Series model results in a high probability of unrealistically high inflow during the drier periods and close to zero probability for larger inflows than 15 000 GWh (as none occurs in 10000 simulations), even though the historical observations (lower right) contains several values above that level. The Bootstrap scenarios can be observed in the lower left plot.

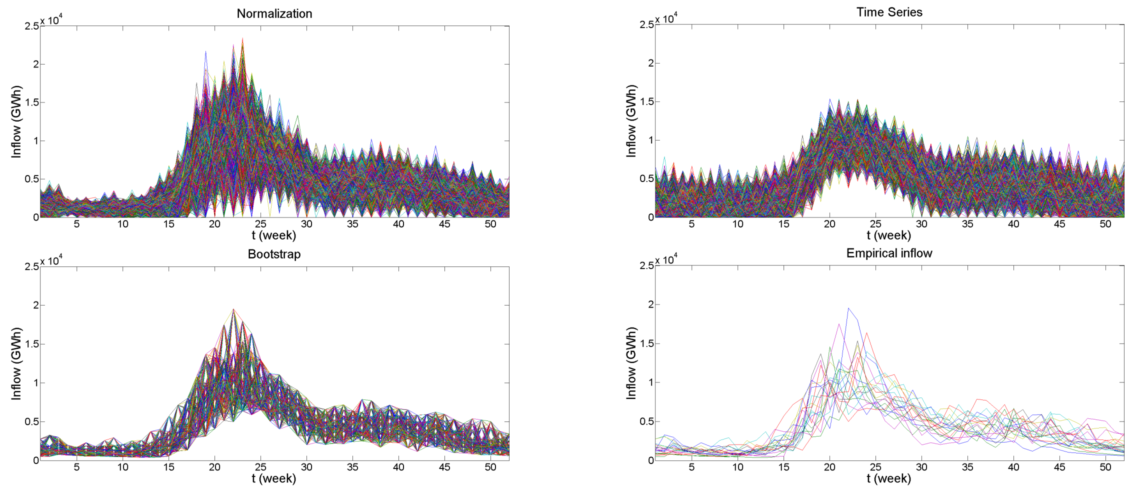

Figure 16: 10 000 simulated scenarios from the independent stochastic method compared to historical inflow. **Upper left**: Normalization distribution method.
Upper right: Time Series distribution method.
Lower left: Bootstrap distribution method.
Lower right: Empirical inflow from the past 19 years.

6.2 Scenario construction

In this section, the different methods of constructing scenarios are discussed. The underlying assumptions of these models are also discussed.

6.2.1 Independent stochastic

In the independent stochastic model, the number of scenarios is chosen by the modeller and is not related to the number of time steps as in the trinomial tree model. Furthermore, this scenario construction technique does not force the scenarios to very extreme outcomes (compared to frequently moving down or up in a scenario tree).

An assumption this method relies on is that there is no correlation between adjacent time periods, since subsequent samples are simulated independently. In reality, there is presumably some amount of correlation regarding the weather in the short-term. For example, changes in weather may often times shift gradually in a slow process. This behaviour is not captured by independent sampling methods, which means that some simulated scenarios may include sudden changes between time steps.

6.2.2 Trinomial tree

In the implementation of a tree model, a critical issue is to determine the size of the different steps (low, medium and high) for moving in the possible directions in the scenario tree. In order to mitigate negative effects resulting from choosing these steps in an undesirable way, two different implementations (trinomial deterministic and trinomial stochastic) of the trinomial tree model were used. This approach further increases the potential for comparison between different methods.

Furthermore, another critical issue of this model is the exponential growth of the scenario tree for each additional time step that is added to the time horizon. This feature results in an exponential growth of the computational time that it takes to solve the resulting optimization problem. In this implementation of the trinomial tree model, Matlab runs out of memory if the number of time steps is larger than 14, which sets the upper limit for the total number of scenarios to $n_S \approx 4.8 \cdot 10^6$ scenarios. Thus, in order to be useful in practice, this method is applied with a relatively long time step to enable the modeling of a reasonable time horizon. In this implementation, the optimizations are performed with 10 time steps, where one time step corresponds to four weeks.

In practice, this method can be useful in the areas of stress testing, when the practitioner is interested in investigating a small amount of different outcomes in each time step, and when the time horizon is fairly short. Specifically, when the number of different outcomes in each step can be reduced to 2 (binomial tree) or 3 (trinomial tree). The reason why this approach is suitable for stress testing is that the vast majority of the scenarios will deviate a lot from the medium scenario that is located in the middle of the tree. Specifically, many realizations in the scenario tree will move upwards or downwards far away from the medium path. This enables insights to be drawn from the result of extraordinary scenarios.

With high probability, the trinomial deterministic tree has a larger spread of inflow compared to the trinomial stochastic tree, since the different steps in the model correspond to a quantile that is located rather far out in the tail of the inflow distribution. Thus, the difference in the total amount of inflow for the scenarios is presumably largest in the trinomial deterministic model. However, this might not be captured by the measure $\sigma(\mathbb{S})$ since a large amount of the scenarios are located near the medium path due to the medium step always being the 50% quantile. E.g., in the case of using the Bootstrap distribution, the deterministic tree will be constructed by using the second most extreme historical observation for moving up and down in the tree. At the same time, the stochastic tree will draw a random outcome from the seven most extreme historical observations.

6.3 Results

Comparisons between DE and MS are not very interesting as the $\mathrm{OSS}_{\mathrm{DE}}$ will always be higher than $\mathrm{OSS}_{\mathrm{MS}}$. This means that it will always be more favourable to optimize with perfect foresight compared to multi-stage optimization. Thus, when comparing the different methods, it makes more sense from a scientific perspective to compare the multi-stage methods separately from the deterministic equivalent solutions.

The common assumption would be that VSS should always be positive, as more information generally should lead to better decisions. In practice, the model will be unable to capture the full advantage/disadvantage of extreme scenarios due to the constraints. Thus, due to the rather tight boundary values regarding the constraints, the added benefit of knowing the inflow distribution is reduced. This means that the extreme scenarios will affect the MVS more compared with their individual contribution to the OSS. This has to do with the mean inflow being calculated as if the wet scenarios are able to use all of the inflow for production. Thus, in some cases, this results in the relationship MVS $>$ OSS.

E.g, for favourable scenarios with extremely high inflow, the model will still not be able to produce more than the maximum allowed production in all time steps. This will be the case for many scenarios, which will yield the same solution, and hence will be unable to fully capture all the benefits. This can be observed by comparing the average total production for the scenarios and the total production for the mean value solution. For these types of circumstances, x_{Total}(MVS) is expected to be larger than the mean of x_{Total}(scenarios).

When the value of the measure VSS is negative, it means that the producer would have earned a higher income if the production planning would be in accordance with the expected inflow instead of taking all the scenarios into account. In this case, some production will be scheduled for time periods when the expected inflow is larger than the actual inflow for the scenario. This means that resources are used, which in fact are not available in order to satisfy all constraints. Hence, a drawback of these methods is that they generate a substantial amount of solutions that violate the constraints for the water level in the reservoirs.

This is in contrast to the deterministic equivalent problem where all solutions satisfy the constraints for min/max production and delta production. The constraint for the water level in the reservoirs will be violated only for scenarios with extreme inflow, even though the solution employ max/min production to offset for these unlikely circumstances in order to satisfy the constraints to the extent possible.

EVPI seems to be positively correlated with the standard deviation of the total amount of inflow. This is reasonable since the standard deviation is a measure of uncertainty in the model. The higher the uncertainty, the more a producer would be willing to pay for perfect information.

High inflows during the spring will allow producers to fill their reservoirs close to maximum in order to save as much water as possible for the more expensive periods. Therefore, more evenly distributed inflow, as in the Time Series model, might result in worse income since the producer is unable to save water resources for the future.

Since EVPI is correlated with the standard deviation of total inflow, the goal for a producer is to model the inflow as well as possible, and at the same time minimize the standard deviation of inflow.

6.4 Different actors

Different actors have different goals, and may thus approach the field of hydro power optimization from different perspectives. The different players surrounding the industry, that we deemed relevant for the discussion, are hydro power producers, regulators, speculators, competitors and academia. This section includes a discussion about which goals that may be of interest for different actors.

A long term hydro power producer is arguably interested in the whole sample space of all possible scenarios of inflow in the future, due to their goal of optimizing the income with regards to the total uncertainty. Regulators and functions responsible for the future supply and balance of electricity may be interested in stress testing of extreme scenarios to ensure a satisfactory power supply in rare circumstances. Since electricity is a perishable commodity, it is important that there is a balance in supply and demand at all times. If this is not the case, power outage or waste of electricity may occur, which can have hazardous consequences for a society.

An electricity trader speculating in the electricity market would supposedly be more interested in testing different subjective scenarios to model the available amount of supply in these scenarios. The trader may typically not be as interested in the underlying distribution of inflow, but rather in the meteorological forecast in comparison to the normal level and the general consensus on the market.

Competitors to hydro power are other main power sources, for example nuclear power and coal power. These actors may benefit from obtaining an expected view of the future supply of hydro power in order to schedule their own production in an optimal manner. However, these actors are not as flexible in their production planning as hydro power producers. This fact may lead these actors to focus on long-term planning since they have limited control over sudden changes in the short-term.

The academic sphere may be interested in future research in the field of hydro power optimization. Furthermore, academia can also be interested in possible generalisations to other fields of study that can benefit from similar types of optimization.

7 Conclusion

In this section, the conclusions are presented. This can be regarded as a summary of the key points from Section 6.

As stated earlier, the Time Series model is unable to capture the nature of real weather due to the constant variation of the noise component. Hence, the resulting recommendation is to use one of the other inflow distribution methods, namely Normalization or Bootstrap.

The aim when constructing scenarios is to cover all possible outcomes of inflow a hydro power producer could be faced with. A drawback of the tree models is that the number of scenarios increase exponentially and becomes impossible for Matlab to handle for the implementation of our model as soon as after 14 time steps. This is a problem when optimizing long-term hydro power. Our conclusion is therefore that the construction of scenarios according to the independent stochastic technique is more suitable for long-term hydro optimization.

Different actors may have different overall goals, and may thus approach the field of hydro power optimization from different perspectives. However, some of the players may have fairly mutual interests which can lead to similar recommendations regarding a suitable model. For the purpose of stress testing extreme scenarios, this can be done using both LP and SP depending on the specific goal. If the goal is to test a small amount of subjective scenarios, then LP would be least time consuming and thus most suitable. In the case of trying to simulate an extreme scenario in a population of many scenarios, then SP can be recommended.

Recall the research question of this thesis:

RQ: *How should a model be developed for different actors in the area of hydro power optimization to take advantage of new information regarding uncertain inflow to the reservoirs?*

Finally, the answer to the research question is that the Normalization and Bootstrap methods describe the nature of real weather in a satisfying manner. An implementation of multi-stage optimization can be used to incorporate the ability to take advantage of new information regarding uncertain inflow to the reservoirs. The independent sampling technique can be recommended for scenario creation due to increased flexibility over scenario trees.

8 Suggestions for future research

The working procedure with this thesis has led to various suggestions for future research that may be of interest for further elaboration.

One area that has potential for continued analysis is to model the stochastic inflow. This process can be carried through in several ways. This thesis has focused on a limited set of methods which were deemed the most suitable. Other practitioners may find potential for improvement regarding these methods. One alternative is to model different kinds of parametric distributions, which for instance could possess certain kinds of characteristics in the tails. The student t-distribution is a candidate for further examination if the practitioners can tolerate heavy-tail behaviour in their field of application. An example of a suitable research question can be to seek the optimal way of describing future inflow to the reservoirs.

In order to model the inflow in a more sophisticated manner, an idea is to include the amount of unmelted snow as an input variable to model future inflow. The yearly inflow typically follows a rather fixed pattern where the largest amount of inflow occurs at the time of the annual snow melting. Thus, there is room for improvement in the area of modeling inflow that explicitly takes into account the amount of snow that has melted until the optimization problem starts.

Another suggestion for further research is the construction of scenarios. Within the field of stochastic programming, a crucial issue is to find an appropriate way of creating a set of scenarios that captures the main point of the practitioner's investigation. This thesis has focused on an implementation of independent sampling and two variations of the trinomial tree model. An example of a potential research question could be to study what kind of information that is gained, or lost, when using different methods for creating scenarios. E.g., one test can be to analyze the impact on the quality of the solution when employing the trinomial tree model compared to the binomial tree model.

Regarding the subject of a scenario tree model, a proposal is to investigate the implications of using recombining trees compared to non-recombining trees. Since the computational workload is rather heavy due to the exponential growth of the tree, there are potential improvements to be made in the area of constructing the scenario tree in a more efficient manner. Furthermore, since many of the constructed scenarios in the tree share mutual characteristics, a more efficient optimization method that is able to benefit from the solution of a similar scenario is presumably favorable.

Moreover, a proposal for future research is the development of other methods to perform hydro power optimization with regards to uncertainty. A contribution of this thesis is an implementation of multi-stage optimization which solves several deterministic equivalent problems and then aggregates the solutions corresponding to the first time step.

Finally, one could also generalize the methods outlined in this thesis in order to apply similar concepts to other fields of application, which can benefit from this kind of optimization. The problem formulation of this thesis is located within the family of optimization problems called scheduling problems, which are applied in a variety of industries.

9 References

[1] Albers, Y., *Stochastic Programming Tools with Application to Hydropower Production Scheduling*. ETH Zurich. Centre for Energy Policy and Economics. Department of Management, Technology and Economics. 2011.

[2] Birge, R., Louveaux, F., *Introduction to Stochastic Programming*. Springer. 1997.

[3] Brockwell, P., Davis, R., *Introduction to Time Series and Forecasting*. Springer. 2nd Edition. 2010.

[4] Catalao, J.P.S., Pousinho, H.M.I., Mendes, V.M.F., *Hydro energy systems management in Portugal: Profit-based evaluation of a mixed-integer nonlinear approach*. Energy, Vol. 36, Issue 1, p.500-507. 2011.

[5] Feltenmark, S., Lindberg, P O., *Network Methods for Head-dependent Hydro Power Scheduling*. 1997.

[6] Fleten, S-E., Kristoffersen, T.K., *Stochastic programming for optimizing bidding strategies of a Nordic hydropower producer*. European Journal of Operational Research, Vol. 181, p.916-928. 2007.

[7] Fleten, S-E., Wallace, S.W., *Power Scheduling with Forward Contracts*. 1998.

[8] Fleten, S-E., Keppo, J., Lumb, H., Sollie, J., Weiss, V., *Hydro Scheduling Powered by Derivatives*. Fields Institute. 2009.

[9] Hubbard, D., *How to Measure Anything: Finding the Value of Intangibles in Business*. John Wiley Sons. 2007.

[10] MathWorks. 2015. Solve linear programming problems - MATLAB linprog. [ONLINE] Available at: `http://se.mathworks.com/help/optim/ug/linprog.html`. [Accessed 26 February 2015].

[11] Ribeiro, A.F., Guedes, M.C.M., Smirnov, G.V., Vilela, S., *On the optimal control of a cascade of hydro-electric power stations*. Electric Power Systems Research, Vol. 88, p.121-129. 2012.

[12] Sasane, A., Svanberg, K., *Optimization*. KTH, Department of Mathematics. 2010. (Compendium)

[13] Yildiran, U., Kayahan, I., Tunc, M., Sisbot, S., *MILP based short-term centralized and decentralized scheduling of a hydro-chain on Kelkit River*. Electrical Power and Energy Systems, Vol. 69, p.1-8. 2015.

[14] Zambelli, M., Luna, I., Soares, S., *Long-Term Hydropower Scheduling Based on Deterministic Nonlinear Optimization and Annual Inflow Forecasting Models*. IEEE Bucharest Power Tech Conference. Romania, Bucharest, June 28th - July 2nd, 2009.