Credit Value Adjustment: The Aspects of Pricing Counterparty Credit Risk on Interest Rate Swaps

Martin Hellander

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Master of Science Thesis at the Department of Mathematical Statistics
Royal Institute of Technology
Stockholm, Sweden

Abstract

In this thesis, the pricing of counterparty credit risk on an OTC plain vanilla interest rate swap is investigated. Counterparty credit risk can be defined as the risk that a counterparty in a financial contract might not be able or willing to fulfil their obligations. This risk has to be taken into account in the valuation of an OTC derivative. The market price of the counterparty credit risk is known as the Credit Value Adjustment (CVA). In a bilateral contract, such as a swap, the party's own creditworthiness also has to be taken into account, leading to another adjustment known as the Debit Value Adjustment (DVA). Since 2013, the international accounting standards (IFRS) states that these adjustments have to be done in order to reflect the fair value of an OTC derivative.

A short background and the derivation of CVA and DVA is presented, including related topics like various risk mitigation techniques, hedging of CVA, regulations etc.. Four different pricing frameworks are compared, two more sophisticated frameworks and two approximative approaches. The most complex framework includes an interest rate model in form of the LIBOR Market Model and a credit model in form of the Cox-Ingersoll-Ross model. In this framework, the impact of dependencies between credit and market risk factors (leading to wrong-way/right-way risk) and the dependence between the default time of different parties are investigated.

Keywords: OTC derivatives, Credit Value Adjustment, Debit Value Adjustment, wrong-way risk, interest rate swaps, LIBOR Market Model, Cox-Ingersoll-Ross process.

Kreditvärdighetsjustering: Prissättning av motpartsrisk för en ränteswap

Sammanfattning

I den här uppsatsen har prisättning av motpartsrisk för en OTC ränteswap undersökts. Motpartsrisk kan defineras som risken att en motpart i ett finansiellt kontrakt inte har möjlighet eller viljan att fulföja sin del av kontraktet. Motpartsrisken måste tas med i värderingen av ett OTC-derivat. Marknadspriset på motpartrisken är känt som Credit Value Adjustment (CVA). I ett bilateralt kontrakt, t.ex. som en swap, måste även den egna kreditvärdighet tas med i värderingen, vilket leder till en justering som är känd som Debit Value Adjustment (DVA). Sedan 2013 skall, enligt den internationella redovisningsstandarden (IFRS), dessa prisjusteringar göras vid redovisningen av värdet för ett OTC derivat.

En kort bakgrund samt härledningen av CVA och DVA är presenterade tillsammans med relaterade ämnen. Fyra olika metoder för att beräkna CVA har jämförts, två mer sofistikerade metoder och två approximativa metoder. I den mest avancerade metoden används en räntemodell i form av LIBOR Market Model samt en kreditmodell i form av en Cox-Ingersoll-Ross modell. I den här metoden undersöks även påverkan av CVA då det existerar beroenden mellan marknads- och kreditriskfaktorer samt beroenden mellan kreditvärdigheten för två olika parter.

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Chapter 1

Introduction

In this chapter follows a short introduction to CVA, explaining what it is and why it is needed.

1.1 Counterparty Credit Risk

Counterparty credit risk is closely related to the traditional credit risk, which can be regarded as lending risk. The lending party has a future obligation to repay the other party and there is a risk that the lending party wont meet their obligations, e.g. due to default. This can concern loans, bonds, mortgages and so on. Counterparty credit risk can be defined as the risk that a counterparty in a financial contract might not be able or willing to fulfil the obligations on their side of the contract. However, compared to credit risk there are two sides to counterparty credit risk. The first one being the market risk which influences the exposure over time to a counterparty and the second one being the credit risk of the counterparty. Another big difference compared to traditional credit risk is the presence of bilateral financial contracts. A bilateral financial contract is a contract where each counterparty has a possible future exposure and hence also a risk towards the other party. Counterparty credit risk mainly arises for derivatives contracts in the over-the-counter (OTC) derivatives market, where contracts such as FX forwards, interest rates swaps, credit default swaps etc. are traded. Some reasons why counterparty credit risk mainly arises in the OTC derivatives market are due to the size of the market, many OTC products are long-dated and the existence of unsecured exposure (trades where a counterparty are unable or unwilling to post margin) (Gregory, 2012).

1.2 The Financial Credit Crisis and The Credit Value Adjustment

The U.S. housing crisis in 2007 was followed by the global credit crisis where the whole financial system was set out of balance. Before the financial crisis, the aspect of counterparty credit risk in financial contracts had often been overlooked, especially for trades where collateral where posted. Big counterparties with a good credit rating were considered to be risk-free and the expression "too big to fail" was often used for the largest financial institutions. However when the systems breaking down, the credit spreads of several large firm s spiked and their credit ratings were downgraded. As an effect of this several firms had huge mark-to-market (MtM) losses in their trading books. In 2008 one of the largest investments banks, Lehman Brothers, was put into default and some other huge financial institutions only made it because they were rescued by the government or purchased by other institutions. At the time of default, Lehman Brothers had around half a million derivative contracts with around 8000 different counterparties. Most of these counterparties probably never even considered a scenario where Lehman would default. Although some firms were rescued by the government, the overall "too big to fail" view of large financial institutes was clearly proven to be wrong (Gregory, 2012).

In the wake of the financial crisis it is now common practice to take the counterparty credit risk into account when valuing OTC derivatives. This adjustment of the price is known as the Credit Value Adjustment (CVA). CVA is something that applies to the new accounting standards and the principle of accounting the fair value of a position, in order to reflect the market value. Furthermore, along with the harder regulations in Basel III, there is now also a capital requirement on CVA. These new regulations will be discussed more later on in this chapter.

So from being overlooked before the financial crisis, CVA and counterparty credit risk now plays a big role in valuation, risk management and accounting. Nowadays almost every large bank has a CVA (or XVA) desk. Dorval and Schanz (2011) divides a bank's CVA strategy into the following four levels, where sophistication and cost are increasing for each level.

1. Measuring. A CVA measuring capability is created to calculate and aggregate CVA risks. Accounting and risk management departments will be the principal users of this function. This stage fulfils compliance obligations under accounting and regulatory standards.

2. Advising. The bank will advise its trading desks with risks related to CVA. For example, limits on positions may be set to include CVA.

3. Hedging. Here CVA is moved from the trading desks to a CVA desk. The CVA desk is then responsible for managing the CVA P&L.

4. Trading. Besides hedging its own CVA risks the bank is also taking CVA positions with the goal of making a profit.

1.3 Basel III

Basel III is the latest regulatory framework for banks, developed by the Basel Committee on Banking Supervision (BIS). BIS were established in 1930 and has 60 member central banks, representing countries that together make up about 95% of the world GDP (BIS, 2011a). Basel III aims to strengthen the regulation, supervision and risk management of the banking sector. Furthermore, three main reasons to why Basel exists is according to BIS (2011a).:

- Improve the banking sector's ability to absorb shocks arising from financial and economic stress, whatever the source.
- Improve risk management and governance.
- Strengthen bank's transparency and disclosures.

1.3.1 CVA Capital Requirement

As stated earlier one of the new sections in Basel III, compared to Basel II, is the CVA capital requirement which falls under credit counterparty risk. In Basel II there had only been a capital requirement for the default risk in counterparty credit risk. Potential CVA losses arises from changes in the exposure driven by market risk factors and changes in counterparties or the investor's credit spreads. The new capital requirement originate from the financial crisis where, as stated by BIS (2011b), about two-thirds of the credit losses were due to CVA losses and only one-third due to actual defaults. This gives a very good picture to why banks should keep capital to protect themselves from potential CVA losses.

The regulatory CVA is a value-at-risk (VaR) measure on a 99% level with a time horizon of 1 year. This can be interpreted as: that a bank should have put enough capital aside to be able to handle CVA losses, occurring within one year, with a probability of 0.99. An accurate calculation of CVA VaR is carried out by simulating market risk factors under the real world measure $\mathbb{P}$, up to the time horizon. In each scenario a CVA is calculated by taking expectations under the risk-neutral measure $\mathbb{Q}$, thus receiving a distribution of CVA at the time horizon. From the distribution the VaR can easily be computed as a quantile at the chosen probability of the distribution (Brigo et al., 2013). Using a brute-force simulation technique, this leads to nested simulations.

In Basel III the regulatory CVA VaR is calculated using either a standard or an advanced method. The standard method is based on a variance-type formula and a normal distribution assumption. The advanced method is a simulation-based method similar to the procedure above but with some critical assumptions, see Gregory (2012).

1.4 IFRS

The International Financial Reporting Standards (IFRS) is a set of accounting standards developed by the International Accounting Standards Board (IASB). These accounting standards were introduced as a global business language to achieve understandable and comparable accounting of business affairs. These were introduced as a consequence of the internationalization of business and shareholding (IFRS, 2013).

1.4.1 Fair Value Measurement

In IFRS 13, which became effective in January 13, the concept of fair value measurement was introduced to accounting. This was introduced in order to account the fair value of assets and liabilities. A fair value is defined as the price that would be received when selling an asset, or paid to transfer a liability, in an orderly transaction between market participants at the measurement date. This emphasises that fair value is a market-based measurement, not an entity-specific measurement. Since it is a market-based measurement it implies the valuation of counterparty credit risk, i.e. CVA, in a derivative position (IFRS, 2013). However IFRS does not come with a technical appendix and methods on how to calculate CVA. As an effect of this various methods are used by market participants in order to estimate CVA on OTC derivatives.

1.4.2 The Over-The-Counter Market

The over-the-counter (OTC) market is a market where parties trade with another directly without using a central marketplace such as an exchange. Thus in the OTC market the participants acts as market makers and agrees on a price together. The prices of a trade in the OTC market do not need to be public, making it a less transparent market than an exchange. According to BIS (2014), in June 2014, the size of the OTC market was 691,492 billion US dollars in terms of notional amount and 17,423 billion US dollars in terms of gross market value. From these huge amounts the interest rate derivatives stands for about 81% respectively 77% of the total amounts.

In the OTC market, securities like stocks, bonds, derivatives, commodities, etc. are traded. The OTC market is way more flexible than a standardized exchange, e.g. when it comes to specific volumes or tailored products. In an exchange there is a need of liquidity and the available products need to be highly standardized which places certain restrictions on volumes, quality and conditions. However, in the OTC market two parties can for example agree on a trade with a non-standard quantity or a financial contract with some unusual conditions. This makes the OTC market a very important market in a risk management perspective. Because of its flexibility it is possible to obtain more accurate hedges than through an exchange.

Historically an OTC transaction has been made without a clearing house as in an exchange. However since the financial crisis the OTC market has also been harder regulated via the Dodd-Frank Wall Street Reform and Consumer Protection Act, in July 2010, and via the European Market Infrastructure Regulation (EMIR), which was proposed in September 2010. As an effect of regulations and as will be discussed in Section 1.6 most of the standardized OTC derivative transactions today should go through a clearing house. The clearing house takes the counterparty risk as they guarantee to fulfil the agreements in case of default for one of the parties. Although CCPs decrease counterparty credit risk there are still transactions that take place without being cleared and the prices of these derivatives needs to be adjusted for the counter party credit risk (Gregory, 2012).

1.5 EMIR

The European Market Infrastructure Regulation (EMIR), which came into force in August 2012, is a regulatory framework for OTC derivatives on the European market. This includes clearing obligations of some OTC derivatives and risk mitigation techniques for non-cleared OTC derivatives. The clearing obligation concerns some standard derivatives like interest rate swaps and index credit default swaps traded by financial counterparties (banks, insurance companies, asset managers, etc.). For non-financial counterparties there are certain clearing thresholds regarding the notional of derivatives used in non-hedging purposes. (ESMA, 2015).

1.6 Central Counterparties

Since the financial crisis a larger and larger part of all transactions in the OTC derivatives market are today being cleared through a central counterparty (CCP). A CCP is a clearing house that processes executed trades and steps in between counterparties to guarantee that each party follow through with their obligations. It can be seen as a CCP acts as buyer to every seller and vice versa. Thus clearing an OTC derivative through a CCP reduces the counterparty credit risk. CCPs handles default losses through methods like netting, margining and loss mutilation. By clearing transactions an investor also increases the possibility of netting contracts. However, the overall intention with CCPs is of course to reduce counterparty and systematic risks (Gregory, 2014).

One of the largest CCPs is LCH.Clearnet. According to SwapClear (2015) in 2014 about 50% of all outstanding and 95% of all cleared OTC plain vanilla interest rate swaps were cleared through LCH.Clearnet's service SwapClear. Since 1999, in total, more than 9.6 million OTC interest rate swaps have been cleared.

However, a lot of derivatives, e.g. cross currency interest rate swaps or exotic derivatives, cannot yet be cleared through a CCP and are thus more exposed to counterparty credit risk.

Chapter 2

Credit Value Adjustment

In this chapter the mathematical concepts of CVA is presented. Different cases are presented when the investor is seen as risk-free and when it is not. This chapter is based on the work of Brigo et al. (2013).

2.1 Credit Value Adjustment

As explained in Chapter 1, CVA can be seen as the market price of counterparty credit risk. Technically speaking, CVA is an option on the residual value of a portfolio, with a random maturity which is given by the default time of the counterparty. Therefore CVA is a strictly positive value which is subtracted from the price. This is very intuitive, since entering into an agreement with a more risky counterparty would lead to a larger CVA and thus a lower price. It is important to point out that since CVA is concerning valuation rather than risk measuring, the calculations are done under the risk-neutral measure following the standard valuation principles.

Calculating CVA is a difficult matter. One of the things that makes it difficult is that CVA introduces model dependence into the valuation. Even for products for which the valuation of the product itself does not require a model, the CVA of the product does require a model. For example, consider a plain vanilla interest rate swap (IRS). The valuation of an IRS only requires the initial term structures and is thus model independent. However when it comes to the CVA, we want to value an option of the residual value of that IRS with the maturity given by the default time of the counterparty. Suddenly we are looking to value a swaption (an option of a swap) with a random maturity, and evidently, valuation of a swaption does require an interest rate model.

Furthermore, CVA is calculated on an aggregated portfolio-level. Thus in order to estimate the CVA impact of a new trade in an existing portfolio one would have to calculate two CVA's of the whole portfolio including and excluding the new trade. The CVA for the new trade is thus the difference between the CVA including the new trades and the CVA of the initial portfolio.

The most basic way to define the CVA, at time t, is as the difference of the default-risk-free price $\Pi(t;X)$ and the true price $\tilde{\Pi}(t;X)$ of a financial contract X, thus

$$\text{CVA} = \Pi(t;X) - \tilde{\Pi}(t;X).$$

A better or more technical definition of CVA is in terms of an expected future loss due to default of a counterparty.

2.1.1 Unilateral CVA

Following the derivation of CVA in Brigo et al. (2013) and starting with a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$, where Ω is the outcome space such that an outcome $\omega \in \Omega$. $\mathcal{G}$ is the σ-algebra and $\mathcal{G}_t$ the filtration up to time t such that for a time $t \leq u$ the following hold $\mathcal{G}_t \subseteq \mathcal{G}_u \subseteq \mathcal{G}$. The filtration $\mathcal{G}_t$ can be interpreted as the total market information up to time t, i.e. $\mathcal{G}_t$ includes all market risk factors and default events up to time t. Furthermore let $\mathcal{F}$ denote the σ-algebra such that the filtration $\mathcal{F}_t$ contains the same information as $\mathcal{G}_t$ but the default events. In other words $\mathcal{F}_t$ is a sub-filtration of $\mathcal{G}_t$, i.e. $\mathcal{F}_t \subseteq \mathcal{G}_t$, where $\mathcal{F}_t$ contains only the risk-free market information. The probability measure $\mathbb{Q}$ is the risk-neutral measure such that under the risk-free bank account, following the process

$$dB_t = B_t r_t dt,\ B_0 = 1,$$

all tradeable assets divided by B_t (i.e. discounted) are martingales.

Suppose that at time t we have a portfolio consisting of derivative contracts with a risky counterparty up to a maturity of T. Following risk-neutral valuation, and denoting a future claim of a derivative at time T_i in the portfolio by X_{T_i}, the risk-free value of the portfolio is

$$V(t,T) = \sum_{T_i \in (t,T]} \mathbb{E}^{\mathbb{Q}}\big[\frac{B_t}{B_{T_i}} X_{T_i} \big| \mathcal{F}_t\big].$$

Lets say that the counterparty defaults at time $\tau < T$ and that we can recover R percent of our exposure, i.e. R is the recovery rate. Then at $t = 0$ we have a discounted loss of

$$\begin{aligned} L &= \mathbb{1}_{\{\tau \leq T\}}(1-R)\frac{B_0}{B_\tau}\Big(\sum_{T_i \in (\tau,T]} \mathbb{E}^{\mathbb{Q}}\big[\frac{B_\tau}{B_{T_i}} X_{T_i} \big| \mathcal{F}_\tau\big]\Big)^+ \\ &= \mathbb{1}_{\{\tau \leq T\}}(1-R)\frac{B_0}{B_\tau}(V(\tau,T))^+ \\ &= \mathbb{1}_{\{\tau \leq T\}}(1-R)D(0,\tau)(V(\tau,T))^+. \end{aligned}$$

Where $D(t,T)$ denotes the discount factor B_t/B_T. CVA is then the risk-neutral expectation of this loss. Thus giving

$$\text{CVA} = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{\tau\leq T\}}(1-R)D(0,\tau)(V(\tau,T))^{+}\big|\mathcal{G}_0\big]. \tag{2.1}$$

By partitioning the time such that $0 = t_0 < t_1 \cdots < t_N = T$ the equation 2.1 can be expressed as a sum of expected values on an interval $(t_i, t_{i+1}]$ such that

$$\text{CVA} = \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{t_{i-1}<\tau\leq t_i\}}(1-R)D(0,t_i)(V(t_i,T))^{+}\big|\mathcal{G}_0\big]. \tag{2.2}$$

An alternative expression for CVA can be derived using the risk-neutral cumulative distribution function for the default time τ, denoted by $Q_D(t)$ such that $Q_D(t) = \mathbb{Q}(\tau \leq t)$. Conditioning on the default time τ, Equation 2.1 can be rewritten as

$$\begin{aligned}\text{CVA} &= \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{\tau\leq T\}}\mathbb{E}^{\mathbb{Q}}[(1-R)D(0,t)(V(t,T))^{+}\big|\tau = t\big]\big|\mathcal{G}_0] \\ &= \int_0^T \mathbb{E}^{\mathbb{Q}}\big[(1-R)D(0,t)(V(t,T))^{+}\big|\tau = t\big]dQ_D(t) \end{aligned}\tag{2.3}$$

By assuming independence between the exposure and the probability of default equation 2.3 can be simplified. Furthermore the recovery rate R will be assumed to be deterministic. The integral is discretized by partitioning the time $(0,T]$ such that t_i for $i = 0, \ldots, N$ where $0 = t_0 < t_1 \cdots < t_N = T$. This yields

$$\text{CVA} = (1-R)\sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[D(0,t_i)(V(t_i,T))^{+}\big|\mathcal{F}_0\big]\big(Q_D(t_i) - Q_D(t_{i-1})\big), \tag{2.4}$$

or by using the t_i-forward measure the discounting can be moved outside the expectation such that

$$\text{CVA} = (1-R)\sum_{i=1}^{N} P(0,t_i)\mathbb{E}^{\mathbb{Q}_{t_i}}[(V(t_i,T))^{+}\big|\mathcal{F}_0]\big(Q_D(t_i) - Q_D(t_{i-1})\big).$$

2.2 Debit Value Adjustment

Debit value adjustment (DVA) is the adjustment of the price made because of a possible default of the own party. This is a strictly positive value which is added to the price. The DVA of a transaction is thus the CVA in the counterparty's point of view.

DVA has quite a strange feature. Imagine a situation where an investor I has a derivative contract with counterparty C, with positive MtM for the investor I. If the credit rating suddenly were to drop for the investor I the DVA of the derivative contract would increase, since it would now be more likely that the investor would default. Hence the investor I's derivative position would increase in MtM due to their own credit rating drop. So if a companies credit quality would decrease their recalculated DVA would give rise to a gain in positive MtM.

2.2.1 Unilateral DVA

Since DVA is the same thing as CVA in the counterparty's point of view, deriving an expression for DVA is straightforward. Let τ_I denote the default time of the investor and R_I the recovery rate from the investor, then DVA can be defined as

$$\mathrm{DVA} = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{\tau_I \le T\}}(1 - R_I)D(0, \tau_I)(-V(\tau_I, T))^+ \big| \mathcal{G}_0\big]. \tag{2.5}$$

Thus $(-V(\tau, T))^+$ is the exposure the counterparty has towards the investor. Or alternative by conditioning on the default time τ_I,

$$\mathrm{DVA} = \int_0^T \mathbb{E}^{\mathbb{Q}}\big[(1 - R_I)D(0, t)(-V(t, T))^+ \big| \tau_I = t\big] dQ_D^I(t), \tag{2.6}$$

where $Q_D^I(t)$ denotes the the risk-neutral cumulative distribution function of the default time τ_I. Discretization of the equations 2.5 and 2.6 can be done in the same way as above for CVA.

2.3 Bilateral Credit Value Adjustment

Bilateral credit value adjustment (BCVA) is used when both parties in a bilateral transaction are regarded as risky, i.e. there is a probability of default of both parties. BCVA is defined as the difference between CVA and DVA. BCVA is thus a positive or negative value which is subtracted to the risk-free price of a derivative.

However, it is not as simple as calculating CVA and DVA as above in equations 2.1 and 2.5. When calculating CVA and DVA one has to use a "first to default" approach, otherwise we would in some way be double counting the risk.

In a bilateral contract between two non-default-free parties, an investor I and a counterparty C, there are a few possible scenarios during the lifetime of the contract. Let τ_I be the default time for the investor and τ_C be the default time for the counterparty. Let $\tau = \min\{\tau_I, \tau_C\}$ be the first default time of either the investor or the counterparty. Furthermore it is assumed that simultaneously default is not possible, i.e. $\mathbb{Q}(\tau_I = \tau_C) = 0$. Then the following scenarios are possible.

- The counterparty defaults before or at the maturity T of the contract and before the investor, i.e. $\tau = \tau_C \le T, \tau = \tau_C < \tau_I$.
  - If the value of the contract at time τ is positive for the investor only a recovery rate R_C of the exposure is paid to the investor.
  - If the value of the contract at time τ is negative for the investor the full exposure is paid to the defaulted counterparty.

- The investor defaults before or at the maturity T of the contract and before the counterparty, i.e. $\tau = \tau_I \leq T, \tau = \tau_I < \tau_C$.
  - If the value of the contract at time τ is positive for the defaulted investor the full exposure is paid to the defaulted investor.
  - If the value of the contract at time τ is negative for the defaulted investor only a recovery rate R_I of the exposure is paid to the counterparty.
- No party defaults before or at the maturity T of the contract.
  - If the value of the contract at time T is positive for the investor the full exposure is paid to the investor.
  - If the value of the contract at time T is negative for the investor the full exposure is paid to the counterparty.

Thus in the valuation all these scenarios need to be taken into consideration.

In order to derive an expression for BCVA, lets look at the price of a contract X with the default-risk-free price $V(0,T) = \Pi(0;X)$ at time $t = 0$. Under bilateral counterparty risk the price of the contract becomes

$$\begin{aligned}\tilde{\Pi}(0;X) =& V(0,T) + \mathbb{E}^{\mathbb{Q}}\big[(1-R_I)\mathbb{1}_{\{\tau_I \leq T, \tau_I < \tau_C\}} D(0,\tau_I)(-V(\tau_I,T))^+ \big| \mathcal{G}_0\big] \\ & - \mathbb{E}^{\mathbb{Q}}\big[(1-R_C)\mathbb{1}_{\{\tau_C \leq T, \tau_C < \tau_I\}} D(0,\tau_C)(V(\tau_C,T))^+ \big| \mathcal{G}_0\big].\end{aligned}$$

Thus for a bilateral derivative contract the CVA is given by

$$\mathrm{BCVA} = \mathrm{CVA} - \mathrm{DVA}, \tag{2.7}$$

where

$$\mathrm{CVA} = \mathbb{E}^{\mathbb{Q}}\big[(1-R_C)\mathbb{1}_{\{\tau_C \leq T, \tau_C < \tau_I\}} D(0,\tau_C)(V(\tau_C,T))^+ \big| \mathcal{G}_0\big]$$

and

$$\mathrm{DVA} = \mathbb{E}^{\mathbb{Q}}\big[(1-R_I)\mathbb{1}_{\{\tau_I \leq T, \tau_I < \tau_C\}} D(0,\tau_I)(-V(\tau_I,T))^+ \big| \mathcal{G}_0\big].$$

2.4 Collateral

Collateral is posted in an OTC derivatives transaction in order to decrease the counterparty credit risk and leave some security for possible future obligations to the other party. Collateralization of OTC derivatives transactions is similar to the collateralization of lending risk, where collateral is posted to reduce the credit exposure. However, the management of collateral is way more complex in OTC derivatives transactions, because of the fluctuation of exposure and the possible bilateral nature of a contract. Since the exposure to a counterparty in a derivative often is volatile with daily changes, collateral needs to be managed frequently in order to have the exposure under control. Collateral is posted by the party whose MtM is negative and it is generally posted in form of risk-free cash flows or of assets. The party whom receives the collateral, pays an interest rate back to the party posting the collateral. This interest rate paid is often an overnight rate such as the EONIA rate, described in Section 3.1.2.

The collateral agreements in a transaction is generally specified under a Credit Support Annex (CSA) to the ISDA Master Agreement, presented later in Section 2.7. The CSA specifies what type of assets and in which currencies that collateral is accepted. It also includes how the collateral should be managed, including frequencies, margin levels etc..

In order to calculate CVA in the presence of collateral, the exposure has to be adjusted. Let $C(t)$ denote the posted collateral at time t. The exposure at time t of a portfolio with the value $V(t,T)$ is with the presence of collateral $(V(t,T) - C(t))^+$. The unilateral CVA thus becomes

$$\mathrm{CVA} = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{\tau \leq T\}}(1-R)D(0,\tau)(V(\tau,T) - C(t))^+ \big| \mathcal{G}_0\big]. \tag{2.8}$$

and is in the same way extended to DVA (Brigo et al., 2013).

2.5 Re-Hypothecation

Re-hypothecation means that the party who receives collateral have the right to re-allocate the collateral into an investment or as collateral to another trade. This can increase the counterparty credit risk.

Consider a situation where an investor posts collateral in the form of cash, to a counterparty. The counterparty uses the collateral to post collateral in another trade where they have a negative MtM. A short period later the MtM of the first trade changes in favour of the investor, and before the next collateral adjustment margining date the counterparty defaults. The counterparty is now not only unable to fulfil their obligations in the contract, but neither are they able to repay the posted (and re-invested) collateral. Thus the investor suffer from a loss, both from the initial trade and the posted collateral (Brigo et al., 2013).

2.6 Netting

The exposure on a counterparty-level can be greatly reduced by the means of netting. A netting agreement is an agreement between two counterparties, that allows aggregation of all trades between the two counterparties in the case of default. This means that the exposure towards a defaulted counterparty is checked at a netted portfolio level rather than for each trade. More generally there can exist several netting agreements between two counterparties, creating different netting sets (Pykhtin and Zhu, 2007).

Consider a portfolio $V_{tot}(t,T) = \sum_{i=1}^{N} V_i(t,T)$ consisting of N different financial contracts. It is the same counterparty to each of the financial contracts $V_i(t,T)$. If the portfolio can be netted, the exposure at time $s \geq t$ is greater or equal to the sum of each exposure. Writing this with equations we have

$$(V_{tot}(s,T))^+ \leq \sum_{i=1}^{N} (V_i(s,T))^+.$$

This is a trivial relationship, however this also holds for CVA, i.e.

$$\mathrm{CVA}_{tot} \leq \sum_{i=1}^{N} \mathrm{CVA}_i.$$

Thus the CVA of a netted portfolio $V_{tot}(t,T)$ is smaller or equal to the CVA of a sum of the portfolios $V_i(t,T)$.

A very simple case is when an investor has entered into one payer swap and one receiver swap with the same counterparty. Both interest rate swaps are equivalent regarding conditions such as swap rate, notional, maturity, etc.. If the portfolio can be netted the CVA is equal to zero, since the net exposure to the counterparty is also equal to zero. However, if the portfolio can not be netted one of the swap will have a positive exposure $V_1(t,T)$ and the other swap a negative exposure of the same size $-V_2(t,T)$. The total exposure then becomes $V_1(t,T)$, which will give rise to a CVA.

One of the biggest advantages when it comes to CCPs is that the whole portfolio of contracts with different counterparties can be netted in order to reduce the exposure.

2.7 The ISDA Master Agreement

The International Swaps and Derivatives Association (ISDA) is an organization for OTC derivative trading. The ISDA master agreement is a legal framework which outlines the terms and conditions between two parties in an OTC derivatives trade. This agreement and the including CSA is used in order to minimize legal uncertainties and to specify standard terms of things like netting, collateral, definition of default etc.. Multiple transactions can be done under the master agreement, which makes parties not have to negotiate the terms of each transaction (Brigo et al., 2013).

2.8 Wrong-Way Risk

Wrong-way risk (WWR) is when the exposure to a counterparty increases with the risk of default of the counterparty.

A classic example of WWR is when an investor has taken a long position in a put option on the underlying stock S_C. The counterparty of the option is the same company as of the underlying stock, i.e. company C. Consider a situation where company C runs into financial trouble. The stock is likely to decrease in value, which means that the investor's put option is increasing in value. Because of company C's financial trouble, the risk is increasing that they won't be able to fulfil their obligations to the investor and pay for the put option at maturity. Hence from the investor's point of view both the exposure and risk to counterparty C has significantly increased. This is known as WWR.

Another scenario that might be more likely is when a counterparty post collateral in a trade. As mentioned earlier collateral is posted in order to leave some security of a trade and reduce the counterparty credit risk. However, if a counterparty would post their own assets, like a bond written on themselves or their own stocks, wrong-way risk arises. Even though this does not cause increased exposure in the initial contract between the parties, it explicitly increases the exposure since the value of the posted bond or stock probably would decrease if the default risk of the counterparty would increase.

Vice Versa, so that the risk is decreasing with increasing exposure, it is known as right-way risk (RWR). In order to capture WWR/RWR it is essential to have a model that models the dependence between underlying market risk factors and credit risk factors. So in the first example above, one would want to be able to capture the dependence between the default risk and the stock price of company C. As mentioned above if the counterparty's credit spreads would increase significantly in a company, this is very likely to also effect their stock price (in a negative way).

2.9 Hedging CVA

Nowadays it is very common that large financial institutions have a CVA (or XCA) desk where CVA is calculated and hedged. As discussed in Section 1.3 about two thirds of the credit losses in the financial crisis was actually CVA losses. In order to reduce these potential losses and also reduce capital requirements it is important to be able to hedge CVA.

As discussed earlier, counterparty credit risk and thus also CVA can be reduced by diversification, netting, posting collateral etc.. However in order to accurately hedge CVA, one has to both hedge the default risk and the market risk. One can do so by buying CDS's written on the counterparty combined with hedging the exposure.

An easier way to hedge CVA is using a contingent credit default swap (CCDS) contract. A CCDS is essentially the same as a CDS but with one difference. Instead of buying protection on a notional (fixed amount) as in a CDS the CCDS lets you buy protection on a (random) value of reference security. Hence the CCDS is the perfect hedge for CVA.

DVA can be hedged in the same way as CVA but since it is not possible to sell a CDS (or preferably a CCDS) on yourself, DVA can be proxy hedged by selling CDS contracts on a reference portfolio of similar companies which are highly correlated with the own firm. This can however be considered risky since there is a possibility of default in the reference portfolio. Selling CDS contracts on a reference firm or index thus proxy hedges the credit spread risk but not the jump to default risk.

Another important aspect to bear in mind is the potential counterparty credit risk arising from the hedge itself (Brigo et al, 2013).

2.10 Data Challenges

In this chapter, CVA has been defined and some complications that comes with CVA has been presented. One of the biggest challenges in the procedure of calculating CVA is finding the correct data in order to estimate models. Since CVA is calculated under the risk-free measure $\mathbb{Q}$, the risk factor models needs to be calibrated under $\mathbb{Q}$.

One point that is needed in a CVA calculation is the risk-neutral probabilities of default. These probabilities can be stripped down from either Credit Default Swap (CDS) or corporate bond data, but for many counterparties it does not exist a liquid CDS or even a quoted bond. In these cases an option is to approximate with a CDS index or a corporate bond written on another company similar to the counterparty.

An even harder task may be finding correlations, also specified under $\mathbb{Q}$, between market risk factors and credit spreads. These correlations are important in order to be able to model WWR. An alternative here is to use historical correlations, i.e. under the historical probability measure $\mathbb{P}$. This is of course not ideal, since it is measured under the wrong measure, but it can still give better information than just guessing (Brigo et al., 2013).

2.11 Funding Value Adjustment

The Funding Value Adjustment (FVA) is another adjustment made to the price of an OTC derivative transactions in order to reflect the funding costs and benefits. Whether to make an FVA or not has been a debated thing in the financial world. For instance, Hull and White (2014) claims that an FVA could lead to arbitrage opportunities.

After the financial crisis, the funding for a bank has become much more expensive. Earlier a bank could lend money at the LIBOR rate, but today the lending for many banks are done at LIBOR plus a spread. A trader needs funding in order to be able to hedge the initial trade and to be able to post collateral.

A collateralized trade is said to be funded by the collateral. If a party receives the collateral, this can be used to fund the collateral for the hedge of the initial trade. Vice versa, if a party needs to post collateral for the initial trade, they can fund this using the received collateral from the hedge.

For an uncollateralized trade, the funding of the hedge has to be done at the banks average funding cost. The FVA is done in order to reflect this funding cost, or benefit, e.g. for an uncollateralized trade with positive MtM (Hull and White, 2014).

FVA is just mentioned and will not be analysed in this thesis. For further reading, see Hull and White (2014) or Brigo et al. (2013).

Chapter 3

Introduction to Interest Rate and Credit Modelling

In this chapter some concepts of interest rate and credit modelling are introduced.

Following the notations in Chapter 2 we will work on a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$, where Ω is the outcome space such that an outcome $\omega \in \Omega$. $\mathcal{G}$ is the σ-algebra and $\mathcal{G}_t$ the filtration up to time t such that for a time $t \leq u$ the following hold $\mathcal{G}_t \subseteq \mathcal{G}_u \subseteq \mathcal{G}$. The filtration $\mathcal{G}_t$ can be interpreted as the total market information up to time t, i.e. $\mathcal{G}_t$ includes all market risk factors and default events up to time t. Furthermore let $\mathcal{F}$ denote the σ-algebra such that the filtration $\mathcal{F}_t$ contains the same information as $\mathcal{G}_t$ but the default events. In other words $\mathcal{F}_t$ is a sub-filtration of $\mathcal{G}_t$, i.e. $\mathcal{F}_t \subseteq \mathcal{G}_t$, where $\mathcal{F}_t$ contains only the risk-free market information. The probability measure $\mathbb{Q}$ is the risk-neutral measure such that under the risk-free bank account, following the process

$$dB_t = B_t r_t dt,\ B_0 = 1,$$

all tradeable assets divided by B_t (i.e. discounted) are martingales.

3.1 Brief Interest Rate Theory

Below follows a short interest rate theory presentation of concepts used in this report. Starting by defining a continuously compounded forward rate $F(t; S, T)$ at time t for a period $[S, T]$ as

$$e^{F(t;S,T)(T-S)} = \frac{P(t,S)}{P(t,T)}.$$

Furthermore introducing an instantaneous forward rate such that

$$f(t,T) = \lim_{\Delta \to 0} F(t; T, T+\Delta).$$

The continuously compounded short rate is defined as $r(t) = f(t,t)$, such that a risk-free bank account evolves according to $B_t = \exp\{\int_0^t r(s)ds\}$. For simplification the discount factor at time t for a cash flow received at time T will be denoted by

$$D(t,T) = \frac{B_t}{B_T} = \exp\{-\int_t^T r(s)ds\}.$$

The price of a risk-free zero-coupon bond at time t received for sure at time T is given by

$$P(t,T) = \exp\{-\int_t^T f(t,s)ds\},$$

where $f(t,s)$ is the instantaneous forward rate, or equivalent using the short rate

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\big[\exp\{-\int_t^T r(s)ds\}\big|\mathcal{F}_t\big].$$

3.1.1 LIBOR Rates

The London Interbank Offered Rate (LIBOR) is a reference rate for a specific maturity for which leading banks in London would be charged if borrowing money from other banks. The LIBOR rates are calculated each day as the average of the rates estimated by the banks, excluding the largest and smallest rates, for 5 different currencies and 7 maturities. Since these banks are not risk-free counterparties there is a credit risk in the LIBOR rates.

The LIBOR spot rate is a simple compounded interest rate defined as

$$L(t,T) = \frac{1}{T-t}\Big(\frac{1}{P(t,T)} - 1\Big) \tag{3.1}$$

and the LIBOR forward rate is defined as

$$L(t;T,S) = \frac{1}{T-S}\Big(\frac{P(t,T)}{P(t,S)} - 1\Big). \tag{3.2}$$

An Actual/360 day-count convention is used for the LIBOR rates. For a more convenient notation, the LIBOR forward rate seen at t between between time T_{i-1} and T_i will be denoted

$$L_i(t) = L(t;T_{i-1},T_i).$$

3.1.2 The Overnight Indexed Swap Rate

Overnight Indexed Swaps (OIS) are interest rate swaps where a fixed rate is exchanged for a floating rate, where the floating rate is an overnight index rate. This index rate is a weighted average on the unsecured overnight lending rates in the interbank market. The overnight index rate is for U.S. dollars is the effective federal funds rate, in Euro the Euro Overnight Index Average (EONIA) or in Sterling the Sterling Overnight Index Average (SONIA). Since these index rates are based on overnight lending rates they

3.1.3 Discounting

LIBOR rates and LIBOR swap rates have often been used as proxies for the risk-free interest rates when discounting future cash flows. However when the market started shaking during the credit crisis in 07-08, the LIBOR rates were traded with a large spread compared to treasury bills and OIS rates. The 3-month U.S. dollar LIBOR-Treasury spread, known as the TED spread, had historically before the credit crisis been less than 50 basis points, but in the middle of the crisis it spiked to an all-time high at over 450 basis points. Similarly the EURIBOR-OIS spread, which historically had been around 10 basis points under normal market conditions, rose to about 350 basis points after the Lehman bankruptcy announcement (Hull and White, 2013). This clearly indicates that the LIBOR rates are poor proxies for the risk-free rates, at least in a stressed market environment.

Hull and White (2013) says that the practice among market participants is to discount collateralized cash flows with the OIS rates and uncollateralized cash flows with the LIBOR rates. The reason for using the OIS rates when discounting a collateralized derivative is that this trade is funded by the collateral, which often pays the overnight interest rate and hence is linked to the OIS rates. But for a non-collateralized derivative the LIBOR rates are better proxies for the dealers cost of funding, and hence the reason to an FVA as discussed in Section 2.11. However Hull and White (2013) argue for instead using the OIS rates for the purpose of discounting in all situations.

3.2 Multi-Curve Framework

Together with the discounting, another area that has changed post-crisis is the interest curve methodology. The classical approach has been to build a single LIBOR yield curve bootstrapped using money market instruments, FRA's and swap rates. From this yield curve, using arbitrage arguments combined with an assumption of no credit risk, all forward curves for different tenors such as 1-month, 3-month, 6-month etc. was derived. As discussed in the section above, the same yield curve was used for discounting.

Nowadays the market has adopted a multi-curve framework. The credit risk in the LIBOR rates are no longer ignored, and thus there is a difference between a 3-months and a 6-months curve. With the presence of credit risk, it becomes less risky to roll over the lending at a 3-months rate compared to lending at a 6-month rate. This can be observed in the markets from the quoted spreads of basis swaps. A basis swap is a swap where two floating rates of different tenors are exchanged, e.g. one party receives a 3-month LIBOR rate quarterly and pays a 6-month LIBOR rate semi-annually to the counterparty. Before the financial crisis these spreads for different spreads were very small, but during the crisis these spreads rose significantly and have since then the spreads have stayed at levels which are too large to be ignored.

The procedure of building multiple yield curves is often done by building one curve for the most liquid tenor and then building the curves for the other tenors by using quotes of basis swaps. E.g. for the EUR market, a typical approach is to build a 6-month curve by using the 6-month EURIBOR rate, FRA's on the underlying 6-month EURIBOR rate and fixed-for-floating swaps with a 6-month tenor on the floating leg. From this 6-month curve, the 3-month curve can be derived by using the quoted spreads of 3-month vs 6-month basis swaps etc..

However, in this thesis a single curve framework will be used and an simple approach of building a single yield curve is described in Section 3.5 (Ametrano and Bianchetti, 2013).

3.3 Interest Rate Swaps

The most common interest rate swap is the fixed-for-floating interest rate swap, which is known as a plain vanilla swap. A pays a floating rate L (typically a 3- or 6-month LIBOR rate) to B, which in turn pays a predetermined fixed (swap) rate S (typically a 6-month or a 1-year rate) , to A. The direction of the swap is often referred to as either a payer or a receiver swap. In the case above, A has a receiver swap (A receives the fixed rate) and B has a payer swap (B pays the fixed rate).

Let payment of the fix leg occur at the dates $T_1, \dots, T_n$, payment of the floating leg at the dates $T_1', \dots, T_N'$ and the initiation of the swap at T_0. Furthermore let the notional amount of the swap be K and let $\Delta T_i = T_i - T_{i-1}$ and $\Delta T_i' = T_i' - T_{i-1}'$. Then at time T_i' A will pay

$$KL(T_{i-1}', T_i')\Delta T_i'$$

and at time T_i receive

$$KS\Delta T_i.$$

If $T_i = T_i'$ and $\Delta T_i = \Delta T_i'$, the net cash flow at time T_i is

$$K\Delta T_i\big(L(T_{i-1}, T_i) - S\big).$$

3.3.1 Swap Rate

A swap can be regarded as two bonds. With this view A has bought a fixed rate bond with return S issued by B and receives coupons based on the fix rate. B has bought a floating rate bond issued by A and receives coupons according to the prevailing spot rate. Usually the swap rate S is determined such that the initial value of the swap is zero. With the swap starting at T_0 and payments of the fix leg occurring at $T_1 < \dots < T_n$ and payments of the floating leg at $T_1' < \dots < T_N'$. Again let $\Delta T_i = T_i - T_{i-1}$ and $\Delta T_i' = T_i' - T_{i-1}'$. In order to have a fair swap, a swap rate S should be determined such that at time $t \leq T_0$

$$\Pi(t; swap) = P_{fixed}(t) - P_{floating}(t) = 0.$$

Following risk-neutral valuation

$$KS\sum_{i=1}^{n} P(t,T_i)\Delta T_i - KB_t\sum_{i=1}^{N}\mathbb{E}^{\mathbb{Q}}\big[\frac{1}{B_{T'_i}}L(T'_{i-1};T'_{i-1},T'_i)\Delta T'_i\big|\mathcal{F}_t\big] = 0$$

and using that the LIBOR forward rate $L(T'_{i-1};T'_{i-1},T'_i)$ is a martingale under the $Q^{T'_i}$-measure

$$KS\sum_{i=1}^{n} P(t,T_i)\Delta T_i - K\sum_{i=1}^{N} P(t,T'_i)L(t;T'_{i-1},T'_i)\Delta T'_i = 0$$

and using the definition of the LIBOR forward rate in equation 3.2

$$KS\sum_{i=1}^{n} P(t,T_i)\Delta T_i - K\sum_{i=1}^{N} P(t,T'_i)\big(\frac{P(t,T'_{i-1})}{P(t,T'_i)} - 1\big) = 0.$$

Simplifying yields,

$$KS\sum_{i=1}^{n} P(t,T_i)\Delta T_i - KP(t,T_0) + KP(t,T'_N) = 0.$$

Solving this for S yields the fair swap rate

$$S = \frac{P(t,T_0) - P(t,T'_N)}{\sum_{i=1}^{n} P(t,T_i)\Delta T_i}. \tag{3.3}$$

3.3.2 Valuing Interest Rate Swaps

Valuing an IRS in between coupons is a bit different. Let t be such that $T_{i-1} < t < T_i < \cdots < T_n$ and $T'_{i-1} < t < T'_i < \cdots < T'_n$. The IRS can be seen as a fix cash flow plus a forward swap. Using that the cash flow at time T_i is known at time t the value of a receiver swap at time is

$$\Pi(t;swap) = KS\sum_{j=i}^{n} P(t,T_j)\Delta T_j + KP(t,T_n) - KP(t,T'_i)\big(1 + \Delta T'_i L(T'_{i-1},T'_i)\big). \tag{3.4}$$

3.3.3 PV01

PV01 (or DV01) is a common risk measure of interest rate swaps. It measures the price sensitivity to a parallel shift in the underlying yield curve. More precise, PV01 measures the change in present value of the swap when the yield curve moves with a parallel shift of 1 basis point (bp), i.e. 0.01%. Since the underlying yield curve does not just move in parallel shifts, this measure is often combined with measures of the price sensitivities to shifts of 1bp in different parts of the yield curve.

3.4 Swaptions

A swaption is simply an option on a swap. From the risk-neutral valuation and using equation 3.3, the price of a payer swaption at time t with maturity T of a swap starting at $T_\alpha > T$ and maturing at $T_\beta > T_\alpha$ with swap rate K is

$$\begin{aligned}\Pi\big(t; swaption(T_\alpha, T_\beta, K, T)\big) &= B_t \mathbb{E}^{\mathbb{Q}}\big[\frac{1}{B_T}\big(\text{swap}(T; T_\alpha, T_\beta, K)\big)^+ \big|\mathcal{F}_t\big] \\ &= B_t \mathbb{E}^{\mathbb{Q}}\big[\frac{1}{B_T} A_{\alpha,\beta}(T)\big(S_{\alpha,\beta}(T) - K\big)^+ \big|\mathcal{F}_t\big]. \qquad (3.5)\end{aligned}$$

Where $A_{\alpha,\beta}(T) = \sum_{i=\alpha+1}^{\beta} \Delta T_i P(T, T_i)$ is the accrual term and $S_{\alpha,\beta}(T)$ the forward swap rate at time T.

3.5 Yield Curve

The initial term structure of the current interest rates is often referred to as the yield curve. A yield curve or a zero-coupon bond curve is needed in order to price swaps and calibrate an interest rate model. The following method on bootstrapping a yield curve is described by Davis (2010). Note that this is the classical approach of building a single yield curve. However, the bootstrapping methodology is also adaptable to the multi-curve framework.

Starting from a set of LIBOR rates for maturities up to 1 year and with a set of LIBOR swap rates with maturity up to 10 years a yield curve and thus also a discount curve can be constructed.

Starting with the LIBOR rates and by inverting equation 3.1 the zero-coupon bond prices are given by

$$P(t, T_i) = \frac{1}{1 + L(t, T_i)(T_i - t)}.$$

Since the LIBOR rates only are available for maturities up to 1 year the rest of the curve will be constructed using swap rates. Assuming swap rates of swaps that pays coupons semi-annually and also assuming the presence of a swap rate for each coupon date. Then the zero-coupon bond prices can be solved recursively using equation 3.3 such that

$$P(t, T_{i+1}) = \frac{1 - S_{i+1} A_i}{1 + \Delta T_{i+1} S_{i+1}}$$

where $A_i = \sum_{k=0}^{i} \Delta T_k P(t, T_k)$ is the accrual term and $\Delta T_{i+1} = T_{i+1} - T_i$ is the time between coupons.

However the assumption of having a swap rate with maturity at each coupon date is unrealistic and as a solution to this interpolation is used. To determine an interpolated value of $P(t, T_k)$ between the known values $P(t, T_{k-1}) = e^{-r_{k-1}T_{k-1}}$ and $P(t, T_{k+1}) =$

$e^{-r_{k+1}T_{k+1}}$ one can use linear interpolation of the rates r_{k-1} and r_{k+1}. By writing the time T_k as a linear combination such that $T_k = (1-\lambda)T_{k-1} + \lambda T_{k+1}$ one can get an interpolated value $\hat{P}(t, T_k) = e^{r_{k-1}(1-\lambda)T_{k-1} + r_{k+1}\lambda T_{k+1}}$.

So by using a binary search algorithm and interpolation we can still determine each zero-coupon bond. Assuming that we have determined $P(t, T_k)$ and that the next swap rate available is S_{k+j}. Furthermore let $i = 1, \ldots, j-1$ and $\hat{P}(t, T_i)$ be the interpolated zero-coupon bond price between the known $P(t, T_k)$ and the unknown value p of the zero-coupon bond maturing at time T_{k+1}. The swap rate at time T_{k+1} can be written as

$$\hat{S}_{k+1} = \frac{1-p}{A_k + \Delta T_{k+1}\hat{P}(t, T_{k+1}) + \cdots + \Delta T_{k+j-1}\hat{P}(t, T_{k+j-1}) - \Delta T_{k+j}p}.$$

Since the true swap rate S_{k+1} is known a value of p can be found such that $|S_{k+1} - \hat{S}_{k+1}| < \epsilon$ for a small value ϵ. Following this approach the following binary search algorithm can be used.

1. Set $l = 0$ and $r = 1$
2. Set $p = (l + r)/2$
3. If $\hat{S}_{k+1} < S_{k+1}$ set $r = p$ else set $l = p$
4. Repeat 2 and 3 until $|S_{k+1} - \hat{S}_{k+1}| < \epsilon$
5. Return $P(t, T_{k+j}) = p$

3.6 Interest Rate Model

The market practice has for a long time been to price simple interest rate derivatives such as caps, floors and swaptions by using an extension of Black's formula. In order to have logical and arbitrage free models which prices caps, floors and swaptions in a Black's formula-form the market models were developed. The LIBOR Market Model (LMM) models the LIBOR forward rates as lognormal random variables and prices caps and floors with Black's formula. Similarly the Swap Market Model (SMM) models the forward swap rate as a lognormal random variable and prices swaptions with Black's formula (Björk, 2009).

3.6.1 LIBOR Market Model

Usually in an interest rate model either the short rate r_t or the instantaneous forward rate $f(t, T)$ is modelled. However none of these two rates are observable in the market. In the LMM instead the observable forward LIBOR rates $L(t; T_{i-1}, T_i)$, i.e. the simple compounded rate seen at time t between time T_{i-1} and T_i, are modelled.

Furthermore the LIBOR forward rate $L_i(t) = L(t; T_{i-1}, T_i)$ is a martingale under its own forward measure $\mathbb{Q}^{T_i}$, which is the measure associated with $P(t, T_i)$ as numeraire. Under $\mathbb{Q}^{T_i}$ the LIBOR forward rate has the following dynamics

$$\frac{dL_i(t)}{L_i(t)} = \sigma_i(t) dW_i^{\mathbb{Q}^{T_i}}(t) \quad i = 1, \dots, N.$$

Here $\sigma_i(t)$ is a scalar and represents the instantaneous volatility of forward rate $L_i(t)$ and $W_i^{\mathbb{Q}^{T_i}}(t)$ is a $\mathbb{Q}^{T_i}$-Brownian Motion. Using this model to price a cap with the payoff

$$\big(L_i(T_{i-1}) - K\big)^+ \Delta T_i$$

leads to Black's formula for caps.

One is often looking to model the future LIBOR forward rates of a finite set of times $T_1, \dots, T_N$. Using a change-of-numeraire technique one can derive the dynamics under a fix forward measure Q^{T_k}, with $P(t, T_k)$ as numeraire, such that the LIBOR forward rates have the following dynamics

$$\frac{dL_i(t)}{L_i(t)} = \begin{cases} -\big(\sum_{j=i+1}^{k} \frac{\Delta T_j L_j(t) \sigma_i(t) \sigma_j(t) \rho_{i,j}}{1+\Delta T_j L_j(t)}\big) dt + \sigma_i(t) dW_i^{\mathbb{Q}^{T_k}}(t) & \text{if } i < k \\ \sigma_i(t) dW_i^{\mathbb{Q}^{T_k}}(t) & \text{if } i = k \\ \big(\sum_{j=k}^{i-1} \frac{\Delta T_j L_j(t) \sigma_i(t) \sigma_j(t) \rho_{i,j}}{1+\Delta T_j L_j(t)}\big) dt + \sigma_i(t) dW_i^{\mathbb{Q}^{T_k}}(t) & \text{if } i > k. \end{cases}$$

Where $\rho_{i,j}$ denotes the correlation between forward rate $L_i(t)$ and $L_j(t)$.

A third specification of the LMM is under the spot measure, which is a measure similar to the usual risk-neutral measure. Since the modelling is done for a finite set of forward rates there is no point of using the usual continuous bank account numeraire $B_t = \exp\{\int_0^t r_s ds\}$. Instead a discrete bank account $B^*(t)$ can be introduced such that

$$B^*(t) = P(t, T_{\eta(t)-1}) \prod_{i=0}^{\eta(t)-1} \big(1 + \Delta T_i L_i(T_i)\big).$$

Where $\eta(t)$ is the index of the first forward rate that has not expired by t, i.e. $\eta(t) = m$ if $T_{m-2} < t \leq T_{m-1}$. The measure associated with $B^*(t)$ as numeraire is called the spot measure and will be denoted by $\mathbb{Q}^*$. Under the spot measure the forward LIBOR rates has the following arbitrage-free dynamics

$$\frac{dL_i(t)}{L_i(t)} = \Big(\sum_{j=\eta(t)}^{i} \frac{\Delta T_j L_j(t) \sigma_i(t) \sigma_j(t) \rho_{i,j}}{1 + \Delta T_j L_j(t)} \Big) dt + \sigma_i(t) dW_i^{\mathbb{Q}^*}(t), \quad i = 1, \dots, N. \tag{3.6}$$

(Brigo and Mercurio, 2012).

3.6.2 Swap Market Model

The Swap Market Model (SMM) is the market model designed for swaption pricing. Imagine a swap with start at T_α and with payments occurring at $T_{\alpha+1}, \ldots, T_\beta$. Then the SMM models the forward swap rate $S_{\alpha,\beta}$ as a lognormal random variable and especially $S_{\alpha,\beta}$ is a martingale under the swap measure $\mathbb{Q}^{\alpha,\beta}$. The swap measure $\mathbb{Q}^{\alpha,\beta}$ is the measure with the accrual term $A_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \Delta T_i P(t,T_i)$. Under $\mathbb{Q}^{\alpha,\beta}$ the forward swap rate has the following dynamics

$$dS_{\alpha,\beta}(t) = \sigma^{\alpha,\beta}(t) S_{\alpha,\beta}(t) dW^{\mathbb{Q}^{\alpha,\beta}}(t). \tag{3.7}$$

Consider a payer swaption with the setting above. From equation 3.5 the price is

$$\Pi\big(t; swaption(T_\alpha, T_\beta, K, T)\big) = B_t \mathbb{E}^{\mathbb{Q}}\big[\frac{1}{B_T} A_{\alpha,\beta}(T)\big(S_{\alpha,\beta}(T) - K\big)^+ \big| \mathcal{F}_t\big].$$

Changing measure using Girsanov's Theorem with the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{\alpha,\beta}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{A_{\alpha,\beta}(t)B(0)}{A_{\alpha,\beta}(0)B(t)}$$

yields

$$\begin{aligned}
\Pi\big(t; swaption(T_\alpha, T_\beta, K, T)\big) &= A_{\alpha,\beta}(t)\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\big[\frac{1}{A_{\alpha,\beta}(T)} A_{\alpha,\beta}(T)\big(S_{\alpha,\beta}(T) - K\big)^+ \big| \mathcal{F}_t\big] \\
&= A_{\alpha,\beta}(t)\mathbb{E}^{\mathbb{Q}^{\alpha,\beta}}\big[\big(S_{\alpha,\beta}(T) - K\big)^+ \big| \mathcal{F}_t\big].
\end{aligned}$$

This leads to Black's formula for swaptions and at $t = 0$ the price is

$$\Pi\big(0; swaption(T_\alpha, T_\beta, K, T)\big) = A_{\alpha,\beta}(0)\big(S_{\alpha,\beta}(0)N(d_1) - KN(d_2)\big) \tag{3.8}$$

where

$$\begin{aligned}
d_1 &= \frac{\log(S_{\alpha,\beta}(0)/K) + 0.5 v_{\alpha,\beta}^2(T)T}{v_{\alpha,\beta}(T)\sqrt{T}} \\
d_2 &= d_1 - v_{\alpha,\beta}(T)\sqrt{T}
\end{aligned}$$

where $v_{\alpha,\beta}(T)$, called the Black's volatility, is the average volatility such that

$$v_{\alpha,\beta}(T) = \frac{1}{T}\int_0^T \big(\sigma^{\alpha,\beta}(t)\big)^2 dt. \tag{3.9}$$

3.6.3 Pricing Swaptions in LMM

Instead of using the SMM, swaptions can be priced in the LMM using Monte Carlo simulation or by using an analytical approximative formula. Starting with the Monte Carlo technique, the forward rates involved in the swaption are simulated and using the relationship

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) L_i(t) \tag{3.10}$$

where

$$w_i(L_{\alpha+1}(t), \ldots, L_\beta(t)) = \frac{\Delta T_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\Delta T_j L_j(t)}}{\sum_{k=\alpha+1}^{\beta} \Delta T_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\Delta T_j L_j(t)}}.$$

By calculating swap rates in each simulation one can compute swaption prices as the average discounted payoff $A_{\alpha,\beta}(T)(S_{\alpha,\beta}(T) - K)^+$.

The analytical approximative formula for swaptions in the LMM is a modified version of Black's formula obtained using a so called "freezing coefficients" technique. Starting with the Black's swap volatility from equation 3.9 in the SMM

$$v_{\alpha,\beta}^2(T_\alpha) = \frac{1}{T_\alpha} \int_0^{T_\alpha} \left(\sigma^{\alpha,\beta}(t)\right)^2 dt.$$

Using the "freezing coefficients" technique and equation 3.10 one can show that an approximation of the Black's swap volatility in the LMM is given by

$$\left(v_{\alpha,\beta}^{\text{LMM}}(T_\alpha)\right)^2 = \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(0) L_i(0) L_j(0) \rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) dt. \tag{3.11}$$

By using this proxy and Black's formula one have an approximative analytical formula for swaption prices also in the LMM.

3.6.4 Calibration of the LMM to Swaptions

In general a calibration procedure is an optimization problem where a function which describes the distance between market and model prices is minimized. In this section it will be described how the LMM can be calibrated to swaption prices, or rather swaption volatilities, since the market practice is to quote swaption prices with the corresponding Black's volatility.

Brigo and Mercurio (2012) describe the so called analytical cascade calibration method. This calibration method is based on a piecewise-constant parametrization of the volatilities such that $\sigma_1(t) = \sigma_{1,1}$ on the interval $t \in (0, T_0]$ and where the volatility is "dead" after T_0. The volatility is "dead" after T_0 since the forward rate $L_1(t)$ is deterministic at $t = T_0$. The whole volatility matrix will look like

$$\Sigma = \begin{pmatrix} \sigma_{1,1} & 0 & 0 & \cdots & 0 \\ \sigma_{2,1} & \sigma_{2,2} & 0 & \cdots & 0 \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{m,1} & \sigma_{m,2} & \sigma_{m,3} & \cdots & \sigma_{m,n} \end{pmatrix}. \tag{3.12}$$

Using the approximation in equation 3.11 and the assumption of piecewise-constant volatilities the following equation is received

$$\left(v_{\alpha,\beta}^{\mathrm{LMM}}(T_\alpha)\right)^2 = \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(0) L_i(0) L_j(0) \rho_{i,j}}{S_{\alpha,\beta}(0)^2} \sum_{h=0}^{\alpha} (T_h - T_{h-1}) \sigma_{i,h+1} \sigma_{j,h+1}. \tag{3.13}$$

Rewriting equation 3.13 yields

$$\begin{aligned} T_\alpha S_{\alpha,\beta}(0)^2 \left(v_{\alpha,\beta}^{\mathrm{LMM}}(T_\alpha)\right)^2 = & \sum_{i,j=\alpha+1}^{\beta-1} w_i(0) w_j(0) L_i(0) L_j(0) \rho_{i,j} \sum_{h=0}^{\alpha} (T_h - T_{h-1}) \sigma_{i,h+1} \sigma_{j,h+1} \\ & + 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta(0) w_j(0) L_\beta(0) L_j(0) \rho_{\beta,j} \sum_{h=0}^{\alpha-1} (T_h - T_{h-1}) \sigma_{\beta,h+1} \sigma_{j,h+1} \\ & + 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta(0) w_j(0) L_\beta(0) L_j(0) \rho_{\beta,j} (T_\alpha - T_{\alpha-1}) \sigma_{\beta,\alpha+1} \sigma_{j,\alpha+1} \\ & + w_\beta(0)^2 L_\beta(0)^2 \sum_{h=0}^{\alpha-1} (T_h - T_{h-1}) \sigma_{\beta,h+1}^2 \\ & + w_\beta(0)^2 L_\beta(0)^2 (T_\alpha - T_{\alpha-1}) \sigma_{\beta,\alpha+1}^2. \end{aligned} \tag{3.14}$$

Using equation 3.14, the following calibration algorithm can be constructed.

1. Select the number s of rows in the swaption matrix that are of interest for the calibration.
2. Set $\alpha = 0$.
3. Set $\beta = \alpha + 1$.
4. Solve equation 3.14 in $\sigma_{\beta,\alpha+1}$.
5. Set $\beta = \beta + 1$. If $\beta \leq s$ go back to step 4, otherwise set $\alpha = \alpha + 1$.

6. If $\alpha < s$ go back to step 3, otherwise stop.

Thus for a given swaption volatility matrix and given correlations between the forward rates the volatility of the model can be calibrated. However since this approach is based on the solution of a second degree equation, there are possible solutions which are either negative or imaginary. As investigated in Brigo and Mercurio (2012), the output volatilities of this cascade calibration method are often negative or imaginary when a non-smooth input swaption volatility matrix is used. A purposed solution to this problem is to use a parametric function to describe the input swaption matrix in order to remove misaligned market quotations corresponding to illiquid swaptions.

Smooth Input Swaption Matrix

As proposed by Brigo and Mercurio (2012), a smooth version of the input swaption volatility matrix will, in most cases, remove negative or imaginary volatilities received from the calibration routine above. They also propose the following parametric function

$$\hat{v}(S,T) = \gamma(S) + \Big(\frac{\exp\{f\log(T)\}}{eS} + D(S)\Big)\exp\big\{-\beta\exp\{p\log(T)\}\big\} \tag{3.15}$$

where

$$\gamma(S) = c + (\exp\{h\log(S)\}a + d)\exp\big\{-b\exp\{m\log(S)\}\big\}$$

and

$$D(S) = (\exp\{g\log(S)\}q + r)\exp\big\{-s\exp\{t\log(S)\}\big\} + \delta.$$

Using the implied swaption volatilites from the market $v(S,T)$, where S represents the maturity of the option and T is the tenor of the underlying swap, the parametric function $\hat{v}(S,T)$ can be estimated using an optimization algorithm which minimizes $\sum_i \Big(v(S_i,T_i) - \hat{v}(S_i,T_i)\Big)^2$.

An estimated volatility surface can be seen beneath in Figure 3.1.

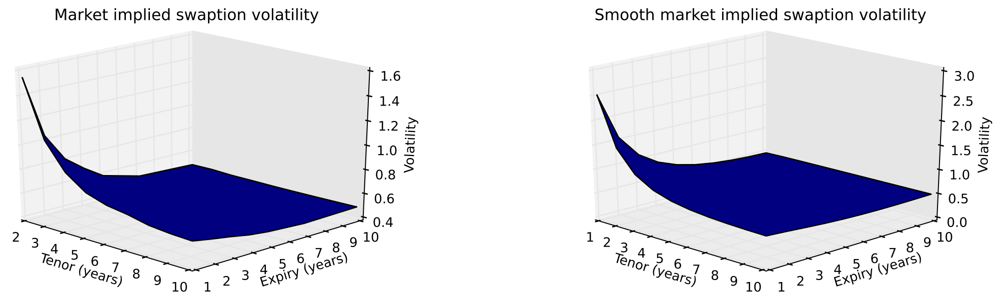

Figure 3.1: *Left:* Market implied swaption volatility from Bloomberg 7/5/2015. *Right:* Smooth market implied volatility from the parametric function in equation 3.15.

Calibration with Optimization Routine

Another way to ensure real and positive output volatilities is to use a simple optimization routine. Again assuming a piecewise linear specification of the LMM volatilities and the presence of market swaption prices and correlations between forward rates. Consider the following optimization problem.

$$\begin{aligned} \underset{\sigma}{\text{minimize}} \quad & ||\mathbf{v} - \mathbf{v}(\sigma)^{\text{LMM}}|| \\ \text{subject to} \quad & \sigma_{i,j} \geq 0,\ i = 1, \ldots, m,\ j = 1, \ldots, n. \end{aligned}$$

This thus becomes a least squares problem, where $\mathbf{v}$ represents a vector with implied volatilites from market quoted swaptions and $\mathbf{v}(\sigma)^{\text{LMM}}$ the approximative Black volatilites from equation 3.13 with the LMM volatility matrix σ in equation 3.12 as input. A solution to this problem guarantees positive and real volatilites $\sigma_{i,j}$ in the LMM model. This optimization problem can easily be solved using a numeric least squares optimization algorithm.

3.6.5 Historical Correlation

Following the calibration algorithm above one of the assumptions is that the correlation between forward rates is known. One way to specify the correlation is simply using historical correlation, i.e. calibrate the correlation under $\mathbb{P}$. However since the volatility is calibrated to market data, the model will still reproduce market prices and in other words be calibrated under $\mathbb{Q}$.

Following Brigo and Mercurio (2012) and assuming that the daily log returns of the forward rates $L_1(t), \ldots, L_n(t)$ have a multivariate normal distribution such that

$$\Big[\log\Big(\frac{L_1(t+\Delta t)}{L_1(t)}\Big), \ldots, \log\Big(\frac{L_n(t+\Delta t)}{L_n(t)}\Big)\Big] \sim N(\mu, \mathbf{\Sigma}).$$

The mean vector μ and the covariance matrix $\mathbf{\Sigma}$ are estimated using the standard sample approach. Given data from $N+1$ trading days with corresponding time vector $t_0, \ldots, t_N$ the estimates are

$$\hat{\mu}_i = \frac{1}{N}\sum_{k=0}^{N-1} \log\Big(\frac{L_i(t_k+\Delta t)}{L_i(t_k)}\Big)$$

and

$$\hat{\mathbf{\Sigma}}_{i,j} = \frac{1}{N-1}\sum_{k=0}^{N-1}\Big[\Big(\log\Big(\frac{L_i(t_k+\Delta t)}{L_i(t_k)}\Big) - \hat{\mu}_i\Big)\Big(\log\Big(\frac{L_j(t_k+\Delta t)}{L_j(t_k)}\Big) - \hat{\mu}_j\Big)\Big].$$

The correlation matrix ρ can then be estimated as

$$\hat{\rho}_{i,j} = \frac{\hat{\mathbf{\Sigma}}_{i,j}}{\sqrt{\hat{\mathbf{\Sigma}}_{i,i}\hat{\mathbf{\Sigma}}_{j,j}}}. \tag{3.16}$$

By using historical LIBOR and swap rates and following the bootstrapping procedure in Section 3.5 historical discount curves are received. From each historical discount curve the LIBOR forward rates $L_1(t), \ldots, L_n(t)$ can be calculated and the correlation in 3.16 can be estimated.

Smooth Correlation Matrix

From an estimated historical correlation matrix $\hat{\rho}$ it is convenient to fit a function $g(T_i, T_j) \approx \hat{\rho}_{i,j}$ to describe how a forward rate $L_i(t) = L(t; T_i, T_i+\Delta T_i)$ is correlated to another forward rate $L_j(t) = L(t; T_j, T_j + \Delta T_j)$. This will be done by fitting the following function

$$g(T_i, T_j) = \alpha + (1-\alpha)e^{-\beta(|T_j - T_i|)}, \tag{3.17}$$

where α and β are constants. Using this parametrization one assures a correlation of 1 if $T_j = T_i$ and the constant α can be seen as the limit correlation such that when $|T_j - T_i|$ is large the correlation converges towards α. This function can be estimated using least squares estimation such that $\sum_{i,j}\big(\hat{\rho}_{i,j} - g(T_i, T_j)\big)^2$ is minimized. This results in a smooth version of the historical correlation matrix (Brigo and Mercurio, 2012).

In Figure 3.2 both the historical correlation $\hat{\rho}_{i,j}$ in equation 3.16 and the smooth version $g(T_i, T_j)$ in equation 3.17 are shown. The correlation for the LIBOR forward rates

$L(t; T_{i-1}, T_i)$ for $i = 1, \ldots, 10$, where $T_i = i$ years, are estimated using one year of historical data.

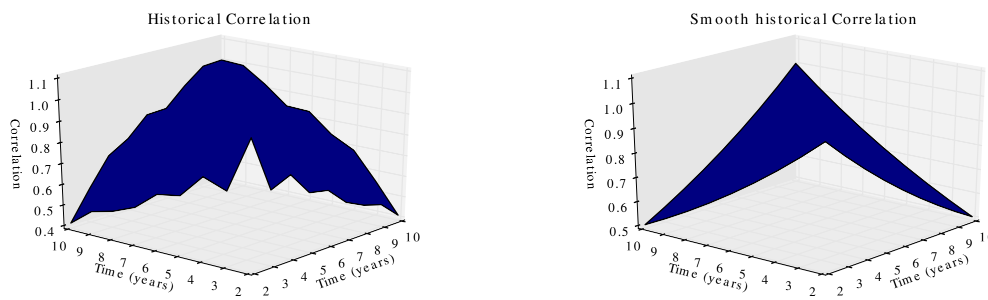

Figure 3.2: *Left:* Historical correlation *Right:* Smooth historical correlation.

3.6.6 Simulation of the LMM

With a set of forward rates $L_1(t), \ldots, L_n(t)$ the aim is to generate future paths of each forward rate for the later use of Monte Carlo pricing. Using an Euler scheme on $\log L_k(t)$ as done in Glasserman (2003), the forward rate $L_k(t + \Delta t)$ under the spot measure, see equation 3.6, can be simulated as

$$\log L_k(t + \Delta t) = \log L_k(t) + \sigma_k(t)\Delta t \sum_{j=\eta(t)}^{k} \frac{\rho_{k,j}\Delta T_j \sigma_j(t) L_j(t)}{1 + \Delta T_j L_j(t)} - \frac{\sigma_k(t)^2}{2}\Delta t + \sigma_k(t)\sqrt{\Delta t} Z_k. \tag{3.18}$$

In this discretization Δt is the time step, $\Delta T_j = T_j - T_{j-1}$ the time between maturities of forward rates, $\rho_{k,j}$ the correlation between the forward rates $L_k(t)$ and $L_j(t)$ and Z_k a standard normal random variable.

The forward rate $L_k(t)$ is simulated until $t + \Delta t = T_k$, since after T_k this forward rate is deterministic.

3.7 Credit Model

In order to be able to describe the future dynamics of probabilities of default for a company a credit model is needed. Historically firm value (or structural) models have been a popular credit model. In a firm value model the firm life is modelled as the firms ability to pay back its debt. These kinds of models are based on the work of Merton (1974). Usually the firms total value V, which is the sum of the firms equity S and its debt D, is modelled as a Geometric Brownian Motion (GBM). When the firms total value V is less or equal to its debt D, the firm has defaulted.

Another more convenient type of credit model is an intensity (or reduced form) model. These models describe the default time τ as the first jump time of an inhomogeneous Poisson process, with intensity $\lambda(t)$. Over an infinitesimal time period dt the probability of default at time τ is given by

$$\mathbb{P}(t < \tau \leq t + dt | \tau > t) = \lambda(t)dt.$$

This kind of model is suitable to model credit spreads and can quite easily be calibrated to corporate bond or CDS data.

From a default intensity model we are looking for risk-neutral probabilities of defaulting, i.e. $\mathbb{Q}(\tau \leq t) = 1 - \mathbb{Q}(\tau > t)$ for a given time t. First defining the strictly increasing, cumulated intensity as

$$\Lambda(t) := \int_0^t \lambda(s)ds.$$

Using the fact that for an inhomogeneous Poisson, the transformation of the time until a jump τ according to its own cumulated intensity $\Lambda(t)$ leads to an exponential random variable, hence

$$\Lambda(\tau) = \xi \sim \text{Exp}(1).$$

So by inverting this equation an expression for the default time is given by

$$\tau = \Lambda^{-1}(\xi).$$

Using the cumulative distribution function of a exponential distribution and that $\Lambda(t)$ is a strictly increasing function, an expression for the risk-neutral survival probabilities is given by

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\Lambda(\tau) > \Lambda(t)) = \mathbb{Q}(\xi > \Lambda(t)) = \exp\{-\int_0^t \lambda(s)ds\}.$$

Which is the same expression as for a discount factor with a continuous compounded spot rate. If the default intensity $\lambda(t)$ is stochastic we get

$$\mathbb{Q}(\tau > t) = \mathbb{E}^{\mathbb{Q}}\big[\exp\{-\int_0^t \lambda(s)ds\}\big|\mathcal{G}_t\big]. \tag{3.19}$$

Hence the same expression as the price of a zero-coupon bond. This is very convenient since an interest rate model can be used to model the default intensity $\lambda(t)$ and the default probabilities can then be easily calculated using analytical formulas (Brigo et al., 2013).

3.7.1 CIR++ Model

The Cox-Ingersoll-Ross (CIR) model was introduced in 1985 as an extension of the Vasicek model and is often used as a model for the short rate $r(t)$. The CIR++ model is an extension of the CIR model and under the risk neutral measure $\mathbb{Q}$ the CIR++ process follows

$$\begin{aligned} dx(t) &= \kappa[\theta - x(t)]dt + \sigma_x\sqrt{x(t)}dW^x(t),\ x(0) = x_0 \\ \lambda(t) &= x(t) + \varphi(t) \end{aligned} \tag{3.20}$$

where the parameters κ, θ and σ correspond to speed of adjustment, mean and volatility. $W^x(t)$ is a $\mathbb{Q}$-Brownian Motion and $\varphi(t)$ is a deterministic function used in order to fit the initial term structure. The model has a mean reversion property towards θ and one of the biggest advantages of the CIR++ model is that it implies a positive $\lambda(t)$ as long as the condition $2\kappa\theta \geq \sigma^2$ and $\varphi(t) \geq 0$ hold. Because of this feature the CIR model is suitable for modelling default intensities and also for stochastic volatilities.

The distribution of future values of a CIR process follows a non-central chi-squared distribution with $d = 4\kappa\theta/\sigma^2$ degrees of freedom and a non-centrality parameter $\eta(t,T) = x(t)4\kappa e^{-\kappa(T-t)}/\sigma^2(1-e^{-\kappa(T-t)})$. Thus $x(t)$ has the following cumulative distribution function

$$F_x(z; d, \eta) = e^{-\eta/2}\sum_{i=0}^{\infty}\frac{(\eta/2)^i}{i!2^{d/2+i}\Gamma(d/2+i)}\int_0^z y^{d/2+i-1}e^{-z/2}dy \tag{3.21}$$

(Brigo and Mercurio, 2012)

3.7.2 Probability of Default

From a standard CIR process

$$dx(t) = \kappa[\theta - x(t)]dt + \sigma_x\sqrt{x(t)}dW^x(t),\ x(0) = x_0$$

the zero-coupon bond price is given by

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\big[\exp\{-\int_t^T x(s)ds\}\big|\mathcal{G}_t\big].$$

This can be determined by computing the moment generating function of the integrated CIR process such that

$$\mathbb{E}^{\mathbb{Q}}\big[\exp\{-u\int_t^T x(s)ds\}\big|\mathcal{G}_t\big] = A(t,T,u)e^{-B(t,T,u)x(t)}$$

where

$$\begin{aligned}
A(t,T,u) &= \left[\frac{2h_u \exp\{(\kappa+h_u)(T-t)/2\}}{2h_u + (\kappa+h_u)(\exp\{h_u(T-t)\}-1)}\right]^{2\kappa\theta/\sigma^2} \\
B(t,T,u) &= \frac{2(\exp\{h_u(T-t)\}-1)}{2h_u + (k+h_u)(\exp\{h_u(T-t)\}-1)} \\
h_u &= \sqrt{\kappa^2 + 2u\sigma^2}.
\end{aligned}$$

By setting $u = 1$ and denoting $A(t,T,1) = A(t,T)$ and $B(t,T,1) = B(t,T)$, the zero-coupon bond price can be written

$$P_{\mathrm{CIR}}(t,T) = A(t,T)e^{-B(t,T)x(t)}. \tag{3.22}$$

Using this and the equation 3.19, the risk-neutral survival function of the default time τ at time 0 is given by

$$\begin{aligned}
\mathbb{Q}(\tau > t) &= \mathbb{E}^{\mathbb{Q}}\big[\exp\{-\int_0^t \lambda(s)ds\}\big|\mathcal{G}_0\big] = \mathbb{E}^{\mathbb{Q}}\big[\exp\{-\int_0^t x(s)ds - \int_0^t \varphi(s)ds\}\big|\mathcal{G}_0\big] \\
&= \exp\{-\int_0^t \varphi(s)ds\}A(0,t)e^{-B(0,t)x(0)}. \qquad (3.23)
\end{aligned}$$

(Brigo et al., 2013)

3.7.3 Calibration of the CIR++ Model

Before using an intensity model as the CIR++ model it needs to be calibrated such that it reproduces the probabilities of default, implied by the market CDS term structure, and such that it has the accurate risk-neutral dynamics under the $\mathbb{Q}$-measure.

Calibration to the CDS Term Structure

As stated above the aim is for the model to imply the same default probabilities as the market. From equation 3.23, the survival probability in the CIR++ intensity model is given by

$$\mathbb{Q}(\tau > t)_{\mathrm{model}} = \exp\{-\int_0^t \varphi(s)ds\}A(0,t)e^{-B(0,t)x(0)}.$$

Let the survival probability implied by the market be given by

$$\mathbb{Q}(\tau > t)_{\mathrm{market}} = e^{-\Gamma(t)} = e^{-\int_0^t \gamma(s)ds}.$$

Where $\gamma(t)$ is the implied hazard rates from the market. In order for the model to imply the market default probability the following must hold

$$\mathbb{Q}(\tau > t)_{\text{model}} = \mathbb{Q}(\tau > t)_{\text{market}}.$$

Or equivalent

$$\exp\{-\int_0^t \varphi(s)ds\}A(0,t)e^{-B(0,t)x(0)} = e^{-\int_0^t \gamma(s)ds}.$$

Solving for $\varphi(t)$ yields

$$\int_0^t \varphi(s)ds = \log\left(\frac{A(0,t)e^{-B(0,t)x(0)}}{e^{-\int_0^t \gamma(s)ds}}\right)$$

or equivalent

$$\int_0^t \varphi(s)ds = \log\left(\frac{P_{\text{CIR}}(0,t)}{\mathbb{Q}(\tau > t)_{\text{market}}}\right). \tag{3.24}$$

Thus using the implied market hazard rates $\gamma(t)$ or the implied survival probabilities, the model in equation 3.20 can be fitted to the initial term structure (Brigo and Alfonsi, 2005). In the sections beneath follows a method on how the market implied survival probabilities can be obtained.

Credit Default Swaps

The most popular way to determine implied risk-neutral probabilities of default is by stripping them down from credit default swap (CDS) prices. A single name CDS is a financial contract that protects the buyer from the risk that a specific party will default. Similar to an interest rate swap, a CDS consist of two legs, one premium leg and one protection leg. Also similar to an interest rate swap, the interest rate in the premium leg is often determined such that the initial value of the contract is equal to zero.

Consider a company A that buys protection, hence a CDS, from another company B. The event that A buys protection on is the event of default of a third company C. The contract offers protection on a predetermined time interval $[T_0, T_n]$ and on a predetermined notional K. Thus if C would default at time $\tau_C \in [T_0, T_n]$, then B would pay the amount $(1 - R)K$ to A at time τ_C, where R denotes the recovery rate. In return for getting this protection A has to pay coupons with a rate of $s(T_n)$ on the amount $(1-r)K$ to B at a set of predetermined times $[T_1, \ldots, T_n]$, let's denote this set of times by $\{T_i\}$ and $\Delta T_i = T_i - T_{i-1}$. Following the derivation in Brigo and Alfonsi (2005), the arbitrage free price of a CDS at time $t \in [T_{\beta(t)-1}, T_{\beta(t)}]$ with the notional $K = 1$ is

$$\begin{aligned}
\mathrm{CDS}(t, \{T_i\}, T_n, s(T_n), (1-R)) =& \mathbb{1}_{\tau>t}\Big[\mathbb{E}^{\mathbb{Q}}\big[s(T_n)D(t,\tau)(\tau - T_{\beta(\tau)-1})\mathbb{1}_{\tau<T_n}\big|\mathcal{G}_t\big] \\
&+ \mathbb{E}^{\mathbb{Q}}\big[\sum_{i=\beta(t)}^{n} s(T_n)D(t,T_i)\Delta T_i \mathbb{1}_{\tau>T_i}\big|\mathcal{G}_t\big] \\
&- \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\tau<T}D(t,\tau)(1-R)\big|\mathcal{G}_t\big]\Big].
\end{aligned} \tag{3.25}$$

Furthermore assuming independence between the interest rate and the default time and assuming $\tau > t$, i.e. dropping $\mathbb{1}_{\tau>t}$, the price of this CDS is equal to

$$\begin{aligned}
\mathrm{CDS}(t, \{T_i\}, T_n, s(T_n), (1-R)) =& s(T_n)\int_t^{T_n} P(t,u)(u - T_{\beta(u)-1})d\mathbb{Q}(\tau \le u) \\
&+ s(T_n)\sum_{i=\beta(t)}^{n} P(t,T_i)\Delta T_i \mathbb{Q}(\tau > T_i) \\
&- (1-R)\int_t^{T} P(t,u)d\mathbb{Q}(\tau \le u).
\end{aligned} \tag{3.26}$$

By letting the risk-neutral survival function be given by $\mathbb{Q}(\tau > t) = e^{-\Gamma(t)}$ and $\mathbb{Q}(s < \tau \le t) = e^{-\Gamma(s)} - e^{-\Gamma(s)}$ the CDS price in equation 3.26 can be written as

$$\begin{aligned}
\mathrm{CDS}(t, \{T_i\}, T_n, s(T_n), (1-R); \Gamma(.)) =& s(T_n)\int_t^{T_n} P(t,u)(T_{\beta(u)-1} - u)d_u(e^{-(\Gamma(u)-\Gamma(t))}) \\
&+ s(T_n)\sum_{i=\beta(t)}^{n} P(t,T_i)\Delta T_i e^{\Gamma(t)-\Gamma(T_i)} \\
&+ (1-R)\int_t^{T} P(t,u)d_u(e^{-(\Gamma(u)-\Gamma(t))}).
\end{aligned} \tag{3.27}$$

The J.P. Morgan Model

The J.P. Morgan Model is a discrete CDS valuation model which also assumes independence between interest rates and default probabilities. This model can be used to estimate a hazard rate curve from CDS spreads using a piecewise linear assumption as described by Castellacci (2008).

The CDS price at time $t = 0$ is here written as

$$\mathrm{CDS}(t=0, \{T_i\}, T_n, s(T_n), (1-R); \Gamma(.)) = PV_{\text{Premium}}(T_n) - PV_{\text{Protection}}(T_n) \tag{3.28}$$

The present value of the premium leg can be approximated as

$$PV_{\text{Premium}}(T_n) = s(T_n)\sum_{i=1}^{n}\Delta T_i P(0,T_i)\mathbb{Q}(T_i < \tau) + \frac{s(T_n)}{2}\sum_{i=1}^{n}\Delta T_i P(0,T_i)\mathbb{Q}(T_{i-1} < \tau \leq T_i) \tag{3.29}$$

$$= s(T_n)\sum_{i=1}^{n}\Delta T_i P(0,T_i)e^{-\Gamma(T_i)} + \frac{s(T)}{2}\sum_{i=1}^{n}\Delta T_i P(0,T_i)\big(e^{-\Gamma(T_{i-1})} - e^{-\Gamma(T_i)}\big) \tag{3.30}$$

$$= s(T_n)\sum_{i=1}^{n}\Delta T_i P(0,T_i)\frac{e^{-\Gamma(T_{i-1})} + e^{-\Gamma(T_i)}}{2}, \tag{3.31}$$

where again $\mathbb{Q}(\tau > t) = e^{-\Gamma(t)} = e^{-\int_0^t \gamma(s)ds}$ is the risk-neutral survival function. The present value of the protection leg can be approximated as

$$PV_{\text{Protection}}(T_n) = (1-R)\sum_{i=1}^{n} P(0,T_i)\mathbb{Q}(T_{i-1} < \tau \leq T_i) \tag{3.32}$$

$$= (1-R)\sum_{i=1}^{n} P(0,T_i)\big(e^{-\Gamma(T_{i-1})} - e^{-\Gamma(T_i)}\big). \tag{3.33}$$

The key assumption in bootstrapping the CDS term structure is assuming piecewise linear hazard rates $\gamma(t)$. By partitioning the time $0 = T_0 < \cdots < T_n$, the intensity rate

$$\gamma(t) = \gamma_i,\ \forall t \in (T_{i-1}, T_i].$$

From this assumption the survival function can be written

$$\mathbb{Q}(\tau > t)_{\text{market}} = S(t) = \exp\Big\{-\sum_{i=1}^{\eta(t)}\gamma_i \Delta T_i + \gamma_{\eta(t)+1}\big(t - T_\eta(t)\big)\Big\},$$

where $\eta(t) := \max\{i \leq n : T_i \leq t\}$ and $\Delta T_i = T_i - T_{i-1}$.

The implied hazard rates are then found by observing the CDS spreads $s(T_i)$ at time $t = 0$, i.e. today, for a set of maturities $T_1, \ldots, T_i, \ldots, T_n$ and then solving for $\Gamma(.)$ such that the CDS price in equation 3.28 satisfies $\text{CDS}(0, \{T_i\}, T_i, s(T_i), (1-R)K; \Gamma(.)) = 0$. This can be solved using a Newton-Raphson method.

Calibration to Market Prices

Calibrating the CIR++ model to have the correct dynamics under the risk-neutral measure $\mathbb{Q}$, require some market price data of credit derivatives which are priced with a credit model, e.g. options on CDS contracts. However, derivatives like CDS single name options are very rare and illiquid, i.e. there is a very small possibility of finding a liquid set of CDS options prices for different maturities of the options and the underlying single name CDS contracts. This makes it very difficult to calibrate an credit intensity model for a certain firm.

A more likely approach would be to use a set of CDS index options and calibrate the model for each party to the same set of prices.

An alternative approach, as suggested by Brigo et al. (2013), is to calibrate the model to a set of hypothetical (i.e. made up) CDS single name option prices.

3.7.4 Simulation of the CIR Model

Producing sample paths of a square-root process is a more complicating issue. The intuitive way would be using a Euler-Maruyama scheme, but then one runs into a problem since the discretized version of the CIR model won't guarantee a positive $x(t)$ and thus the value of $\sqrt{x(t)}$ becomes imaginary.

Exact Simulation

However since the future distribution of the CIR process is known, see equation 3.21, it can be simulated from this distribution. Using that a non-central chi-squared distribution can be seen as a normal chi-squared distribution with a Poisson-distributed degrees of freedom. The following algorithm from Andersen et al. (2010) can be used to generate sample paths from the CIR model.

1. Draw a Poisson random variable N, with $\mathbb{E}[N] = \frac{1}{2}x(t)\eta(t, t + \Delta t)$.
2. Given N, draw a regular chi-square random variable χ^2_ν with $\nu = d + 2N$ degrees of freedom.
3. Set $x(t + \Delta t) = \chi^2_\nu \frac{e^{-a\Delta t}}{\eta(t,t+\Delta t)}$

This is though a very time-consuming simulation technique and not suitable for a pricing framework where one would want to simulate a large number of samples.

Modified Euler Scheme

A fast way to generate samples of the CIR process is by using a modified version of the Euler-Maruyama scheme. This modified version assures positive values of $x(t)$ simply by adding a $x(t)^+ = \max(x(t), 0)$. The process can then be simulated using the following discretized equation

$$x(t + \Delta t) = x(t) + \kappa(\theta - x(t)^+)\Delta t + \sigma\sqrt{x(t)^+}\sqrt{\Delta t}Z.$$

Where Z is a standard normal random variable.

Quadratic-Exponential Scheme

Another scheme described in Andersen et al. (2010) is the Quadratic-Exponential (QE) scheme. The QE scheme is a biased, moment matching scheme. This scheme is more advanced than the modifier Euler or the Milstein scheme but it guarantees positive values for the CIR process for a larger set of the input parameters (κ, θ, σ).

The idea behind the QE scheme is that the density function of $x(t+\Delta t)$ conditioning on $x(t)$ can be well approximated by simple distributions. If $x(t)$ is large let

$$x(t+\Delta t) = a(b+Z)^2, \tag{3.34}$$

where a and b are constants determined by moment matching and Z a standard normal variable. If $x(t)$ is small let

$$x(t+\Delta t) = \begin{cases} 0 & \text{if } U \leq p \\ \\ \frac{1}{\beta}\log\left(\frac{1-p}{1-U}\right) & \text{if } U > p. \end{cases} \tag{3.35}$$

where p and β are constants and U a standard uniformly distributed variable.

From moment matching the conditional mean is

$$m = \mathbb{E}[x(t+\Delta t)|x(t) = \hat{x}(t)] = \theta + (x(t) - \theta)e^{-\kappa\Delta t}$$

and the conditional variance is

$$\begin{aligned} s^2 &= \text{var}\big(x(t+\Delta t)|x(t) = \hat{x}(t)\big) \\ &= \frac{x(t)\sigma^2 e^{-\kappa\Delta t}}{\kappa}(1 - e^{-\kappa\Delta t}) + \frac{\theta\sigma^2}{2\kappa}(1 - e^{-\kappa\Delta t})^2. \end{aligned}$$

Let $\phi = s^2/m^2$ and the constants (a, b, p, β) above be determined such that

$$a = \frac{m}{1+b^2}, \quad b^2 = \frac{2}{\phi} - 1 + \sqrt{\frac{2}{\phi}(\frac{2}{\phi} - 1)}$$

and

$$p = \frac{\phi - 1}{\phi + 1}, \quad \beta = \frac{2}{m(\phi+1)}$$

If $\phi \leq \phi_c$ simulation is done using equation 3.34 and else simulation is done using equation 3.35. According to Andersen et al. (2010) the choice of ϕ_c is non-critical and they suggest $\phi_c = 1.5$.

In order to compare the different schemes the future simulated distributions of $x(t)$ one year from now are compared. In Figure 3.3 both simulations from the modified Euler scheme and from the QE scheme are compared to the exact simulation.

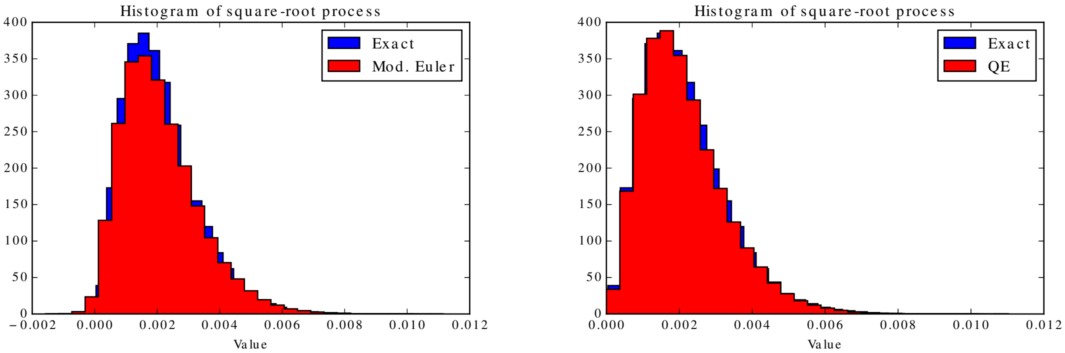

Figure 3.3: Histogram of simulated values of a square-root process. *Left:* Modified Euler scheme. *Right:* QE scheme.

In Table 3.1 some statistics comparing the schemes are presented. By comparing the histograms and the first two moments of the schemes the QE scheme is to prefer.

Scheme	$\hat{\mathbb{E}}[x(t)]$	$\hat{\text{var}}(x(t))$
Exact	0.0025	1.19e-05
Mod. Euler	0.0024	1.42e-05
QE	0.0025	1.18e-05

Table 3.1: Statistics of the three different schemes.

Chapter 4

CVA Frameworks

In this chapter several different CVA frameworks are presented. Both cases where the investor is considered risk-free (unilateral CVA) and the more realistic case where both parties are considered risky (BCVA) are concerned. Note that for all methods the recovery rate R is taken to be deterministic. For notations and definitions see Chapter 2 and Chapter 3.

4.1 Expected Future Exposure Approach

The expected future exposure CVA approach is probably the most used and known framework. The key features of this framework is that it uses simulation of market risk factors to capture the future exposure of a portfolio and that it assumes independence between the exposure and probability of default. This means that the method ignores potential WWR/RWR.

By partitioning the time, CVA can be approximated as a sum of discounted future exposures weighted by the risk-neutral probabilities of default in each time bucket. The future exposure is simulated by valuing a portfolio at a future time, based on simulated market risk factors. Using a Monte Carlo approach, based on N_s simulations, the discounted expected positive exposure at time t_i

$$\mathbb{E}^{\mathbb{Q}}\big[D(0,t_i)(V(t_i,T))^+\big|\mathcal{F}_0\big]$$

is approximated by

$$EPE(t_i) = \frac{1}{N_s}\sum_{j=1}^{N_s} D_j(0,t_i)(V_j(t_i,T))^+,\ V(0,T) = v_0 \tag{4.1}$$

and the discounted expected negative exposure by

$$ENE(t_i) = \frac{1}{N_s}\sum_{j=1}^{N_s} D_j(0,t_i)(-V_j(t_i,T))^+,\ V(0,T) = v_0. \tag{4.2}$$

Here j denotes the simulation index. From the law of large numbers this will converge to the expected value when N_s is large.

4.1.1 UCVA

Starting with the case where the investor is considered to be risk-free. The setup of the probability space and filtrations are the same as in Section 2.1.1. As shown in Section 2.1.1, by assuming independence between the exposure and the probability of default and by partitioning the time $(0,T]$ such that t_i for $i = 0,\dots,N$ where $0 = t_0 < t_1 \cdots < t_N = T$ we have

$$\text{CVA} = (1-R)\sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[D(0,t_i)(V(t_i,T))^+ \big| \mathcal{F}_0\big]\big(\mathbb{Q}(\tau \le t_i) - \mathbb{Q}(\tau \le t_{i-1})\big),$$

where $V(t_i,T)$ is the future portfolio value at time t_i and $\mathbb{Q}(\tau \le t_i)$ is the risk-neutral cumulative distribution function of the default time τ of the counterparty. By using the Monte Carlo approximation in equation 4.1 CVA can be approximated by

$$\text{CVA} \approx (1-R)\sum_{i=1}^{N} EPE(\bar{t}_i)\big(\mathbb{Q}(\tau \le t_i) - \mathbb{Q}(\tau \le t_{i-1})\big)$$

where $\bar{t}_i = (t_{i-1}+t_i)/2$ is the midpoint. A midpoint valuation is chosen in order to get a better convergence of the discretized integral.

4.1.2 BCVA

In Section 2.3 we derived an expression for bilateral CVA from an investor I's point of view with a risky counterparty C as

$$\begin{aligned}\text{BCVA} =& \mathbb{E}^{\mathbb{Q}}\big[(1-R_C)\mathbb{1}_{\{\tau_C \le T, \tau_C < \tau_I\}} D(t,\tau_C)(V(\tau_C,T))^+ \big| \mathcal{G}_t\big] \\ &- \mathbb{E}^{\mathbb{Q}}\big[(1-R_I)\mathbb{1}_{\{\tau_I \le T, \tau_I < \tau_C\}} D(t,\tau_I)(-V(\tau_I,T))^+ \big| \mathcal{G}_t\big].\end{aligned}$$

Again by assuming independence between the exposure and the default times we end up with the discretized equation

$$\begin{aligned}\text{BCVA} =& (1-R_C)\sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[D(0,t_i)(V(t_i,T))^+ \big| \mathcal{F}_0\big]\big(\mathbb{Q}(t_{i-1} < \tau_C \le t_i, \tau_I > t_i)\big) \\ &- (1-R_I)\sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[D(0,t_i)(-V(t_i,T))^+ \big| \mathcal{F}_0\big]\big(\mathbb{Q}(t_{i-1} < \tau_I \le t_i, \tau_C > t_i)\big)\end{aligned}$$

By assuming independence between the default time of the investor τ_I and the default time of the counterparty τ_C, a bivariate distribution can be constructed such that the cumulative distribution function is

$$\mathbb{Q}(\tau_I \le t_I, \tau_C \le t_C) = \mathbb{Q}(\tau_I \le t_I)\mathbb{Q}(\tau_C \le t_C)$$

Using this and the expected positive exposure (EPE) and the negative expected exposure (ENE) calculated via the Monte Carlo approximations in equations 4.1 and 4.2 the bilateral CVA can be approximated as

$$\begin{aligned}\text{BCVA} \approx &(1 - R_C) \sum_{i=1}^{N} EPE(\bar{t}_i)\mathbb{Q}(\tau_I > t_i)\big(\mathbb{Q}(\tau_C \le t_i) - \mathbb{Q}(\tau_C \le t_{i-1})\big) \\ &- (1 - R_I) \sum_{i=1}^{N} ENE(\bar{t}_i)\mathbb{Q}(\tau_C > t_i)\big(\mathbb{Q}(\tau_I \le t_i) - \mathbb{Q}(\tau_I \le t_{i-1})\big)\end{aligned}$$

where once again $\bar{t}_i = (t_{i-1} + t_i)/2$ is the midpoint time.

4.1.3 Model

In order to be able to simulate future exposure of interest rate swaps we need to model the underlying risk factors, i.e. the interest rates.

Interest Rates

Starting with a model for the interest rate model we will use the LIBOR Market Model presented in Section 3.6.1. In the LMM the forward rate $L_i(t)$ is a log normal model under its own forward measure $\mathbb{Q}^{T_i}$. Furthermore the LMM has the following dynamics under the so called spot measure $\mathbb{Q}^*$,

$$\frac{dL_i(t)}{L_i(t)} = \Big(\sum_{j=\eta(t)}^{i} \frac{\Delta T_j L_j(t)\sigma_i(t)\sigma_j(t)\rho_{i,j}}{1 + \Delta T_j L_j(t)}\Big)dt + \sigma_i(t)dW_i^{\mathbb{Q}^*}(t), \quad i = 1, \dots, N.$$

In the LMM with this setting a T-claim X is priced using the following risk-neutral formula

$$\Pi(t; X) = B^*(t)\mathbb{E}^{\mathbb{Q}^*}[\frac{1}{B^*(T)}X(T)\big|\mathcal{F}_t]$$

where

$$B^*(t) = P(t, T_{\eta(t)-1}) \prod_{i=0}^{\eta(t)-1} \big(1 + \Delta T_i L_i(T_i)\big)$$

is the discrete bank account acting as numeraire of the spot measure and $\eta(t) = m$ if $T_{m-2} < t \le T_{m-1}$.

4.2 Default Model Approach

The default model CVA framework is a more sophisticated framework which includes both credit and market risk factor models. This framework lets us model dependence between exposure and probability of default, which captures potential WWR/RWR. This is thus a more complex framework than the one in Section 4.1 and also a more computer intensive framework, since it demands more simulation.

The methodology is to simulate default times τ^j, for each scenario j and value the portfolio in each scenario at each time τ^j. Again using a Monte Carlo approach CVA is approximated by the average portfolio value of these scenarios.

4.2.1 UCVA

Again the setup of the probability space and filtrations are the same as in Section 2.1.1. With a risk-free investor and a risky counterparty C, we have just as in Equation 2.1 the expression for CVA as

$$\text{CVA} = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{\tau \leq T\}}(1-R)D(0,\tau)(V(\tau,T))^{+}\big|\mathcal{G}_0\big]. \tag{4.3}$$

and with a partition of time time $0 = t_0 < \cdots < t_i < \cdots < t_N = T$,

$$\text{CVA} = \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{t_{i-1} < \tau \leq t_i\}}(1-R)D(0,t_i)(V(t_i,T))^{+}\big|\mathcal{G}_0\big].$$

Let j denote the simulation index,

$$\text{CVA} \approx \frac{1}{N_s}\sum_{j=1}^{N_s}\Big(\sum_{i=1}^{N}\mathbb{1}_{\{t_{i-1}<\tau^j \leq t_i\}}(1-R)D_j(0,\bar{t}_i)(V_j(\bar{t}_i,T))^{+}\Big).$$

Where the portfolio is valued at a midpoint time $\bar{t}_i = (t_{i-1}+t_i)/2$.

4.2.2 BCVA

Consider an investor I and a counterparty C, where both are non-default-free parties.

$$\text{BCVA} = \text{CVA} - \text{DVA}.$$

using a partition of the time such that $0 = t_0 < \cdots < t_i < t_N = T$ and taking into account which party that defaults first we get

$$\text{CVA} = \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{t_{i-1}<\tau_I \leq t_i, \tau_I < \tau_C\}}(1-R_I)D(0,t_i)(V(t_i,T))^{+}\big|\mathcal{G}_0\big]$$

and

$$\text{DVA} = \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{t_{i-1}<\tau_C\leq t_i, \tau_C<\tau_I\}}(1-R_C)D(0,t_i)(-V(t_i,T))^{+}\big|\mathcal{G}_0\big].$$

Using a Monte Carlo approach based on N_s number of simulations, CVA and DVA is approximated by

$$\text{CVA} \approx \frac{1}{N_s}\sum_{j=1}^{N_s}\Big(\sum_{i=1}^{N}\mathbb{1}_{\{t_{i-1}<\tau_I^j\leq t_i, \tau_I^j<\tau_C^j\}}(1-R_C)D_j(0,\bar{t}_i)(V_j(\bar{t}_i,T))^{+}\Big).$$

and

$$\text{DVA} \approx \frac{1}{N_s}\sum_{j=1}^{N_s}\Big(\sum_{i=1}^{N}\mathbb{1}_{\{t_{i-1}<\tau_C^j\leq t_i, \tau_C^j<\tau_I^j\}}(1-R_I)D_j(0,\bar{t}_i)(-V_j(\bar{t}_i,T))^{+}\Big).$$

Again j denotes the simulation index and the portfolio is valued at a midpoint time $\bar{t}_i = (t_{i-1}+t_i)/2$.

4.2.3 Model

In order to be able to simulate default times and future exposure of interest rate swaps some models are needed.

Interest Rates

As in the expected future exposure approach the interest rates will be modelled using the LIBOR Market Model, see Sections 3.6.1 and 4.1.3.

Credit

In order to simulate default times a credit model is needed. We will here use a default intensity model, where the default intensity is modelled by a CIR process and a deterministic function. This model is presented in Section 3.7.1. For both the investor I and the counterparty C the following model will be used

$$\begin{aligned} dx^k(t) &= \kappa^k[\theta^k - x^k(t)]dt + \sigma_k\sqrt{x^k(t)}dW^{x^k}(t), \\ \lambda_k(t) &= x^k(t) + \varphi^k(t), \quad k \in (I,C). \end{aligned} \tag{4.4}$$

Set $\Lambda_k(T) = \int_0^T \lambda_k(s)ds$, which is a strictly increasing function. Default times τ_k for each party can then be simulated by the inverse mapping

$$\tau_k = (\Lambda_k)^{-1}(\xi_k), \quad k \in (I,C) \tag{4.5}$$

where ξ_k is a standard exponential random variable.

Dependence

Dependence between interest rate and credit models are modelled with a linear correlation between the Brownian Motions such that

$$dW_i(t)dW^{x^k}(t) = \rho_k dt \ \ \forall i, \ \ k \in (I, C). \tag{4.6}$$

Dependence between default times of each counterparty k can be modelled either with a copula method between ξ_k with correlation or with linear correlation in credit models $dW_t^{x^I} dW_t^{x^C} = \rho_{x^I,x^C} dt$. However according to Brigo et al. (2010), default correlation between ξ_k creates a larger dependence than correlation in the intensity models does.

4.2.4 Copula Distributions

In order to model the dependence between the two exponential variable ξ_I and ξ_C two different copula distributions will be used. The copula distributions comes from two important properties of distribution. The first one is the probability transform. It says that if X is a random variable with a continuous distribution function F, then $F(X)$ is uniformly distributed on $(0, 1)$. The second property is the quantile transform. The quantile transform says that if U is uniformly distributed on $(0, 1)$ and G is any distribution function, then $G^{-1}(U)$ has a G-distribution.

The distribution function C of a random vector $\mathbf{U}$ whose components U_k are uniformly distributed on the interval $(0, 1)$ is called a copula, hence

$$C(u_1, \dots, u_d) = P(U_1 \leq u_1, \dots, U_d \leq u_d).$$

(Hult et al., 2010)

Gaussian copula

The Gaussian copula is one of the most used copula distributions. For a d-dimensional vector of random variables the Gaussian copula function is

$$C_{\mathbf{R}}^{Ga}(\mathbf{u}) = \Phi_{\mathbf{R}}^d(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)).$$

Student's t_ν copula

Again for a d-dimensional vector of random variables the student's t copula with ν degrees of freedom is

$$C_{\nu,\mathbf{R}}^t(\mathbf{u}) = t_{\nu,\mathbf{R}}^d(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d)).$$

In Figure 4.1 one can see the difference between the default times of two companies when modelling the dependence from a Gaussian copula and from a Student's t copula with correlation $\rho_{\xi_I,\xi_C} = 0.5$.

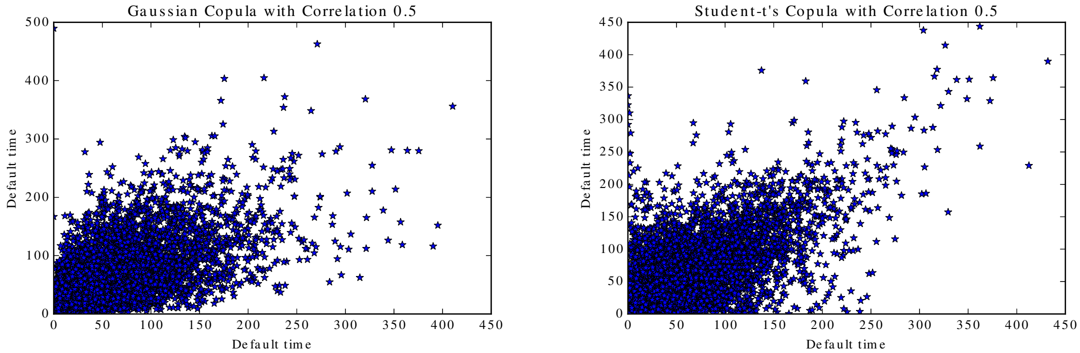

Figure 4.1: *Left:* Gaussian copula. *Right:* Student's t copula with degrees of freedom equal to 3.

4.3 Variable Exposure Approach

An easier approach to calculate CVA, described in EY (2014), is the variable exposure approach. This method requires no simulation and thus ignores both future potential exposures and possible WWR. This approach estimates CVA as the cost of buying credit default protection for the remaining future exposure. If we consider a series of future cash flows which are forecasted by the current forward prices, then at each cash flow date the sum of the remaining cash flows is calculated. CVA is then approximated by the cost of buying CDS protection for each cash flow date on a notional that equals the sum of all remaining cash flows. Each CDS protection starts at time t_i and expires at time t_{i+1}. Using the notations from EY (2014), CVA can be expressed as

$$\text{CVA} = \sum_i PV_{\text{Premium}}(CDS(t_i)).$$

This approach also takes into account bilateral derivative contracts and in case the remaining cash flow is negative one applies the investors credit spreads instead of the counterparty.

Since this method forecasts the future exposure with current forward prices, the potential future exposure, is not taken into consideration. Thus in the case of a very volatile future exposure this method would underestimate the unilateral CVA. However due to the same reason this method is easy to implement and not at all time consuming.

The methodology in this method is that CVA should be equal to the cost of hedging the counterparty credit risk. However as known from Section 2.9, in order to accurately hedge the counterparty credit risk one needs to hedge both the credit and the market risk. This approach however only hedges the credit risk for an expected cash flow and, as stated earlier, is thus underestimating the hedging cost for the unilateral case. When it comes to a bilateral derivative the hedging cost of the market risk cannot be disregarded either.

As will be seen later the expected positive and the expected negative future exposure are often not equal. So depending on the future exposure profile this method will overestimate or underestimate BCVA.

4.3.1 IRS Example

Considering CVA of a plain vanilla receiver swap on a notional K and swap rate S which is initiated at time t with payments of the fix leg occurring at the times $T_1, \dots, T_i, \dots, T_n$ and payments of the floating leg at the times $T'_1, \dots, T'_i, \dots, T'_N$.

Introducing a new time partition of all payments $\tilde{T}_1, \dots, \tilde{T}_i, \dots, \tilde{T}_{\tilde{N}}$ such that either the fix, floating or both payments occur at time $\tilde{T}_i$. At time t one should buy CDS protection with an expiry at time $\tilde{T}_1$ on all cumulative future cash flows of the swap, and at time $\tilde{T}_1$ one should buy CDS protection with an expiry at time $\tilde{T}_2$ on all cumulative future cash flows of the swap minus the cash flow at time $\tilde{T}_1$ etc.. Thus at time $\tilde{T}_{i-1}$ one should buy CDS protection with expiry at time $\tilde{T}_i$ on the notional

$$C_i(\tilde{T}_i) = \sum_{j=\beta(\tilde{T}_i)}^{n} P(\tilde{T}_i, T_j) K \Delta T_j S - \sum_{j=\eta(\tilde{T}_i)}^{N} P(\tilde{T}_i, T_j) K \Delta T'_j L_j(\tilde{T}_i)$$

where $\beta(\tilde{T}_i) = k$ if $T_{k-1} < \tilde{T}_i \leq T_k$ and $\eta(\tilde{T}_i) = k$ if $T'_{k-1} < \tilde{T}_i \leq T'_k$. CVA is then the present value of the cost of buying protection at each payment date.

Denoting the investor by I and the counterparty by C and denoting the CDS spread with expiry T of the investor by $s_I(T)$ and of the counterparty by $s_C(T)$. Furthermore at time t the expected future cumulative cash flow at time $\tilde{T}_i$ will be denoted by $C_i(t)$ such that

$$C_i(t) = \sum_{j=\beta(\tilde{T}_i)}^{n} P(t, T_j) K \Delta T_j S - \sum_{j=\eta(\tilde{T}_i)}^{N} P(t, T'_j) K \Delta T'_j L_j(t).$$

From Section 3.7.3, using equation 3.29, the premium leg of the CDS and thus the bilateral CVA is

$$\begin{aligned}
\text{BCVA} &= \sum_{i=1}^{\tilde{N}} \text{PV}_{\text{Premium}}(C_i(\tilde{T}_i)) \\
&= \sum_{i=1}^{\tilde{N}} \Delta \tilde{T}_i C_i(t) \Big(s_C(\tilde{T}_i) \mathbb{Q}(\tilde{T}_i < \tau_C) \mathbb{1}_{\{C_i(t)>0\}} + s_I(\tilde{T}_i) \mathbb{Q}(\tilde{T}_i < \tau_I) \mathbb{1}_{\{C_i(t)<0\}} \Big) \\
&+ \frac{1}{2} \sum_{i=1}^{\tilde{N}} \Delta \tilde{T}_i C_i(t) \Big(s_C(\tilde{T}_i) \mathbb{Q}(\tilde{T}_{i-1} < \tau_C \leq \tilde{T}_i) \mathbb{1}_{\{C_i(t)>0\}} \\
&+ s_I(\tilde{T}_i) \mathbb{Q}(\tilde{T}_{i-1} < \tau_I \leq \tilde{T}_i) \mathbb{1}_{\{C_i(t)<0\}} \Big).
\end{aligned} \tag{4.7}$$

Depending on a positive or a negative cumulative cash flow at each time the counterparty or the investors own credit spreads and default probabilities are used.

4.4 Discounted Cash Flow Approach

The fourth and the most simple approach considered is the discounted cash flow approach. This approach described in EY (2014) takes a future projected cash flow and compares the risk-free value of that cash flow with the risky value. The risky value of the cash flow is calculated by adjusting discount rates with an additional credit spread. Using the notations from EY (2014), CVA can be expressed as

$$\text{CVA} = FV_{\text{Risk free}} - FV_{\text{Credit adjustment}}.$$

There are some variations on this method regarding when the investor's or the counterparty's credit spread should be added to the discount factors. In this thesis we will use the counterparty's credit spread when each individual future cash flow is an asset and the investor's credit spreads when each individual future cash flow is a liability.

Using the risk-free zero-coupon bond price $P(t,T)$ and assuming independence between a default time τ and interest rates yields the default-able zero-coupon bond price $\tilde{P}(t,T)$ as

$$\begin{aligned}
\tilde{P}(t,T) &= \mathbb{E}^{\mathbb{Q}}\big[D(t,T)\mathbb{1}_{\{\tau>T\}} + RD(t,\tau)\mathbb{1}_{\{\tau\leq T\}}\big|\mathcal{G}_t\big] && (4.8)\\
&= P(t,T)\mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{\tau<T\}}\big|\mathcal{G}_t\big] + R\mathbb{E}^{\mathbb{Q}}\big[P(t,\tau)\mathbb{1}_{\{\tau\leq T\}}\big|\mathcal{G}_t\big] && (4.9)\\
&= P(t,T)\mathbb{Q}(\tau>T) + R\int_t^T P(t,u)d\mathbb{Q}(\tau\leq u). && (4.10)
\end{aligned}$$

So the CVA is approximated by the difference of the cashflow when discounted using risk-free zero-coupon bonds and the default-able version in equation 4.8.

Neither this method consider the potential future exposure or possible WWR, which of course is a disadvantage. However, since simulation is not needed, it is an efficient method which is easy to implement. Furthermore, the only data needed, with the exception of own portfolio data, are credit spreads of the counterparty and the own firm.

4.4.1 IRS Example

Considering a plain vanilla receiver swap on a notional K and swap rate S which is initiated at time t with payments of the fix leg occurring at the times $T_1,\dots,T_i,\dots,T_n$ and payments of the floating leg at the times $T'_1,\dots,T'_i,\dots,T'_N$.

Introducing a new time partition of all payments $\tilde{T}_1,\dots,\tilde{T}_i,\dots,\tilde{T}_{\tilde{N}}$ such that either the fix, floating or both payments occur at time $\tilde{T}_i$.

Using the LIBOR forward rates $L_i(t)$ with and the IRS theory in Section 3.3, the future cash flow of the fixed leg at time T_i is $K\Delta T_i S$ and the future cash flow of the floating leg at time T_i' is $K\Delta T_i' L_i(T_{i-1})$. The projected cashflow of the receiver swap at time $\tilde{T}_i$ can be divided up in three different cases such that

$$c_i = \begin{cases} K\Delta T_j S & \text{if } \tilde{T}_i = T_j \\ \\ -K\Delta T_k' L_k(t) & \text{if } \tilde{T}_i = T_k' \\ \\ K\Delta T_j S - K\Delta T_k' L_k(t) & \text{if } \tilde{T}_i = T_j \text{ and } \tilde{T}_i = T_k'. \end{cases}$$

Using the default-able zero-coupon bond price in equation 4.8, the bilateral CVA calculation becomes

$$\text{BCVA} = \sum_{i=1}^{\tilde{N}} P(t, \tilde{T}_i) c_i - \sum_{i=1}^{N} c_i \Big(\mathbb{1}_{\{c_i > 0\}} \tilde{P}_C(t, \tilde{T}_i) + \mathbb{1}_{\{c_i < 0\}} \tilde{P}_I(t, \tilde{T}_i) \Big) \tag{4.11}$$

where discount factors are adjusted according to counterparty or own survival probabilities depending on a positive or negative cash flow.

Chapter 5

Results

In this chapter the results of the different CVA frameworks in Chapter 4 are presented. Unilateral and bilateral CVA are calculated of a plain vanilla interest rate swap. Furthermore the impact of different dependencies in the default model approach are investigated.

5.1 Model Settings

In order to do a proper calculation with real market data we will take Nokia as the investor I and Handelsbanken as the counterparty C. The CDS spreads of contracts written on Handelsbanken and Nokia can be seen in the left plot in Figure 5.1. The CDS spreads have a quarterly frequency with an actual/360 day count convention. Furthermore the spreads are interpolated using a cubic spline interpolation.

Following the bootstrapping procedure in Section 3.7.3, the risk neutral default probabilities are received as shown in the right plot of Figure 5.1. It can be seen that the market considers Nokia (the investor) as more risky, i.e. more likely to default, than Handelsbanken (the counterparty).

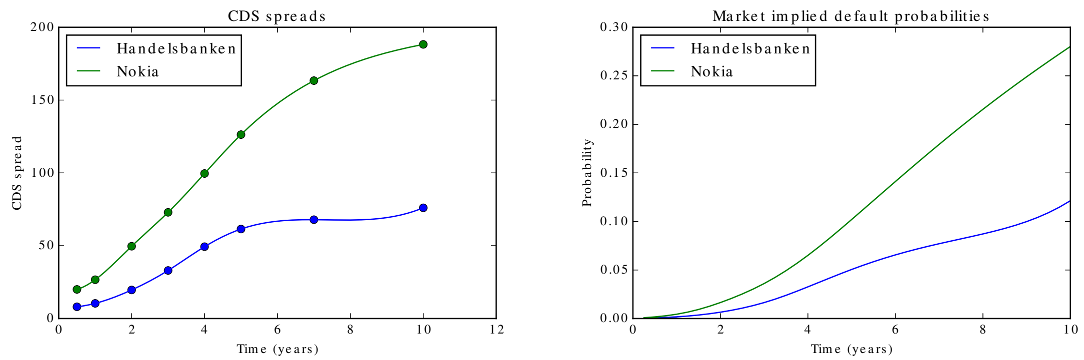

Figure 5.1: *Left:* CDS spreads of Handelsbanken and Nokia in basis points from Bloomberg 6/5/2015. *Right:* Market implied default probabilites from the CDS spreads of Handelsbanken and Nokia.

The EUR yield curve observed at the 6th of May 2015 can be seen below in Figure 5.2. The yields are bootstrapped from EURIBOR, FRA and swap rates with maturities from 6 months up to 10 years. The yields are interpolated using a cubic spline interpolation. It can be seen that the yield curve is an upward sloping curve.

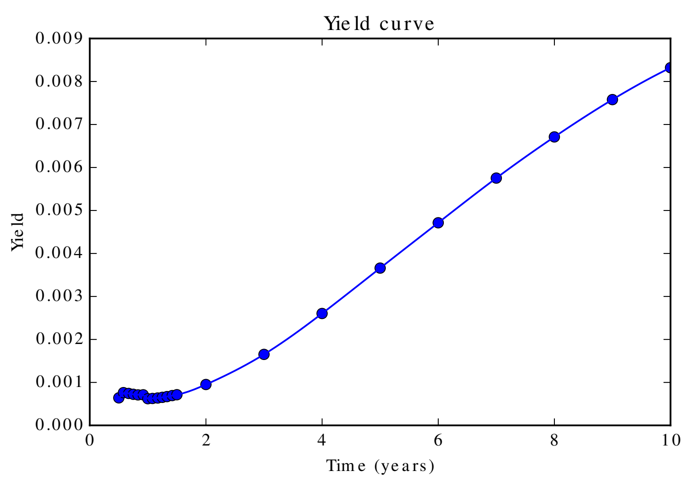

Figure 5.2: EUR yield curve with annual compounding from Bloomberg 6/5/2015.

Recovery Rate

In all results presented below the recovery rate is taken to be 40%, i.e. $R = 0.4$. This may seem to be taken out of the blue, but it is a quite standard assumption in credit risk modelling and which also have some support from historical default data. Looking at the average recovery rate across all debt seniorities, it can be seen that the average recovery rate have been around 30-50% (Gregory, 2012).

LIBOR Market Model

For the EFE approach and the default model approach an interest rate model is needed in order to simulate the future exposure of an IRS. Since we will be looking at a swap with a 6-month floating rate, these forward rates will be modelled. Note that in this thesis we will use a single-curve valuation, i.e. we will not discount cash flows using OIS rates as discussed in Section 3.1.3. For each scenario, and using equation 3.18, the following rates will be simulated

$$\begin{pmatrix} L_1(T_0) & 0 & 0 & \cdots & 0 \\ L_2(T_0) & L_2(T_1) & 0 & \cdots & 0 \\ L_3(T_0) & L_3(T_1) & L_3(T_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_n(T_0) & L_n(T_1) & L_n(T_2) & \cdots & L_n(T_{n-1}) \end{pmatrix}$$

where T_0 is the initiation date of the swap and $T_1, \ldots, T_n$ the cash flow dates of the swap.

The correlations between forward rates are required inputs in the calibration of the LMM volatilities, and they are estimated using 1 year historical data, following the procedure in Section 3.6.5. From historical correlations the parametric function in equation 3.17 is estimated with the following parameters in Table 5.1.

α	β
-0.156	0.070

Table 5.1: Estimated parameters of the function in equation 3.17.

The LMM volatilites are calibrated to Black's at-the-money swaption volatilities, taken from Bloomberg, according to the calibration procedure explained in Section 3.6.4. Before doing the calibration, the parametric function in equation 3.15 is fitted to the input swaption volatilities in order to remove possible misaligned market quotations. The following parameters in Table 5.2 are received.

a	b	c	d	e	f	g	h
-0.0166	1.0288	0.5943	-0.0287	0.0211	0.004	-0.9308	3.1778

m	p	q	r	s	t	β	δ
0.7210	0.7102	-0.2922	-0.1301	-4.6918	-0.0631	1.1478	4.7305

Table 5.2: Estimated parameters of the function in equation 3.15.

Using both these estimated parametric functions together with the yield curve in Figure 5.2 the calibration of the LMM volatilities can be carried out. Beneath in Figure 5.3 both the smooth historical correlation and the smooth market implied swaption volatilities are visualized.

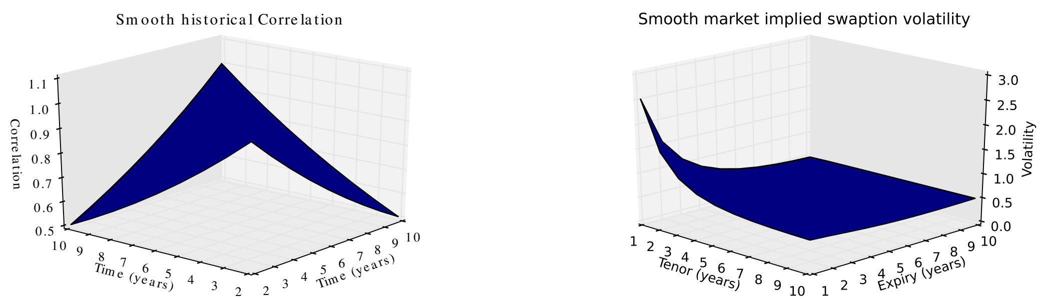

Figure 5.3: *Left:* Smooth historical correlation. *Right:* Smooth market implied swaption volatilites.

CIR++ Model

For the default model approach a credit model is needed in order to simulate default times. Each credit model in equation 4.4 is calibrated to each CDS term structure according to the procedure in Section 3.7.3. The free parameters in the CIR process are set arbitrarily with the constraints that $2\kappa\theta \geq \sigma^2$ and $\varphi(t) \geq 0$ in order for $\lambda(t)$ to stay positive and $\Lambda(t) = \int_0^t \lambda(s)ds$ to be a strictly increasing function. The CIR processes are simulated using the QE scheme in Section 3.7.4.

The following parameters in Table 5.3 are used for the default intensity of the counterparty C.

x_0^C	κ^C	θ^C	σ_C
0.001	1	0.0025	0.07

Table 5.3: Parameter setting in CIR++ model of the counterparty.

The following parameters in Table 5.4 are used for the default intensity of the investor I.

x_0^I	κ^I	θ^I	σ_I
0.002	1.2	0.007	0.12

Table 5.4: Parameter setting in CIR++ model of the investor.

Comparing the parameters in Tables 5.3 and 5.4 one can see that the counterparty's credit model has a less volatile setting than the investor's credit model. Comparing the implied market default probabilities in Figure 5.1 this seems reasonable. In order for the conditions $2\kappa\theta \geq \sigma^2$ and $\varphi(t) \geq 0$ to hold both models above have very low volatilities. However later in Section 5.2.4 we will see the impact of larger volatilities when the condition $2\kappa\theta \geq \sigma^2$ is dropped.

5.1.1 Dependence

In the default model approach in Section 4.2 there are three free parameters in order to model dependence. The first two parameters are the correlations ρ_I and ρ_C. These are linear correlations between the rates $L_i(t)$ and the credit spreads of the investor I and counterparty C. The third parameter is the correlation ρ_{ξ_I,ξ_C} which models the correlation in the copula distributions between the exponential random variables ξ_I and ξ_C. In all CVA calculations below in Tables 5.6, 5.7, 5.8 and 5.9 these three parameters are set to zero, i.e. $\rho_I = \rho_C = \rho_{\xi_C,\xi_I} = 0$. Later on these parameters will be varied and the resulting CVA will be shown in plots.

5.1.2 Sample Size

In order to determine a sufficient simulation sample size the BCVA is plotted against the number of simulations for the EFE and the default model method. The plot is shown below in Figure 5.4. It can be seen, as expected, that the default model method requires a lot more samples than the EFE method in order to get a sufficient convergence. This is because of the need to generate a sufficient number of default times such that $\tau < T$ where T is the maturity of the contract. The probability of surviving past the expiry of the contract is often much greater than the probability of defaulting before the expiry of the contract. Let's say that there is a 5% default probability before the maturity of the contract, then using a sample size of $N_s = 10^6$ will only generate about $5 \cdot 10^4$ scenarios where there has been an default and the portfolio is valued. A sample size of $N_s = 10^5$ seems to be enough for the EFE approach and a sample size of $N_s = 10^6$ for the default model approach. For consistency throughout this report, all results from the EFE and the default model approach have a sample size of $N_s = 10^6$.

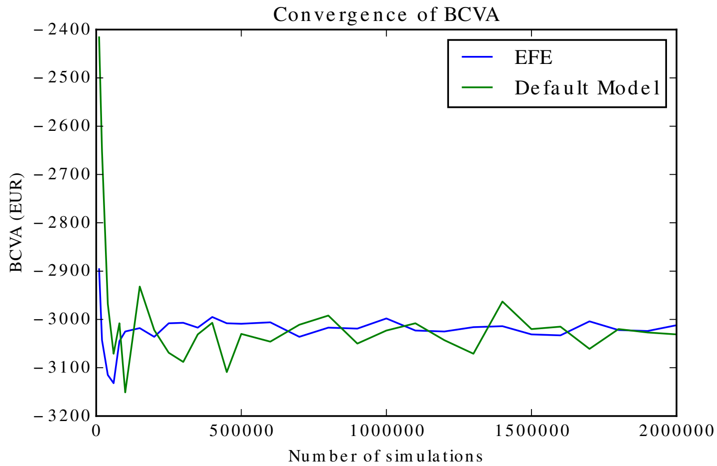

Figure 5.4: Convergence of BCVA for EFE approach and for default model approach.

5.2 Results

An plain vanilla interest rate swap is investigated. The swap has the following setting

- Currency: EUR
- Notional: 10 Million EUR
- Tenor: 5 year
- Initiated: 6/5/2015
- Floating rate: 6-month EURIBOR paid semi-annually with an actual/360 day count convention.
- Fix rate: Fair swap rate paid annually with an 30/360 day count convention.

Using the initial EUR yield curve in Figure 5.2 and equation 3.3 the fair swap rate $S = 0.365\%$ is received. The projected cash flow of the receiver swap can be seen in Table 5.5.

$T_1 = 0.5$	$T_2 = 1$	$T_3 = 1.5$	$T_4 = 2$	$T_5 = 2.5$
-3 199.5	33 445	-4 490.6	28 131.8	-12 777.9

$T_6 = 3$	$T_7 = 3.5$	$T_8 = 4$	$T_9 = 4.5$	$T_{10} = 5$
18 715.4	-24 017.4	5 918.0	-36 593.7	-5 739.7

Table 5.5: Projected cash flow of a receiver swap.

5.2.1 PV01

The PV01 for the payer swap is approximately 4950 EUR (-4950 EUR for the receiver swap). This means that if the market moves, resulting in a parallel shift upwards in the yield curve of 10bps, the owner of the payer swap would have a PnL of approximately 49500 EUR. Having these numbers in mind, it is easier to get an idea of the magnitude of the resulting CVA of the swap.

5.2.2 Payer Swap

In Figure 5.5 the discounted expected positive and negative exposures for the payer swap are shown. The payer swap is valued in each simulated scenario in between of each cash flow. It can be seen that the expected positive exposure is greater than the expected negative exposure. As explained in Pykhtin and Zhu (2007) when then yield curve is upward sloping, as our curve in Figure 5.2, the exposure is greater for a payer swap than the receiver swap. The opposite is true when the yield curve is downward sloping.

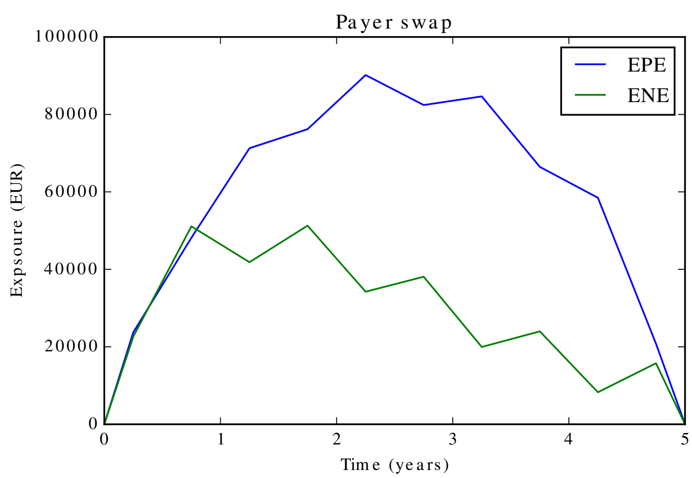

Figure 5.5: Future expected positive and negative exposure of a payer swap.

The results from the unilateral CVA calculations, when the investor is considered default-risk-free, are shown in Table 5.6. The default model approach and the expected future exposure approach should converge to the same CVA when there is no dependence in the default model approach. It can be seen that both the variable exposure approach and the discounted cash flow approach underestimates the unilateral CVA. As discussed in Sections 4.4 and 4.3 this was expected, since fluctuations of the exposure are ignored using these methods.

Method	CVA	Diff
EFE Approach	1930	0%
Default Model Approach	1922	-0.4%
Variable Exposure Approach	519	-73.1%
Discounted Cash Flow Approach	1193	-38.2%

Table 5.6: Unilateral CVA of a payer swap and difference in percentage of CVA compared to the EFE approach.

The result from the BCVA calculations, when both parties are considered risky, are shown i Table 5.7. All methods show a positive BCVA which means that the investor should receive money from the counterparty. Once again the default model approach and the expected future exposure approach should converge to the same values here since no dependence is modelled in the default model approach. Again, as expected the variable exposure and the discounted cash flow approach underestimates the DVA. However, compared to the EFE method, the resulting BCVA is overestimated by the variable exposure approach and underestimated by the discounted cash flow approach.

Method	CVA	DVA	BCVA	Diff
EFE Approach	1764	1474	328	0%
Default Model Approach	1824	1501	323	-1.5%
Variable Exposure Approach	519	4	515	57.0%
Discounted Cash Flow Approach	1193	1026	167	-49.1%

Table 5.7: CVA, DVA and BCVA of a payer swap and difference in percentage of BCVA compared to the EFE approach.

Dependencies in the default model approach

In Figure 5.6 the unilateral CVA and BCVA are plotted as a function of the correlation between forward rates in the LMM model and credit spreads in the CIR++ models. For BCVA both correlations are varied simultaneously, i.e. $\rho_I = \rho_C$ in equation 4.6.

For the unilateral CVA in the left plot it can be seen that when the correlation ρ_C increases the CVA increases. It can also be seen that CVA is more sensitive when $\rho_C > 0$. This is because the value of the payer swap increases at the same time as the default risk of the counterparty increases, i.e. there is a case of WWR. When $\rho_C < 0$ the value of the payer swap decreases simultaneously as the default risk of the counterparty C increases, hence there is a case of RWR.

For the bilateral CVA in the right plot it can be seen that BCVA increases almost as a linear function with the correlations $\rho_I = \rho_C$. Remember the reasoning for CVA above. For DVA when $\rho_I < 0$, the value of the reverse position in the payer swap (i.e. a receiver swap) increases simultaneously as the default risk of the investor I increases, i.e. there is WWR. When $\rho_I > 0$, the value of the reverse position in the payer swap decreases simultaneously as the default risk of the investor I increases, i.e. there is RWR. So the DVA will have the reverse characteristics as the unilateral CVA in the left plot, hence when $\rho_I < 0$ is decreasing the DVA will also increase and when $\rho_I > 0$ is increasing not so much will happen to the DVA.

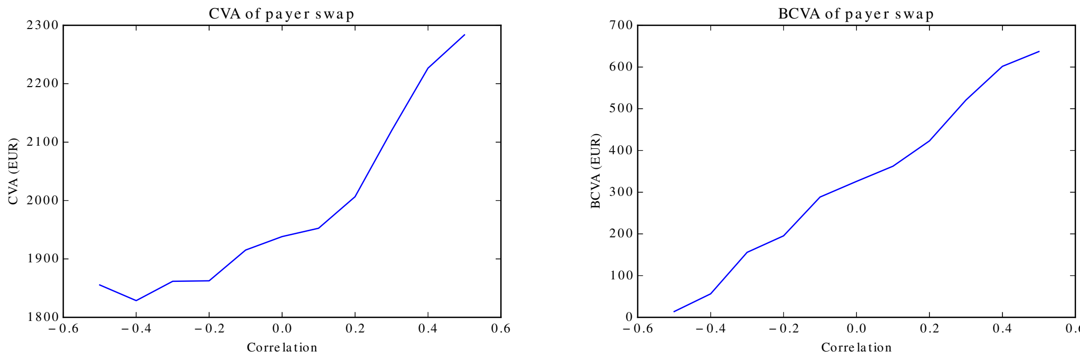

Figure 5.6: *Left*: Unilateral CVA with varying correlation ρ_C between forward rates and credit spread. *Right*: BCVA with varying correlations $\rho_I = \rho_C$ between forward rates and credit spreads.

The surface of the bilateral CVA when both varying ρ_I and ρ_C can be seen in Figure 5.7. As pointed out before there is wrong-way risk when $\rho_C > 0$ and $\rho_I < 0$. When $\rho_C > 0$ and ρ_C increases the BCVA also increases and when $\rho_I < 0$ and $\rho_I < 0$ decreases the BCVA also decreases. However in absolute values the effect of ρ_C is greater. This is because of the expected positive exposure towards the counterparty C is greater than the expected negative exposure towards the investor I.

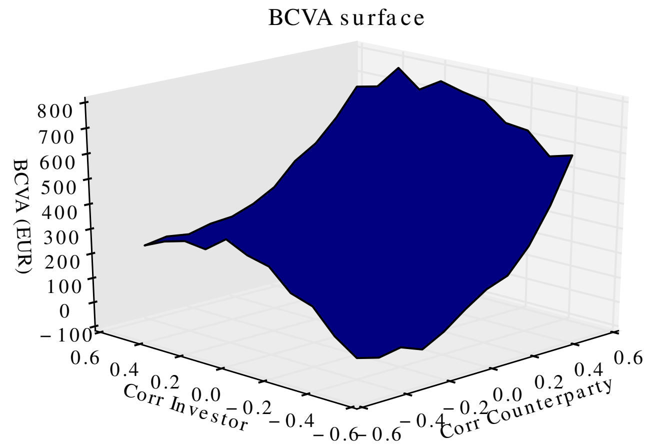

Figure 5.7: The surface of BCVA when varying the correlations ρ_I and ρ_C.

In Figure 5.8 the resulting bilateral CVA is plotted as a function of the correlation between default times ρ_{ξ_I,ξ_C} in a Gaussian and a Student's t copula distribution. It can be seen that BCVA decreases with a larger correlation. Furthermore BCVA decreases faster when using a Student's t copula, which is because of the more extreme scenarios occurring with a Student's t-distribution, see Figure 4.1.

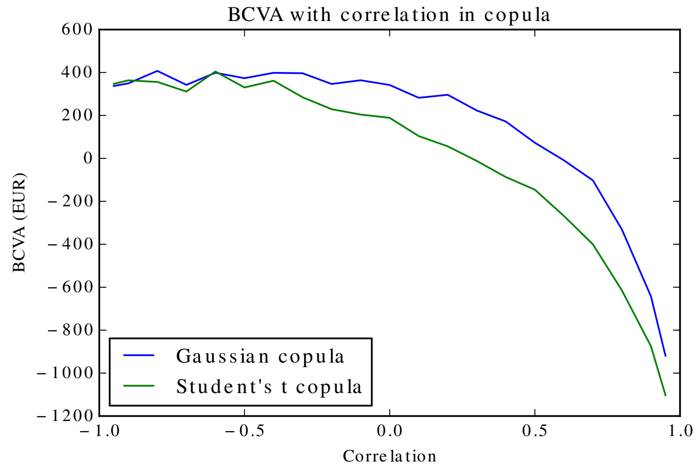

Figure 5.8: BCVA with varying ρ_{ξ_I,ξ_C}, which describe the correlation between ξ_I and ξ_C in a Gaussian and a Student's t copula.

5.2.3 Receiver Swap

In Figure 5.9 the discounted expected positive and negative exposures for the receiver swap are shown. The receiver swap is valued in each simulated scenario in between each cash flow. This is simply the opposite of the exposures for the payer swap in Figure 5.5. So here the expected negative exposure is greater than the expected positive exposure.

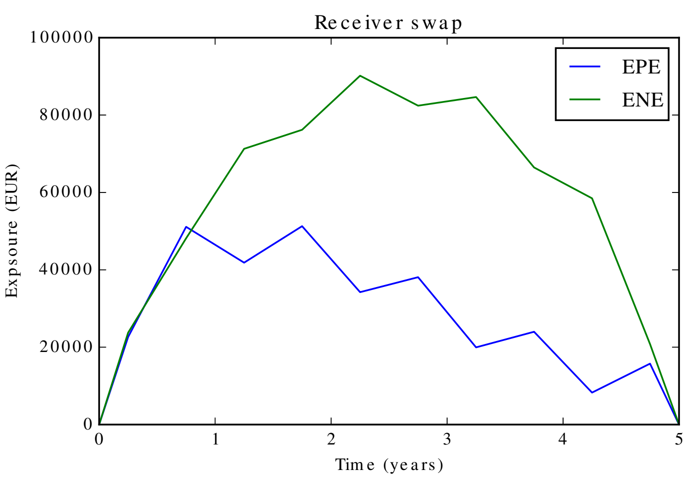

Figure 5.9: Future expected positive and negative exposure of a receiver swap.

The results from the unilateral CVA calculations, when the investor is considered default-risk-free, are shown in Table 5.8. Since the default model approach is modelled with no dependence it is very close to the value received from the expected future exposure approach. As stated earlier, both the other methods underestimates the unilateral CVA. However for the variable exposure approach the resulting unilateral CVA is equal to 2 which is way off compared to the other methods. This is because the sum of the future projected cash flow is negative for all dates except for $T_1 = 0.5$ and this method approximates CVA as the cost of buying CDS protection on this projected future exposure. The projected cash flow can be seen in Table 5.5.

Method	CVA	Diff
EFE Approach	741	0%
Default Model Approach	744	0.4%
Variable Exposure Approach	2	-99.7%
Discounted Cash Flow Approach	470	-36.6%

Table 5.8: Unilateral CVA of a receiver swap and difference of CVA compared to the EFE approach.

The results from the bilateral CVA calculations, when both parties are considered risky, are shown i Table 5.9. All methods give a negative BCVA, which means that the counterparty should receive money from the investor. Compared to the payer swap, the BCVA has changed sign and is a lot larger. This is because the expected negative exposure is larger for the receiver swap and since the investor is more likely to default than the counterparty. Once again, since no dependence is modelled in the default model approach the BCVA of the first two methods converges to the same value. Both the variable exposure and the discounted cash flow approach underestimates BCVA. Again the discounted cash flow approach is closer to the value from the EFE approach.

Method	CVA	DVA	BCVA	Diff
EFE Approach	684	3644	-3024	0%
Default Model Approach	709	3732	-3023	0.0%
Variable Exposure Approach	2	1092	-1090	-64.0%
Discounted Cash Flow Approach	470	2244	-1774	-41.3%

Table 5.9: CVA, DVA and BCVA of a receiver swap and difference in percentage of BCVA compared to the EFE approach.

Dependencies in the default model approach

In Figure 5.10 the unilateral CVA and BCVA are plotted as a function of the correlation between forward rates in the LMM model and credit spreads in the CIR++ models. For

BCVA both correlations are varied simultaneously, i.e. $\rho_I = \rho_C$ in equation 4.6.

From the left plot it can be seen that the unilateral CVA increases with a larger negative correlation ρ_C, but almost nothing happens when $\rho_C > 0$. This is because of the presence of WWR when $\rho_C < 0$ and RWR when $\rho_C > 0$. When $\rho_C < 0$ the receiver swap is increasing in value with lower rates at the same time as the probability of default is increasing with increasing credit spreads. When $\rho_C > 0$ the receiver swap is decreasing in value with larger rates at the same time as the probability of default is increasing with increasing credit spreads.

For the bilateral CVA in the right plot it can be seen that BCVA decreases with the correlations $\rho_I = \rho_C$. Remember the reasoning above for the unilateral CVA. For DVA wrong-way risk is occurring when $\rho_I > 0$. The reverse position in a receiver swap (i.e. a payer swap) increases with larger rates and simultaneously the default risk of the investor I is increasing. When $\rho_I < 0$ there is right way risk and thus the effect on DVA is small.

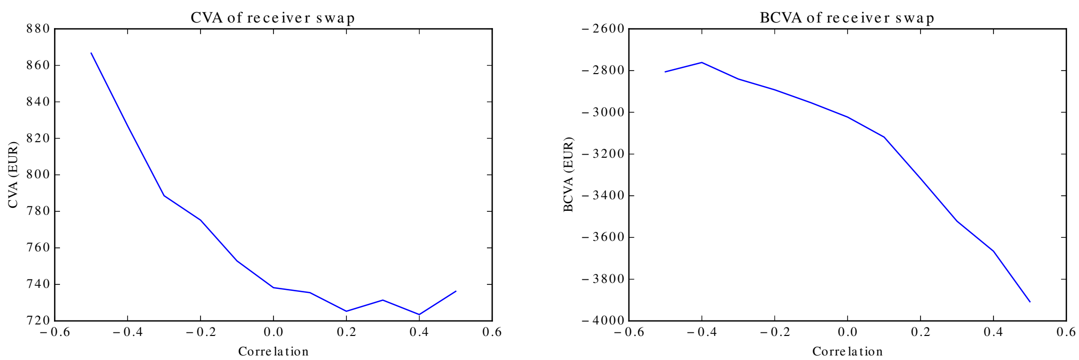

Figure 5.10: *Left*: Unilateral CVA with varying correlation ρ_C between forward rates and credit spread. *Right*: BCVA with varying correlations $\rho_I = \rho_C$ between forward rates and credit spreads.

The surface of bilateral CVA when both varying ρ_I and ρ_C can be seen in Figure 5.11. As stated above there is a presence of wrong-way risk when $\rho_C < 0$ and when $\rho_I > 0$. Another important thing to notice is that the bilateral CVA is way more sensitive to the correlation between forward rates and the credit spreads of investor I, i.e. ρ_I. Similar to the payer swap but the other way around this make sense because the expected negative exposure in Figure 5.9 is larger than the expected positive exposure.

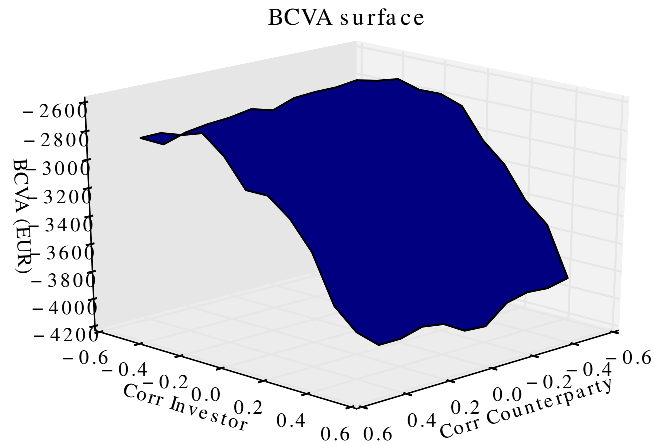

Figure 5.11: The surface of BCVA when varying the correlations ρ_I and ρ_C.

In Figure 5.12 the resulting bilateral CVA is plotted as a function of the correlation between default times ρ_{ξ_I,ξ_C} in the Gaussian copula and the Student-t's copula. It can be seen that BCVA increases with a larger positive correlation but when $\rho_{\xi_I,\xi_C} > 0.9$ BCVA suddenly decreases rapidly. This feature seems very strange and one would perhaps expect a similar behaviour as in Figure 5.8 where BCVA decreases with an increasing correlation, but in the opposite direction. However what happens when the correlation ρ_{ξ_I,ξ_C} is extremely large is that the two exponential random variables is approximately equal, i.e. $\xi_I \approx \xi_C$. Since the default times is given by $\tau_I = (\Lambda_I)^{-1}(\xi_I)$ and $\tau_C = (\Lambda_C)^{-1}(\xi_C)$ the party with larger initial default probabilities, i.e. larger $\Lambda(t)$, will almost always default before the other party. In our case the investor I has a greater default risk, as can be seen in Figure 5.1, thus when $\xi_I \approx \xi_C$ the investor I will almost always default before the counterparty C.

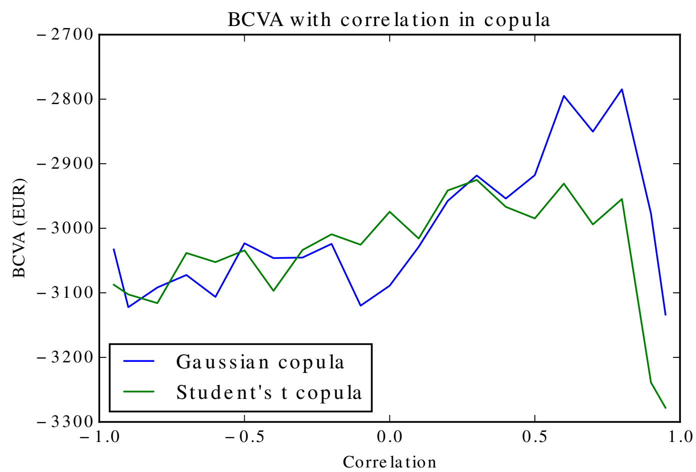

Figure 5.12: BCVA with varying ρ_{ξ_I,ξ_C}, which describe the correlation between ξ_I and ξ_C in a Gaussian and a Student's t copula.

5.2.4 Impact of Credit Model Volatilities

When modelling dependencies in the default model approach the settings in the credit model can be quite crucial. In the results above the parameters in Tables 5.3 and 5.4 were used in each credit model. These parameters were chosen arbitrarily with the constraints that $2\kappa\theta \geq \sigma^2$ and $\varphi(t) \geq 0$. However it is possible to drop the constraint $2\kappa\theta \geq \sigma^2$, which is the constraint that guarantees a positive $x(t)$ in equation 4.4. Using the QE simulation scheme in Section 3.7.4 and keeping the constraint $\varphi(t) \geq 0$, the process $\lambda(t)$ will stay positive anyway. This means that a larger set of the parameters in the credit models are possible to use.

The volatilities σ_C and σ_I in equation 4.4 are the parameters with most impact, if there is a dependence between the exposure and the credit model is used. In Figure 5.13 the unilateral CVA of both the payer and the receiver swap are plotted against the correlation ρ_C for different volatilities σ_C in the credit model. It can be seen that the unilateral CVA changes dramatically when the volatility σ_C is increased. The effect of the WWR increases with a larger volatility.

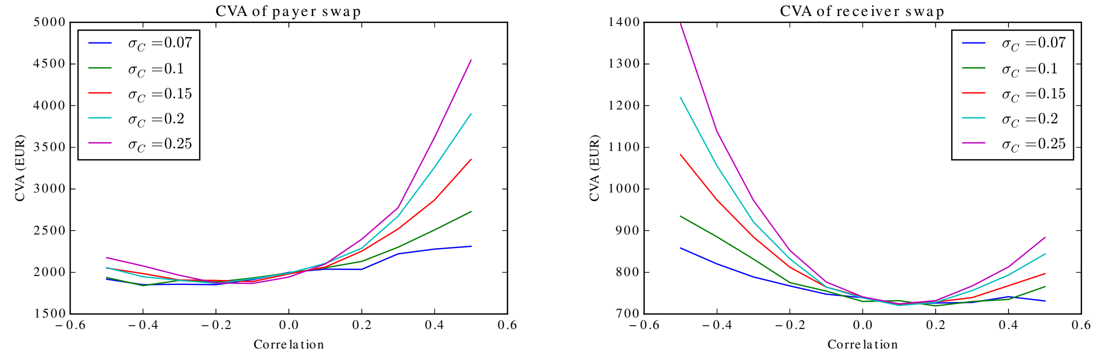

Figure 5.13: *Left*: CVA of a payer swap when varying the correlation ρ_C and with different volatilities σ_C in the credit model. *Right*: CVA of a receiver swap when varying the correlation ρ_C and with different volatilities σ_C in the credit model.

In Figure 5.14 the bilateral CVA of both the payer and the receiver swap is plotted against the correlations $\rho_C = \rho_I$ for different volatilities σ_C and σ_I in both credit models. Again it can be seen that the WWR increases for both CVA and DVA with increasing volatilities. There is however a big difference between the BCVA of payer and the receiver swap. This is due to the expected positive/negative exposure combined with the default risk of the counterparties. For the payer swap there is a larger expected positive exposure which is towards the less risky counterparty. But for the receiver swap there is a larger expected negative exposure which is towards the more risky investor.

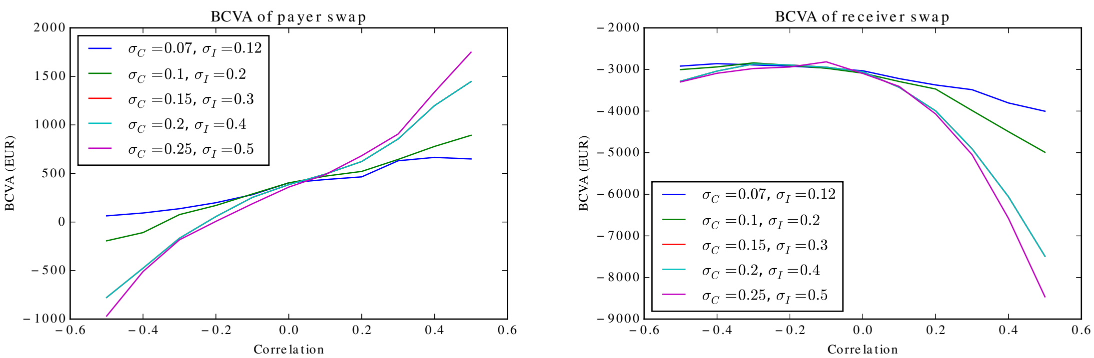

Figure 5.14: *Left*: BCVA of a payer swap when varying the correlations both $\rho_C = \rho_I$ and with different volatilities σ_C and σ_I in both credit models. *Right*: BCVA of a payer swap when varying the correlations both $\rho_C = \rho_I$ and with different volatilities σ_C and σ_I in both credit models.

Chapter 6

Discussion and Conclusions

In this chapter the results and performance of the different methods are discussed. Some further extensions of the models and other methods are also briefly discussed.

6.1 CVA Frameworks

In this thesis four different methods for calculating CVA have been compared. From advanced frameworks based on theoretical models to easier approximative frameworks.

From the derivation of CVA in Chapter 2, it is easy to understand that the EFE and the default model approach are the two most correct approaches from a theoretical viewpoint. The EFE approach is constructed using the critical assumptions of independence between exposure and the default time of a counterparty and independence between default times of different parties. Whereas in the default model approach, both dependencies between the exposure and default times of counterparties and dependencies between the default times of different parties can be modelled. The results in the tables in Chapter 5 are presented using no dependence and one can see that both models converge to the same values. However when adding dependencies in the default model approach, the resulting CVA changes drastically.

Comparing the result of the two easiest frameworks, namely the variable exposure and the discounted cash flow approach we have seen that the discounted cash flow approach seems to produce more accurate results. Compared to the BCVA from the EFE approach the discounted cash flow approach is approximately off by 40%, whereas the variable exposure approach is approximately off by 60% or more. Clearly, what could be seen from the results, and also which was expected, is that these two methods underestimates the absolute value of the unilateral CVA and DVA.

It is however difficult to draw general conclusions since only interest rate swaps have been investigated in this thesis. But in order to achieve an accurate valuation of the counterparty credit risk of a portfolio of interest rate swap, a more sophisticated framework like the EFE or the default model approach is needed. Since these are more advanced methods and more computer intensive they require a bigger knowledge of financial modelling and a more sophisticated IT infrastructure. For an investor with a small portfolio and the need for a simple method, the discounted cash flow method is to prefer, at least for interest rate swaps. This method is both easy to understand and to implement. Furthermore the only required data are credit spreads and forward prices. However using this method one should be aware of its limitations and that the approximation can be quite rough.

6.1.1 Wrong-Way Risk

As wrong-way risk is a key feature in CVA it is important to be able to capture these effects if they exist in a portfolio. In Chapter 5 we have seen that dependence between exposure and the default risk, and also dependencies between default risk of different parties, have a big effect on the CVA calculations. However, the fact that these correlations often are non-observable in the market causes problems. As discussed earlier a possible solution to this would be to use historical correlations between credit spreads and the exposures. Furthermore we have also seen some effects the volatilities in the credit models have on CVA, when the exposure is correlated with the credit spreads. This causes another big problem since, as discussed in Section 3.7.3, the parameters in the credit models are often very hard to specify under the risk-neutral measure. To do an accurate risk-neutral calibration of the credit model one needs to have market prices of instruments like single name CDS options, and these types of derivatives are rare and probably rather illiquid.

From a risk management perspective, it is very important to be able to capture the effect of possible dependencies between the exposure and default risk of a counterparty. Though it is hard, from a theoretical point of view, to capture this effect correct, a trade with WWR would need to be "penalized" with a larger unilateral CVA compared to an equivalent trade with no WWR.

The dependence between the default time of counterparties may be more significant. It seems likely that two companies in the same sector, e.g. two banks, have a larger default correlation than two companies coming from different sectors. Again correlations like these are non-observable in the market and it is also impossible to measure historically since default only occurs once. A possible solution would be to estimate these correlations from historical credit spreads of the two parties.

If one suspects the presence of wrong-way risk in a portfolio, a framework like the default model approach is a preference. However, as has been discussed above it is difficult to calibrate such a framework under the risk-neutral measure. Furthermore one should be aware of the impacts of CVA that the correlations and the parameters in the credit models have and set them with caution.

6.1.2 Performance

Since CVA needs to be calculated at a portfolio level rather than at a trade level, the calculation speed becomes an issue. In this thesis the CVA of a simple IRS has been calculated. However, calculating CVA for a large OTC portfolio which contains thousands of trades towards hundreds of counterparties quickly becomes more demanding and complex. In CVA frameworks as the EFE approach or the default model approach, it is often the valuation that is the most time-consuming part. One way of speeding up the valuation is by using a Least Squares Monte Carlo valuation. This is especially needed when a portfolio contains instruments which needs to be priced using simulation, e.g. path-dependent exotic derivatives, since the CVA calculation of such an instruments otherwise leads to nested simulations.

Both the EFE and the default model framework are very costly when it comes to computational time and computer power since these two methods are based on simulation and Monte Carlo valuation. For CVA as a pricing component this becomes a problem since pricing often needs to be done very fast. However for the purpose of accounting the fair value of assets and liabilities it does not have to be a problem. So depending on the purpose of the CVA calculations different methods may be preferred.

The default model approach is based on simulating default times of companies that often are not very likely to default. This means that even with a very large number of simulations the number of default times before maturity of the contracts in a portfolio is likely to be a few. Since just a few default times are received the Monte Carlo valuation may not be very accurate. As seen in Figure 5.4 one often needs to simulate more than 10^6 scenarios in order to have enough scenarios where a default actually took place in order to get some convergence. However when calculating CVA for a large portfolio, as discussed above, it is the valuation that is most time-consuming. So since the default model only needs to be valued where there has been a default, it requires less valuation points than to the EFE approach. The EFE approach has to perform a valuation in each time step in each simulation. So although a larger initial sample size has to be used for the default model approach, the number of valuations is often less than in the EFE approach.

6.2 Further Work

One thing that is very critical for the calculation of CVA but which has been taken to be deterministic in this thesis is the recovery rate R. An interesting extension of the advanced methods would be to combine these with a model of the recovery rate. However estimating such a model is again very difficult because of the lack of data.

Another thing that has not been taken into consideration in this thesis is the multi-curve valuation methodology. Here, the discounting has been done using LIBOR rates and only one yield curve has been used. As mentioned in Section 3.1.3 and 3.2, a multi-curve framework for valuation of interest rate derivatives is now the market practice. Thus a recommended and more proper valuation of interest rate swaps and hence a more accurate CVA would require a multi-curve framework.

As discussed in Section 6.1.2, pricing often needs to be done fast, a Monte Carlo approach is not always suitable. Therefore approximating analytical or so called semi-analytical formulas for CVA are often desired. These methods often produce more accurate results than methods like the variable exposure and the discounted cash flow approach. For further studies about analytical CVA approximations for interest rate swaps see for example Nisiba (2011) or Cern and Witzany (2014).

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