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On the risk relation between Economic Value of Equity and Net Interest Income

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Abstract

The Basel Committee has proposed a new Pillar 2 regulatory framework for evaluating the interest rate risk of a bank's banking book appropriately called Interest Rate Risk in the Banking Book. The framework requires a bank to use and report two different interest rate risk measures: Economic Value of Equity (EVE) risk and Net Interest Income (NII) risk. These risk measures have previously been studied separately but few models have been proposed to investigate the relationship between them. Based on previous research we assume that parts of the banking book can be approximated using a portfolio strategy of rolling bonds and propose a model for relating the connection between the portfolio maturity structure, EVE risk and NII risk. By simulating from both single- and multi-factor Vasicek models and measuring risk as Expected Shortfall we illustrate the resulting risk profiles. We also show how altering certain theoretical assumptions seem to have little effect on these risk profiles.

Ekonomiskt Värde av Eget Kapital-risk samt Räntenettorisk

och sambandet dem emellan

Sammanfattning

Baselkommittén har föreslagit ett nytt Pelare 2-regelverk för att utvärdera ränterisken i en banks bankbok kallat Interest Rate Risk in the Banking Book. Regelverket kräver att en bank beräknar och rapporterar två olika mått på ränterisk: Ekonomiskt Värde av Eget Kapital-risk (EVE-risk) samt Räntenettorisk (NII-risk). Dessa två mått har tidigare studerats separat men få modeller har föreslagits för att studera relationen dem emellan. Baserat på tidigare forskning så antar vi att delar av bankboken kan approximeras som en rullande obligationsportfölj och föreslår en modell för att relatera sambandet mellan portföljens löptidsstruktur, EVE-risk och NII-risk. Genom att simulera korträntor från Vasicek-modeller med olika antal faktorer så undersöker vi de resulterande riskerna mätt som Expected Shortfall. Vi visar också att vissa av de teoretiska antagandena verkar ha liten påverkan på riskprofilen.

Acknowledgments

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Stockholm, June 2017,
André Berglund and Carl Svensson

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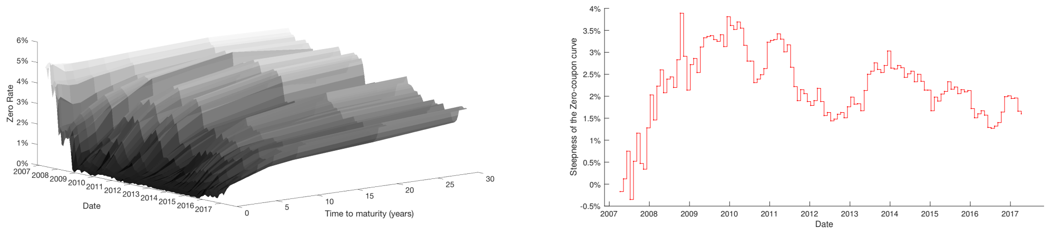
Introduction

> The banks' risk appetite for Interest Rate Risk in the Banking Book should be articulated in terms of the risk to both economic value and earnings.
>
> *BCBS (2016b, p. 6, Principle 3)*

In its traditional role the commercial bank acts as an intermediary between lender and borrower. Since the demand and supply of funds from the bank's customers can differ in aspects such as maturity and credit quality there is an inherent risk associated with the intermediary role. Even given the possibility to exactly match the asset and liability side of the balance sheet, one could not assume with certainty that the bank would want to. As the saying goes, traditionally the bank "borrows short and lends long", which is in reference to the shorter duration of the liability side of the balance sheet. In a somewhat simplified setting we can assume that the bank faces a decision of how it is going to fund every new retail or business loan it underwrites. The funding can be secured using either existing funds, such as retained earnings and equity capital, or by using debt capital that the bank borrows from creditors. Historically, interest rate curves have often been upward sloping, but as can be seen in figure 1.1a, over a ten-year period the zero-coupon curve will often exhibit a vast amount of different shapes. This is also illustrated in figure 1.1b where the steepness[2] of the same curve is plotted over time. By keeping the maturity mismatch between assets and liabilities the bank has been able to increase its earnings due to the lower interest paid on the shorter maturity liabilities. However, the mismatch gives rise to two types of financial risks for the bank, interest rate risk (IRR) and liquidity risk.

IRR for a bank is the current or potential risk to the bank's capital *and* to its earnings that arises from the impact of adverse movements in interest rates (BCBS, 2016b). Due to the different perspective of focusing on either capital or earnings risk there exists two, different but complementary, methods for measuring and assessing IRR. The first being the present value sensitivity of an asset or a liability to changes in interest rates and the second being the short-term expected earnings sensitivity to changes in interest rates. Three subtypes of IRR, gap risk, basis risk and optionality risk, are the main drivers

[2] Measured as the difference between the 1 year and 20 year point on the curve.



(a) The zero-coupon curve (b) Steepness of the zero-coupon curve

Figure 1.1: Evolution of the US zero curve between 2007 and 2017

of these two measurements. Mismatches in the timing of cash flows between assets and liabilities, as in the "borrow short, lend long" strategy mentioned above is an example of gap risk. The risk that occurs when cash flows are sensitive to different interest rate curves is a type of basis risk[1] and optionality risk occurs when there are automatic or behavioral optionality for the bank or its counterparties to alter the level or timing of cash flows. Gap risk, basis risk and option risk can all cause changes to both the present value of instruments and the expected earnings of those instruments. Contemporary examples of crises that, at least partially, were results of banks' exposure to IRR is the Secondary Banking Crisis in the U.K. during the 1970s and the Savings and Loan crisis in the U.S. during the 1980s (English, 2002).

While long-term assets financed with short-term liabilities can cause IRR it also gives rise to liquidity risk when liabilities mature prior to the assets they finance. Unless a bank with short-term liabilities has additional liquid funds, its survival is contingent on the bank's ability to refinance those liabilities. The global financial crisis of 2007-2008 was in part a liquidity crisis where central banks had to provide liquidity support and showed how costly mismanagement of risk in banking could be for society as a whole (Brunnermeier, 2009). Post crisis, several areas of the then-existing banking regulations were put under review and with respect to IRR resulted in the so-called Interest rate risk in the banking book (IRRBB) proposal from the Basel Committee. IRRBB introduces new proposals to ensure that a bank has enough capital to cover the IRR arising in its banking book. The framework requires the bank to understand, compute and report its IRR and requires it to specify its IRR appetite. For that purpose two different IRR measures are computed, Economic Value of Equity (EVE) and Net Interest Income (NII). In essence they are the present value risk and short term earnings risk of IRR previously mentioned and will be more thoroughly presented in section 2.1.

The Basel committee notes that commercial banks tend to focus on managing earnings risk while regulators previously have focused on EVE risk (BCBS, 2016b). Each measure has both advantages and disadvantages compared to the other and neither has yet to prevail as the standard (Alessandri and Drehmann, 2010). The Basel committee furthermore acknowledges an important aspect of the two risk measures, which is that if a bank minimizes its EVE risk it runs the risk of earnings volatility. Hence, the bank is facing a trade-off problem when relating and valuing volatility in NII to EVE. This requires both banks and regulators to gain an understanding of how different balance sheet compositions

[1] An example could be an asset that is sensitive to 3-month LIBOR that is funded with a liability sensitive to 6-month LIBOR

affect NII and EVE volatility.

Banks will – provided national regulators implement it – be required to compute sensitivities of both measures under IRRBB. These new regulatory standards will also require banks to determine and articulate risk limits in both EVE and NII, which combined with current levels and exposures will be made public. There exists a vast amount of literature investigating IRR in banking but to the best of the authors' knowledge only three papers have been written about the interaction between NII and EVE. One of these is Memmel (2014) who using a strategy of rolling par-coupon bonds and historical simulations shows how different interest rate changes affect both NII and EVE. A few methodological choices leads to the proposed NII measurement being fairly different to how it will be measured under IRRBB and the focus is not on risk. However, the conceptual framework of approximating the bank as a rolling bond portfolio is helpful when wanting to analyze IRR in isolation. It is also noteworthy that all of these papers measure NII and EVE in slightly different ways.

The purpose of this thesis, commissioned by Svenska Handelsbanken AB, is to propose a model that can be used to consistently study how varying the maturity structure of a portfolio affects both NII and EVE risk. As in Memmel (2014) the basic building block of the model is a rolling portfolio of non-defaultable coupon bonds. In order to study the resulting risk profile we will also investigate how the risk profile changes with the maturity of the portfolio and how combining different portfolios can change the attainable combinations of NII and EVE risk.

The focus in this thesis will be on the relation between NII risk and EVE risk and we will not try to say anything about the potential trade-off between risk and return. It has been argued by Alessandri and Drehmann (2010) that IRR should be studied in tandem with credit risk. However, in this thesis we will limit our study to IRR in isolation and not its interaction with other types of risk. We will also limit our portfolio to contain only one type of instrument, namely non-defaultable coupon bonds. This means that we will not be able to capture some IRR effects that a bank faces, e.g. basis risk from instruments being sensitive to different interest rate curves and optionality risk from instruments such as demand deposits. The portfolio model should be viewed as an investment strategy and not as a complete banking book. However, parts of the banking book could possibly be approximated by our portfolio model. Memmel (2008) has investigated if German banks' net interest income can be approximated as a combination of several rolling coupon bond portfolios with different maturities, we will not investigate how well this assumption works for Swedish banks. Interest rates will be simulated from a short-rate model, for this purpose parameters will need to be estimated. The estimation scheme will be described but since the purpose of the thesis is not parameter estimation we will not evaluate how well this model performs. We refer the reader who is interested in empirical studies of IRR to the two comprehensive literary reviews written by Staikouras, see Staikouras (2003) and Staikouras (2006).

The outline of this thesis is as follows. Chapter 2 introduces the reader to EVE, NII and IRRBB. This is followed by chapter 3 where we present the mathematical background and review previous research relating EVE and NII to each other. Chapter 4 discusses the assumptions and simplifications of the model and shows its mathematical formulation. Chapter 5 familiarizes the reader with the data we use to estimate the parameters for the short-rate models and presents the simulation scheme used to compute IRR. These simulations are then shown and discussed in chapter 6, where we also investigate the effects of combining several portfolios. Lastly, chapter 7 contains our conclusions and suggestions for further research.

Chapter 2

Preliminaries

This chapter serves as an introduction to the two main classes of IRR measures, NII and EVE. In section 2.1 we will define the measures and by using an illustrative example[2] we will show the somewhat different effects each measure captures. This is followed by section 2.2 which contains a short review of IRRBB focusing on its definitions of NII and EVE.

2.1 EVE and NII measures

Defining the measures

In the literature treating IRR it is a well-known fact that there does not exist a unified measure of IRR (Wolf (1969), Iwakuma and Hibiki (2015), Ozdemir and Sudarsana (2016)). An interest rate sensitive instrument's IRR could be viewed as the risk that the instrument's present value changes due to shifting interest rates. However, the owner of the instrument could also be concerned about the effect that the same interest rates shifts could have on the interest income or expense over a foreseeable short period, e.g. one year. When translated into balance sheet measures of IRR these classes of measures are called Economic Value (EV) measures and Earnings-based measures. The two groups of measures can be used in tandem to reflect the different impacts a change in interest rates can have on both the size and present value of future cash flows.

The EV class of measures can itself be divided into two different classes, Economic Value of Equity (EVE) and earnings-adjusted EV (BCBS, 2016b). To understand the difference we state the classic balance sheet equation that relates equity, assets and liabilities to each other as

$$\text{Assets} = \text{Equity} + \text{Liabilities}.$$

EVE and earnings-adjusted EV differ in the treatment of equity. EVE measures risk as the present value change of assets less the present value change of liabilities when

[2] We thank Magnus Hanson at Svenska Handelsbanken AB for this example.

interest rates changes. This way of treating equity is similar to the way it is treated on a company's balance sheet. However, since the company finances its assets with both non-equity liabilities and the equity liability we could assume that some assets are bought to specifically hedge the equity liability. One could then argue that the equity liability should be included so that the interest rate risk from that part of the asset portfolio is cancelled, this is the earnings-adjusted EV (BCBS, 2016b).

A common earnings-based measure is Net Interest Income (NII), for which a decision about how to treat equity has to be made as well. The NII measure excluding equity (no servicing cost) is often referred to as Commercial NII (Bessis, 2011). In this thesis equity is excluded from both the EV and NII measure, resulting in an EVE measure and a commercial 1-year NII measure, hereafter simply referred to as NII. For NII, we also have to decide if we should discount the cash flows taking the time value of money into account or if this should be disregarded.

When the measures have been defined, a decision has to be made regarding the balance sheet's evolution over time. BCBS (2016b) lists the three possible choices as

(i) a run-off balance sheet where existing assets and liabilities are not replaced as they mature, except to the extent necessary to fund the remaining balance sheet.

(ii) a static balance sheet where total balance sheet size and shape is maintained by assuming like-for-like replacement of assets and liabilities as they run off.

(iii) a dynamic balance sheet where future business expectations are incorporated.

Note that since EVE is defined as the present value of assets less liabilities with no rate or term applied to the equity itself, it is a run-off or gone concern perspective in the sense that only existing assets and liabilities are considered. As was mentioned above, the NII measure is generally focused on shorter time horizons, typically one to three years. Thus, it can be viewed as the short to medium term vulnerability of the bank to IRR, assuming the bank is able to continue operating during the measurement's time horizon, a so-called going concern viewpoint. In contrast to EVE, NII measures may assume any of the three balance sheet types above (BCBS, 2016b).

A simple example

After this short review the reader might ask why different measures are used? To illustrate this we show the problem a hypothetical bank faces when trying to manage NII and EVE simultaneously by comparing two simple balance sheet compositions. We assume a setting in which the hypothetical bank can choose between investing in two assets

(i) an overnight account (O/N), i.e. an account whose interest rate is repriced on a daily basis. This account initially pays a 2% interest rate per annum.

(ii) a perpetual (infinite maturity) bond with a fixed coupon of 6% per annum. The bond initially trades at par, i.e. the current price is equal to the bond's face value.

On the liability side it is assumed that the bank has borrowed USD 80 at the O/N account with an initial interest rate of 2% per annum and USD 20 by issuing equity (disregarded in the EVE and NII calculation), with no possibility of changing this composition. The goal of the balance sheet to the left in figure 2.1, balance sheet #1, is to choose an asset structure that minimizes the volatility in NII. To do this we want all *net* cash flows to be fixed during the first year, this is achieved by investing USD 80 in the O/N account and the remaining USD 20 in the perpetual bond. The goal of the right balance sheet, balance sheet #2, in figure 2.1 is to minimize EVE volatility. This is achieved by investing USD 100 in the O/N account since the present value of the O/N is very insensitive to interest rate changes[1].

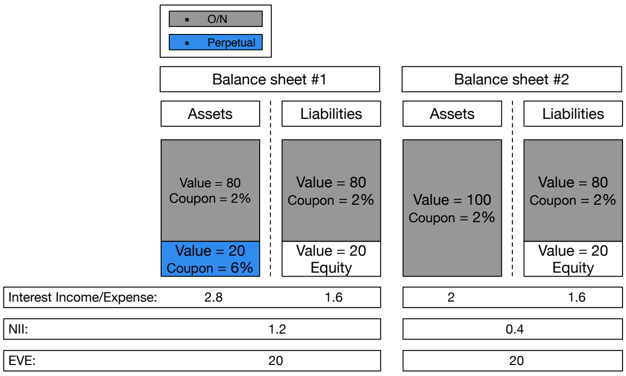


Figure 2.1: Two simple balance sheets for protecting NII & EVE respectively

To illustrate how the balance sheets' NII and EVE are affected by an interest rate change it is assumed that the interest rate curve is shifted +100bps for all maturities. After the shift, all O/N balance sheet items interest rate are repriced, while the perpetual bond's coupon remains unchanged. Thus, balance sheet #1 is not affected in NII due to the equal amount of assets and liabilities that reprice. However, the remaining perpetual bond's discounted value does change leading to a change in EVE[2]. Hence, balance sheet #1 is sensitive in EVE but not in NII. A similar argument holds for balance sheet #2 where USD 20 more assets than liabilities reprice, leading to a change in NII whereas EVE remains unchanged after the interest rate shift. The balance sheets and risk measures after the interest rate curve shift are shown in figure 2.2.

[1]If the bank can choose to withdraw the money each day we can view the O/N account as a bond with a one day maturity, hence the discount factor will be ~ 1 assuming reasonable interest rates

[2]The perpetual bond with a coupon yield of 6% and new discount rate of 7% has a present value of $PV = \int_0^\infty CFe^{-r\times t}dt = \int_0^\infty 0.06e^{-0.07\times t}dt = \frac{0.06}{0.07} \approx 0.857$

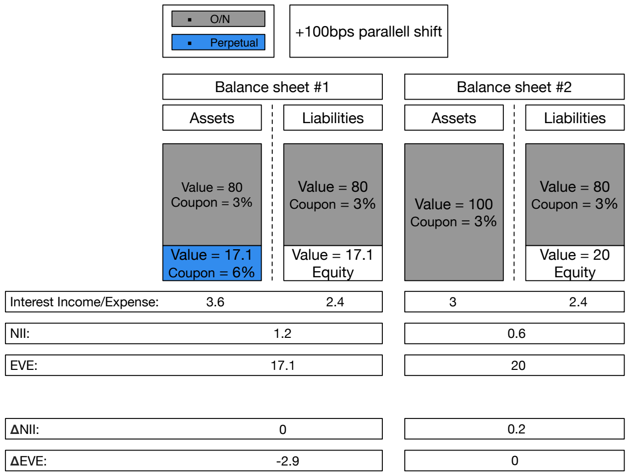


Figure 2.2: Two simple balance sheets for protecting NII & EVE respectively after +100 bps interest rate curve shift

2.2 IRRBB

The Basel Committee for Banking Supervision (BCBS) is part of the Bank for International Settlements (BIS), an organization for central banks. The BCBS proposes standards for the supervision of banks, which most national supervisors[1] then adopt with some local variations. The BCBS is most known for publishing the Basel Accords, which are conveniently named Basel I, II and III. Basel I was published in 1988, proposing a minimum capital ratio for banks in the member countries and was later amended a couple of times during the years following its introduction (BCBS, 2016a). Basel II was introduced in 2004 and contained the three so-called pillars, which can be summarized as

Pillar 1. Defining the minimum capital requirements for banks.

Pillar 2. Practices for how supervisors should review and evaluate banks' compliance with regulation, e.g. by setting standards for internal models.

Pillar 3. Disclosure practices that govern which risk metrics banks have to make publicly available.

Following the financial crisis of 2007-2008 Basel III was introduced, with the first proposals being published in 2010 (BCBS, 2016a). The current IRR standards is part of Basel II but will soon be replaced by the Interest Rate Risk in the Banking Book (IRRBB). IRRBB is expected to be implemented in 2018 and is a part of Basel III (BCBS, 2016b). Unless otherwise mentioned, information in this section refers to the latest version of IRRBB, see BCBS (2016b). Before being able to focus on the specifics of IRRBB it is necessary to explain the concept of a banking book. The banking book and the trading book are accounting definitions that are used to classify different assets and liabilities. Traditional commercial banking products, e.g. deposits and retail loans, are usually classified as belonging to the banking book whereas more actively traded assets and liabilities, e.g.

[1]In Sweden the national supervisor is Finansinspektionen.

equities held for market making, belong to the trading book (Bessis, 2011). Assets and liabilities in the banking book can be difficult to mark-to-market, are generally held to maturity and therefore tend to be valued according to accounting principles or marked-to-model.

The Basel II IRR standards, *Principles for the management and supervision of interest rate risk*, belongs to Pillar 2 and requires banks to be capable of measuring IRR to parallel interest rate shifts using both earnings and EV approaches. A risk threshold exists for EV but not for NII. If a decline in EV from a prescribed interest rate shift of 200bps exceeds the threshold, the national supervisor should take "remedial actions". In the original IRRBB proposal the BCBS suggested changing IRRBB management to a Pillar 1 approach. Thus requiring minimum capital to be held to cover IRR, which would have been computed using a standardized approach. However, this approach was heavily criticized by the industry, which emphasized the heterogeneous nature of banking book instruments, arguing that these are not amenable to standardization[1]. The finalized proposal instead contains an "enhanced" Pillar 2 approach (with some Pillar 3 elements) and a standardized Pillar 1 framework "which supervisors could mandate their banks to follow, or a bank could choose to adopt" (BCBS, 2016b).

Out of interest for this thesis is the trade-off between EVE and NII. In the IRRBB consultative document, the BCBS acknowledges the risk of unintended consequences if the focus is solely on EVE, saying that "there is a trade-off between optimal duration of equity and earnings stability" (BCBS, 2015). In addition to this the committee notes that most commercial banks focus on earnings-based measures for IRRBB management, while regulators tend to focus on economic value measures. The IRRBB proposal outlines several principles that banks should comply with. One principle of particular interest is

> **Principle 3:** "The banks' risk appetite for IRRBB should be articulated in terms of the risk to both economic value and earnings. Banks must implement policy limits that target maintaining IRRBB exposures consistent with their risk appetite".

As we will illustrate later in this thesis, the two measures are interconnected. Thus if a bank chooses a policy limit for one of the two measures, this will imply the limits that are possible for the other measure. Another interesting difference is an emphasis on a wider range of interest rate scenarios than the current standards

> **Principle 4:** "Measurement of IRRBB should be based on outcomes of both economic value and earnings-based measures, arising from a wide and appropriate range of interest rate shock and stress scenarios".

Even if the committee does not require the implementation of the standardized approach it is of interest since the approach is an approved model for a bank to measure IRR and several of its components have been mandated in the Pillar 2 part. Before proceeding with a short description of the standardized approach we note that **Principle 8** of the Pillar 2 part requires banks to

- Exclude equity from the computation.
- Compute EVE risk for a run-off balance sheet, assuming no new business.

[1] The various comments received by the committee can be found on the committee's website.

- Compute NII over a 1-year period assuming a constant balance sheet, where maturing assets are rolled into equivalent new assets.

For the EVE part the banks are also allowed to discount cash flows using a risk-free zero-coupon rate if commercial margins and other spreads are removed from cash flows. With regards to the standardized approach, both the measurement of NII and EVE involves a so-called gap-analysis for quantifying risks. This means that asset and liability cash flows are slotted into time buckets based on the first date the instrument is rate sensitive to, the repricing date. Given a decision regarding commercial margins and other spread components, the EVE calculations, for an interest rate scenario i, are straightforward

$$\begin{aligned} EVE_j =& \sum_{k=1}^{K} CF_j(t_k)e^{-R_j(0,t_k)t_k} = \sum_{k=1}^{K} CF_j(t_k)DF_j(t_k) \\ \Delta EVE_i =& EVE_0 - EVE_i. \end{aligned} \tag{2.2.1}$$

Where $j = 0$ is the current interest rate term structure, $CF_j(t_k)$ is the net cash flow of instruments that reprice in the time bucket t_k, $R_j(0, t_k)$ the interest rate between 0 and t_k, with both being computed under scenario j. There are six prescribed scenarios (the actual sizes of which are dependent on the currency under consideration)

(i) parallel shock up,

(ii) parallel shock down,

(iii) steeper curve shock,

(iv) flatter curve shock,

(v) short rates up,

(vi) long rates down.

IRRBB's NII measure is a *present value* measure of NII. Formulas for NII and NII risk for the unknown NII (during the next year) of an asset that reprices at t_1 is presented below, the period of measurement is t_0 to t_2

$$\begin{aligned} NII(t_0, t_1, t_2) &= A[e^{R(t_0,t_1,t_2)(t_2-t_1)} - 1] \\ &= A[e^{R(t_0,t_2)t_2 - R(t_0,t_1)t_1} - 1] \end{aligned} \tag{2.2.2}$$

$$\begin{aligned} PV(NII(t_0, t_1, t_2)) &= NII(t_0, t_1, t_2)e^{-R(t_0,t_2)t_2} \\ &= A[e^{-R(t_0,t_1)t_1} - e^{-R(t_0,t_2)t_2}]. \end{aligned} \tag{2.2.3}$$

Where A is the cash flow repricing at t_1 and $R(t_0, t_1, t_2)$ is the forward rate at t_0 between t_1 and t_2. When calculating NII after an interest rate shock, a *parallel* interest rate shock of size ΔR is applied

$$PV(NII(t_0, t_1, t_2))_{shocked} = A[e^{-(R(t_0,t_1)+\Delta R)t_1} - e^{-(R(t_0,t_2)+\Delta R)t_2}]. \tag{2.2.4}$$

NII at risk to a shock is expressed as

$$\Delta PV(NII(t_0,t_1,t_2)) = PV(NII(t_0,t_1,t_2))_{shocked} - PV(NII(t_0,t_1,t_2)). \tag{2.2.5}$$

Two things are worth noting, firstly the asset has a known return until t_1, and thus the formulas above only account for the risk in the unknown, expected return and not the present value change of known NII. Secondly, equation 2.2.2 is only correct if parallel interest rate shocks are used and forward rates are assumed to be implied from the zero-coupon curve since it does then not matter if the instrument is rolled over several times or only once to t_2. Whilst not described here, the framework also contains formulas for the computation of IRR for non-standardized instruments such as non-maturing deposits.

Chapter 3

Theory and Previous Research

This chapter introduces the mathematical theory and notation that will be used in the following chapters. We end with section 3.6 in which we present previous research about studying NII and EVE simultaneously. If nothing else is stated, we assume a bond market free of arbitrage and the existence of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$, where $\mathbb{Q}$ denotes the martingale measure. We will denote the physical measure as $\mathbb{P}$, the measure under which we observe the actual realization of bond prices. All interest rates are expressed as continuously compounded. For a more thorough description of continuous time models in finance see Björk (2009) and specifically for their use in fixed income modeling see Brigo and Mercurio (2007).

3.1 Interest rates and bonds

A non-defaultable zero-coupon bond (ZCB) with maturity at time T is a financial contract that, with certainty, pays its owner 1 at time T. The price of the ZCB at time $t < T$ is denoted as $p(t, T)$ and obviously $p(T, T) = 1$. We view the forward rate as the interest rate that can be contracted for a future period today. Using the ZCB we define the (continuously compounded) forward rate between t and T at s as

$$R(s, t, T) = -\frac{\log p(s, T) - \log p(s, t)}{T - t}. \tag{3.1.1}$$

If $t = s$ we call the forward rate the zero-coupon rate and denote it as $R(t, T)$. In this thesis the zero-coupon curve at time t describes the mapping

$$T \mapsto R(t, T), \ t < T.$$

A continuous zero-coupon curve is a theoretical concept since there does not exist quoted bonds for all maturities in the market. Instead it is approximated using available market quotes of e.g. bonds and swaps. Figure 3.1 shows the US zero-coupon curve bootstrapped[2] semi-annually between 2007-04-30 and 2017-04-28 using data from the U.S. Treasury. As

[2]More about this in chapter 5

can be seen in the figure, the zero-coupon curve's relative level, slope and curvature varies over time.

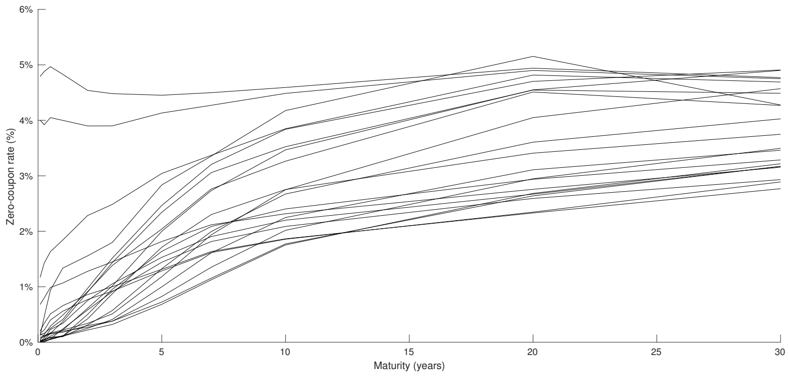


Figure 3.1: Semi-annual zero-coupon curve bootstrapped using data from the U.S. Treasury between 2007-04-30 and 2017-04-28

A fixed-rate coupon bond is a bond that at predetermined points in time, $(t_1, t_2, \ldots, t_n)$, makes predetermined coupon payments, $(c_1, c_2, \ldots, c_n)$. Its price, $\bar{p}_c(t,T)$, at time $t \leq t_i$ is simply a linear combination of ZCB prices and can be written as

$$\bar{p}_c(t,T) = p(t,T) + \sum_{i:t_i \geq t}^{n} c_i p(t,t_i). \tag{3.1.2}$$

Out of particular interest in this thesis is the theoretical construct of a bond paying a continuous coupon. This can be thought of as a bond that pays $\bar{c}(s)ds$ over a small time interval $[s, s+ds]$. Similarly to equation 3.1.2 we have that the price at t, $p_c(t,T)$, of a bond that pays the coupon $\bar{c}(s)$ at $s \in [t,T]$ is

$$p_c(t,T) = p(t,T) + \int_t^T \bar{c}(s) p(t,s) ds. \tag{3.1.3}$$

If the coupon bond is currently trading at a price of 1 it is said to trade at par and we call it a par-coupon bond. Assuming that $\bar{c}(s)$ is constant between t and T we can solve for the constant par coupon, denoted $c(t,T)$, at t as

$$c(t,T) = \frac{1 - p(t,T)}{\int_t^T p(t,s) ds}. \tag{3.1.4}$$

Another concept utilized later on is the *forward* par coupon, $c(u,t,T)$, which we define as the constant coupon we can, without initial cost, contract at $u < t$ to receive between t and T for a guaranteed cost of 1 at t. The coupon could easily be solved for by introducing the t-forward measure but to avoid this we note that we can simply discount the cash flows in equation 3.1.3 to u. Similarly to equation 3.1.4 we get that

$$c(u,t,T) = \frac{p(u,t) - p(u,T)}{\int_t^T p(u,s) ds}. \tag{3.1.5}$$

Before proceeding we note that obviously $c(t,t,T) = c(t,T)$ since $p(t,t) = 1$.

3.2 Short-rate and ATS models

Having defined these simple concepts the natural follow-up question is how do we determine the price of a bond? To do this we define the short rate. The short rate is a theoretical rate which describes the annualized rate in a money account over an infinitesimal period dt. Under the martingale measure $\mathbb{Q}$ the short rate, $r(t)$, is modeled as the solution to a stochastic differential equation (SDE) of the form

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^{\mathbb{Q}}(t), \tag{3.2.1}$$

where $\mu(t, r(t))$ and $\sigma(t, r(t))$ are functions for the drift and diffusion coefficients respectively and $W(t)$ is a $\mathbb{Q}$-Wiener process. The price of a T-maturity ZCB at t is then given by

$$p(t, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right]. \tag{3.2.2}$$

In this thesis we will use a special class of short rate models called affine term structure (ATS) models. ATS models can be preferable to work with since they provide closed-form expressions of bond prices, which is good from a computational point of view. A model is said to have an affine term structure if ZCB prices are given by

$$p(t, T) = F(t, r(t); T),$$

where F can be written as

$$F(t, r; T) = e^{A(t,T)-B(t,T)r}. \tag{3.2.3}$$

It turns out that the model admits an ATS solution if the drift and squared diffusion term in equation 3.2.1 are affine functions that can be written as

$$\begin{aligned} \mu(t, r) &= \alpha(t)r + \beta(t), \\ \sigma^2(t, r) &= \gamma(t)r + \delta(t), \end{aligned} \tag{3.2.4}$$

where α, β, γ and δ are deterministic functions. The deterministic functions $A(t, T)$ and $B(t, T)$ can be found by solving the following system of equations

$$\begin{cases} B_t(t, T) + \alpha(t)B(t, T) - \frac{1}{2}\gamma(t)B^2(t, T) = -1, \\ \qquad\qquad\qquad\qquad\qquad\qquad B(T, T) = 0, \end{cases} \tag{3.2.5}$$

$$\begin{cases} A_t(t, T) = \beta(t)B(t, T) - \frac{1}{2}\delta(t)B^2(t, T), \\ A(T, T) = 0, \end{cases} \tag{3.2.6}$$

where $A_t(.)$ and $B_t(.)$ denotes the derivatives with respect to t (Björk, 2009). The first model we study is the single-factor Vasicek model, which under $\mathbb{Q}$ is specified as

$$dr(t) = \kappa(\bar{\theta} - r(t)) + \sigma dW(t). \tag{3.2.7}$$

The SDE admits the solution[1]

$$r(t)|\mathcal{F}_s \sim \mathcal{N}\left(\bar{\theta}(1 - e^{-\kappa(t-s)}) + e^{-\kappa(t-s)}r(s), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-s)})\right), \; s < t, \tag{3.2.8}$$

[1]Can be solved by using a "trick" and first setting $Y(t) = (\kappa(\bar{\theta} - r(t))$ and then using Itô's lemma on $e^{\kappa t}Y(t)$

where $\mathcal{N}(\mu, \sigma^2)$ denotes the Normal distribution with expected value μ and variance σ^2. We note that

$$\lim_{t\to\infty} r(t)|\mathcal{F}_s \sim \mathcal{N}\left(\bar{\theta}, \frac{\sigma^2}{2\kappa}\right),$$

and thus we can view $\bar{\theta}$ as a long-run mean and κ as a parameter that determines the speed of convergence to the limiting distribution.

Using equations 3.2.5 and 3.2.6 together with equation 3.2.7 we find the explicit expression for the T-ZCB price at t as

$$\begin{aligned} p(t,T) &= e^{A(t,T)-B(t,T)r(t)}, \\ B(t,T) &= \frac{1}{\kappa}(1-e^{-\kappa(T-t)}), \\ A(t,T) &= \frac{\gamma(B(t,T)-(T-t))}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa}, \end{aligned} \tag{3.2.9}$$

where $\gamma = \kappa^2\bar{\theta} - \sigma^2/2$. The Vasicek model has been criticized due to the fact that it allows negative rates with a positive probability. In light of recent years unorthodox monetary policies with market rates in many currencies at levels below zero, this is not necessarily a drawback of the model. Nevertheless, the model has other drawbacks one being that the model is not able to perfectly fit the current term structure, which can be critical if the model is going to be used to price derivatives (Brigo and Mercurio, 2007). For the purposes of this thesis this is of minor importance. However, what might be a more serious disadvantage is the limited types of zero-coupon curve shifts that can be achieved using a single-factor model. To see this we express equation 3.1.1 using equation 3.2.9 as

$$R(t,T) = R(0,t,T) = \frac{B(t,T)r(t) - A(t,T)}{T-t}.$$

We then have that for $s < t < S < T$

$$\begin{aligned} Cov^{\mathbb{Q}}[R(t,T), R(t,S)|\mathcal{F}_s] &= \mathbb{E}^{\mathbb{Q}}[R(t,T)R(t,S)|\mathcal{F}_s] - \mathbb{E}^{\mathbb{Q}}[R(t,T)|\mathcal{F}_s]\mathbb{E}^{\mathbb{Q}}[R(t,S)|\mathcal{F}_s] \\ &= \frac{B(t,T)B(t,S)}{(T-t)(S-t)}(\mathbb{E}^{\mathbb{Q}}[r^2(t)|\mathcal{F}_s] - \mathbb{E}^{\mathbb{Q}}[r(t)|\mathcal{F}_s]^2) \\ &= \frac{B(t,T)B(t,S)}{(T-t)(S-t)} Var^{\mathbb{Q}}(r(t)|\mathcal{F}_s) \\ &= \sqrt{Var^{\mathbb{Q}}(R(t,T)|\mathcal{F}_s)}\sqrt{Var^{\mathbb{Q}}(R(t,S)|\mathcal{F}_s)}, \end{aligned}$$

and thus the correlation between $R(t,T)$ and $R(t,S)$ is equal to 1. By looking at figure 3.1 one could deduct that this does not seem to be true for the actual zero-coupon curve. Brigo and Mercurio (2007) contends that single-factor models can still be useful for some risk management purposes, examples being payoffs depending on rates that are highly correlated, e.g. the 3-month and 6-month rate. To avoid perfect correlations between different maturities one can use a multi-factor model. In multi-factor models the short rate r is modeled as being driven by several sources of randomness, here called state variables. As in Bolder (2001) we denote these as $y_1, ..., y_n$, with the short rate r given by

$$r(t) = \sum_{i=1}^{n} y_i(t), \tag{3.2.10}$$

where the SDEs governing the state variables are modeled as

$$\begin{aligned} dy_1(t) &= \mu_1(t, y_1)dt + \sum_{i=1}^{n} \rho(y_1, y_i, t)dW_1^{\mathbb{Q}}(t), \\ &\vdots \\ dy_n(t) &= \mu_n(t, y_n)dt + \sum_{i=1}^{n} \rho(y_n, y_i, t)dW_n^{\mathbb{Q}}(t). \end{aligned} \tag{3.2.11}$$

where $W_i^{\mathbb{Q}}(t)$ is a scalar $\mathbb{Q}$-Weiner process and $d\langle W_i, W_j \rangle = 0$. This leads to two natural questions, how many state variables should we use and how should we model $\mu_i(.,.)$ and $\rho(.,.)$? Numerous studies have been done on this topic and a common result is that at least three factors are needed (Van Deventer et al., 2013). Remember from section 2.2 that the prescribed interest rate shifts for NII and EVE were different in IRRBB's standardized model. For NII only parallel shifts are used, whereas for EVE six scenarios corresponding to three different types of shifts are used. Therefore, we limit our investigation to single-, two- and three-factor models in this thesis.

3.3 The multi-factor Vasicek model

A natural extension of the single-factor Vasicek model defined in equation 3.2.7 is the multi-factor Vasicek model. Using the notation from equation 3.2.11 we set

$$\begin{aligned} \mu_i(t, y_i) &= \kappa_i(\bar{\theta}_i - y_i(t)), \\ \rho(y_i, y_j, t) &= \sigma_{ij}, \end{aligned} \tag{3.3.1}$$

where κ_i, $\bar{\theta}_i$ and σ_{ij} are constants for all i and j in $1, \dots, n$. Thus the n-factor Vasicek model can be written as

$$\begin{aligned} r(t) &= \sum_{i=1}^{n} y_i(t), \\ dy_1(t) &= \kappa_1(\bar{\theta}_1 - y_1(t))dt + \sum_{i=1}^{n} \sigma_{1i} dW_1^{\mathbb{Q}}(t), \\ &\vdots \\ dy_n(t) &= \kappa_n(\bar{\theta}_n - y_n(t))dt + \sum_{i=1}^{n} \sigma_{ni} dW_n^{\mathbb{Q}}(t). \end{aligned} \tag{3.3.2}$$

It can be shown, see e.g. Bolder (2001), that the n-factor Vasicek model is also an ATS model with ZCB prices given by

$$\begin{aligned} p(t,T;y_1,\dots,y_n) &= e^{A(t,T)-\sum_{i=1}^n B_i(t,T)y_i}, \\ B_i(t,T) &= \frac{1}{\kappa_i}(1-e^{-\kappa_i(T-t)}), \\ A(t,T) &= \sum_{i=1}^n \gamma_i \frac{B_i(t,T)-(T-t)}{\kappa_i^2} - \frac{\sigma_i^2 B_i^2(t,T)}{4\kappa_i} \\ &+ \sum_{i\neq j} \frac{\sigma_{ij}}{2\kappa_i\kappa_j}\left(T-t-B_i(t,T)-B_j(t,T)+\frac{1}{\kappa_i+\kappa_j}(1-e^{-(\kappa_i+\kappa_j)(T-t)})\right), \end{aligned} \tag{3.3.3}$$

where $\gamma_i = \kappa_i^2\bar{\theta}_i - \sigma_i^2/2$. As can be seen the solution is similar to the single-factor case in equation 3.2.9, with the double sum in $A(t,T)$ resulting from the correlation between state variables. By assuming non-zero correlation between state variables it is possible to achieve more complicated volatility structures (Brigo and Mercurio, 2007). However, the numerical optimization algorithm used to fit the model becomes more complex and unstable (Bolder, 2001). Since no complex derivatives will be priced in this thesis we choose to assume independence between state variables and thus set $\sigma_{ij}=0$ if $i\neq j$. Nevertheless, having independent state variables does not mean that the correlation structure between different points on the zero-coupon curve remains the same as in the single-factor case. Similarly to the single-factor case we have that

$$R(t,T) = \frac{-A(t,T)+\sum_{i=1}^n B_i(t,T)y_i(t)}{T-t},$$

and for $s<t<S<T$ we get that

$$\begin{aligned} Cov^{\mathbb{Q}}[R(t,T),R(t,S)|\mathcal{F}_s] &= \mathbb{E}^{\mathbb{Q}}[R(t,T)R(t,S)|\mathcal{F}_s] - \mathbb{E}^{\mathbb{Q}}[R(t,T)|\mathcal{F}_s]\mathbb{E}^{\mathbb{Q}}[R(t,S)|\mathcal{F}_s] \\ &= \sum_{i=1}^n \frac{B_i(t,T)B_i(t,S)}{(T-t)(S-t)} Var^{\mathbb{Q}}(y_i(t)|\mathcal{F}_s), \end{aligned}$$

using which we also have that

$$\begin{aligned} Var^{\mathbb{Q}}(R(t,T)|\mathcal{F}_s) &= Cov^{\mathbb{Q}}[R(t,T),R(t,T)|\mathcal{F}_s] \\ &= \sum_{i=1}^n \frac{B_i^2(t,T)}{(T-t)^2} Var^{\mathbb{Q}}(y_i(t)|\mathcal{F}_s). \end{aligned} \tag{3.3.4}$$

Thus, we can see that the correlation between two points on the zero-coupon curve need not be equal to 1 as was the case with the single-factor model. Before proceeding we note that in case of forced independence the state variables distribution will be similar to equation 3.2.8 and we have that

$$y_i(t)|\mathcal{F}_s \sim \mathcal{N}\left(\bar{\theta}_i(1-e^{-\kappa_i(t-s)}) + e^{-\kappa_i(t-s)}y_i(s), \frac{\sigma_i^2}{2\kappa_i}(1-e^{-2\kappa_i(t-s)})\right), \; s<t. \tag{3.3.5}$$

3.4 Model calibration

There exists several methods with which one could estimate the parameters for a given short rate model. For a single-factor model one could assume that a point on the zero-coupon curve approximates the short rate, e.g. the three-month zero-coupon rate. For a multi-factor model this is not enough and one could assume that the other state variables correspond to economically sound variables, e.g. a long-term rate or inflation. If the model is going to be used for derivatives pricing one could use derivatives such as swaptions to make sure that the model replicates the markets volatility structure (Brigo and Mercurio, 2007). In this thesis we will use an alternative method called a Kalman filter to fit our models. The Kalman filter is useful in this setting since it does not force us to specify what each state variable should correspond to. Instead we estimate the parameters by assuming that we observe points on the zero-coupon curve over time and that these are driven by unobservable state variables. If nothing else is mentioned, this section is based on Bolder (2001) who provides a thorough description of the multi-factor Vasicek and CIR model and how estimation can be done using a Kalman filter. Since the algorithm is the same for a single-, two- and three-factor Vasicek model we only show the three factor setup below.

To begin with we note that when observing the actual evolution of the zero-coupon curve we are under the physical measure, $\mathbb{P}$, and when we have specified a short-rate model under $\mathbb{Q}$ we have specified the entire term structure Björk (2009). If we denote λ_i the market risk premium for state variable i and define θ_i as

$$\theta_i = \bar{\theta}_i + \frac{\sigma_i \lambda_i}{\kappa_i}, \quad i = 1, \dots n,$$

we have that under $\mathbb{P}$ state variable i is governed by the dynamics

$$dy_i(t) = \kappa_i(\theta_i - y_i(t))dt + \sigma_i dW_i^{\mathbb{P}}(t).$$

It is then easy to see that under $\mathbb{P}$, y_i will have the following distribution (compare with equation 3.3.5)

$$y_i(t)|\mathcal{F}_s \sim \mathcal{N}\left(\theta_i(1 - e^{-\kappa_i(t-s)}) + e^{-\kappa_i(t-s)}y_i(s), \frac{\sigma_i^2}{2\kappa_i}(1 - e^{-2\kappa_i(t-s)})\right), \ s < t. \quad (3.4.1)$$

We now assume that we at regularly spaced points in time $t_1, \dots, t_N$, with $t_{j+1} - t_j = \Delta t$, have observed the vector $\bar{R}(t_i)$ of points on the zero-coupon curve as

$$\bar{R}(t_i) = [R(t_i, t_i + M_1), R(t_i, t_i + M_2), \dots, R(t_i, t_i + M_p)]^T, \ t_i \in \{t_1, \dots, t_N\}. \quad (3.4.2)$$

To ease notation later on we note that the B_is and A in equation 3.3.3 are time invariant and henceforth we write them as

$$B_i(t_i, t_i + M_j) = B_i(M_j),$$
$$A(t_i, t_i + M_j) = A(M_j).$$

The measurement system describes the relationship between the observed zero-coupon rates and the state variables. Using equations 3.1.1 and 3.3.3 we can express equation 3.4.2 as

$$\bar{R}(t_i) = -\begin{bmatrix} \frac{A(M_1)}{M_1} \\ \vdots \\ \frac{A(M_p)}{M_p} \end{bmatrix} + \begin{bmatrix} \frac{B_1(M_1)}{M_1} & \frac{B_2(M_1)}{M_1} & \frac{B_3(M_1)}{M_1} \\ \vdots & \vdots & \vdots \\ \frac{B_1(M_p)}{M_p} & \frac{B_2(M_p)}{M_p} & \frac{B_3(M_p)}{M_p} \end{bmatrix} \begin{bmatrix} y_1(t_i) \\ y_2(t_i) \\ y_3(t_i) \end{bmatrix} + \begin{bmatrix} \nu_1(t_i) \\ \nu_2(t_i) \\ \nu_3(t_i) \end{bmatrix} \tag{3.4.3}$$
$$= -A + By(t_i) + \nu(t_i),$$

where $\nu(t_i)$ represents a noise term included in our observations and could be thought of relating to e.g. bid-ask spreads or data-entry errors (Bolder, 2001). We assume that $\nu_i(t_k)$ has the following distribution

$$\nu(t_i) \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{bmatrix} \right) = \mathcal{N}(0, R). \tag{3.4.4}$$

We now need to state the transition system that describes the distribution of the state variables under $\mathbb{P}$ between time points t_i and t_{i+1}. Utilizing equation 3.4.1 we have that

$$\begin{bmatrix} y_1(t_{i+1}) \\ y_2(t_{i+1}) \\ y_3(t_{i+1}) \end{bmatrix} = \begin{bmatrix} \theta_1(1 - e^{-\kappa_1 \Delta t}) \\ \theta_2(1 - e^{-\kappa_2 \Delta t}) \\ \theta_3(1 - e^{-\kappa_3 \Delta t}) \end{bmatrix} + \begin{bmatrix} e^{-\kappa_1 \Delta t} & 0 & 0 \\ 0 & e^{-\kappa_2 \Delta t} & 0 \\ 0 & 0 & e^{-\kappa_3 \Delta t} \end{bmatrix} \begin{bmatrix} y_1(t_i) \\ y_2(t_i) \\ y_3(t_i) \end{bmatrix} + \begin{bmatrix} \epsilon_1(t_{i+1}) \\ \epsilon_2(t_{i+1}) \\ \epsilon_3(t_{i+1}) \end{bmatrix}$$
$$= C + Fy(t_i) + \epsilon(t_{i+1}), \tag{3.4.5}$$

where

$$\epsilon(t_{i+1})|\mathcal{F}_{t_i} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sigma_1^2}{2\kappa_1}(1 - e^{-2\kappa_1 \Delta t}) & 0 & 0 \\ 0 & \frac{\sigma_2^2}{2\kappa_2}(1 - e^{-2\kappa_2 \Delta t}) & 0 \\ 0 & 0 & \frac{\sigma_3^2}{2\kappa_3}(1 - e^{-2\kappa_3 \Delta t}) \end{bmatrix} \right)$$
$$\sim \mathcal{N}(0, Q).$$

The Kalman filter works as a recursive algorithm where we make an a priori estimate of the transition system. Once we observe the actual state of the measurement system we update our estimate of the transition system. Using this updated estimate we can compute the following a priori estimate of the transition system. The full algorithm is described below.

Step 0

At t_0 we make an initial educated "guess" of the transition system at t_1. Since no previous information is available we use the unconditional mean and variance of $y(t_1)$

$$\mathbb{E}^{\mathbb{P}}[y(t_1)] = [\theta_1, \theta_2, \theta_3]^T,$$
$$Var^{\mathbb{P}}(y(t_1)) = \begin{bmatrix} \frac{\sigma_1^2}{2\kappa_1} & 0 & 0 \\ 0 & \frac{\sigma_2^2}{2\kappa_2} & 0 \\ 0 & 0 & \frac{\sigma_3^2}{2\kappa_3} \end{bmatrix}. \tag{3.4.6}$$

Step 1

Using the linearity of equation 3.4.3 we get that the conditional prediction of the measurement system and the conditional variance of this prediction is

$$\mathbb{E}^{\mathbb{P}}[\bar{R}(t_i)|\mathcal{F}_{t_{i-1}}] = -A + B\mathbb{E}^{\mathbb{P}}[y(t_i)|\mathcal{F}_{t_{i-1}}],$$
$$Var^{\mathbb{P}}(\bar{R}(t_i)|\mathcal{F}_{t_{i-1}}) = BVar^{\mathbb{P}}(y(t_i)|\mathcal{F}_{t_{i-1}})B^T + R. \tag{3.4.7}$$

Step 2

We can now compute the measurement error of the conditional prediction as

$$v(t_i) = \bar{R}(t_i) - \mathbb{E}^{\mathbb{P}}[\bar{R}(t_i)|\mathcal{F}_{t_{i-1}}]. \tag{3.4.8}$$

We can also compute the so-called Kalman gain matrix

$$K(t_i) = Var^{\mathbb{P}}(y(t_i)|\mathcal{F}_{t_{i-1}})B^T Var^{\mathbb{P}}(\bar{R}(t_i)|\mathcal{F}_{t_{i-1}})^{-1}, \tag{3.4.9}$$

which can be thought of as determining the relative importance of the measurement error in equation 3.4.8 when updating our prediction of the transition system.

Step 3

The a posteriori estimate of the transition system and its variance is found, using equations 3.4.8 and 3.4.9, to be

$$\mathbb{E}^{\mathbb{P}}[y(t_i)|\mathcal{F}_{t_i}] = \mathbb{E}^{\mathbb{P}}[y(t_i)|\mathcal{F}_{t_{i-1}}] + K(t_i)v(t_i),$$
$$Var^{\mathbb{P}}(y(t_i)|\mathcal{F}_{t_i}) = (I - K(t_i)B)\,Var^{\mathbb{P}}(y(t_i)|\mathcal{F}_{t_{i-1}}), \tag{3.4.10}$$

where I is the identity matrix.

Step 4

The final step is making an a priori estimate of the transition system and its variance by

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[y(t_{i+1})|\mathcal{F}_{t_i}] &= C + F\mathbb{E}^{\mathbb{P}}[y(t_i)|\mathcal{F}_{t_i}], \\ Var^{\mathbb{P}}(y(t_{i+1})|\mathcal{F}_{t_i}) &= Var^{\mathbb{P}}(y(t_i)|\mathcal{F}_{t_{i-1}}) - FVar^{\mathbb{P}}(y(t_i)|\mathcal{F}_{t_i})F^T + Q. \end{aligned} \tag{3.4.11}$$

Steps 1 to 4 are then repeated up to t_N. Using the assumption that the prediction errors of the measurement system are normally distributed a log-likelihood function can be constructed as

$$\begin{aligned} \log[\mathcal{L}(\Theta)] = -\frac{Np\log[2\pi]}{2} - \frac{1}{2}\sum_{i=1}^{N} \log[|Var^{\mathbb{P}}(\bar{R}(t_i)|\mathcal{F}_{t_{i-1}})|] \\ - \frac{1}{2}\sum_{i=1}^{N} v^T(t_i)Var^{\mathbb{P}}(\bar{R}(t_i)|\mathcal{F}_{t_{i-1}})^{-1}v(t_i)) \end{aligned} \tag{3.4.12}$$

To find the (in the log-likelihood sense) optimal parameters we maximize equation 3.4.12 numerically in R, a programming package for statistical computing, using a non-linear optimization package called *nlminb*. The implementation in R was built on Goh (2013). Since the main purpose of this thesis is not the statistical estimation of parameters in the Vasicek model we will not investigate how good our estimate is. The interested reader can see e.g. Babbs and Ben Nowman (1999) for a discussion on this topic. However, similarly to Bolder (2001) we investigate how well the numerical optimization works on simulated data. We do this by simulating the evolution of the state variables over a ten-year period and each month computing the zero-coupon curve at the time points 1 month, 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years and 30 years. To simulate the state variables we use equation 3.4.5 with $\Delta t = 1/12^2$, i.e. we discretize each month into 12 parts. To compute the simulated zero-coupon curve we use every 12th simulation of the state variables and add a simulated measurement error with distribution as in equation 3.4.4, setting $r = 0.01^2$. This scheme is repeated 200 times and for every ten-year period the Kalman filter is applied on the simulated zero-coupon curves and the resulting log-likelihood function is maximized. From the 200 estimates we then compute means and standard deviations. The results for the 3-factor model can be seen in table 3.1. As can be seen, it works fairly well for some of the parameters but the θ_is estimates are a bit off, which implies that more than 200 simulations would need to be done for better convergence.

3.5 Expected shortfall

To define Expected Shortfall (ES) we first need to define Value-at-Risk (VaR). For a portfolio with value $V(t)$ at time t and $V(0)$ today, the P&L over the period can be written as $X = V(t) - V(0)$. We set our risk horizon to t and let L denote the loss

Table 3.1: Estimates from 200 simulations

Parameter	Actual Value	Mean	Standard Deviation
θ_1	0.0020	0.0083	0.0134
θ_2	0.0200	0.0121	0.0202
θ_3	0.0030	0.0124	0.0177
κ_1	0.6000	0.6588	0.2006
κ_2	0.0200	0.0201	0.0017
κ_3	0.4000	0.3575	0.1142
σ_1	0.0100	0.0124	0.0085
σ_2	0.0200	0.0193	0.0034
σ_3	0.0300	0.0271	0.0067
λ_1	0.8000	0.6253	0.5029
λ_2	-0.1400	-0.1497	0.1923
λ_3	-0.7100	-0.6399	0.3597

distribution, i.e. $L = -X$. $\text{VaR}_\alpha(X)$ is defined as the $(1-\alpha)$-quantile of L (Bessis, 2011). If the distribution function is strictly increasing and continuous this can be written as

$$\text{VaR}_\alpha(X) = F_L^{-1}(1-\alpha), \tag{3.5.1}$$

where F_L^{-1} is just the regular inverse of L's cumulative distribution function. According to BCBS (2015), VaR for EVE and NII[1] is the risk metric most commonly monitored by regulators for IRR. However, several of VaR's properties have been criticized. One example being that it lacks the subadditivity property, which roughly means that diversification need not always lower risk (Hult et al., 2012). ES is an extension of VaR, where ES_α is the average VaR below α. Formally, we define ES_α as

$$\text{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \text{VaR}_s(X)ds, \tag{3.5.2}$$

where typical levels of α are 1% and 5%. In an IRR setting, ES of EVE is currently monitored by a few regulators (BCBS, 2015). ES is a coherent risk measure, which means that it satisfies the subadditivity, monotonicity, translation invariance and positive homogeneity properties (Hult et al., 2012). This implies that ES is a convex risk measure, meaning that for $\lambda \in [0,1]$ we have that

$$\text{ES}(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\text{ES}(X_1) + (1-\lambda)\text{ES}(X_2). \tag{3.5.3}$$

As was mentioned in section 2.2, IRRBB prescribes the usage of different deterministically determined interest rate scenarios. Instead, to decide on a risk measure we take guidance from FRTB, the corresponding BCBS framework for risk in the trading book. In the new FRTB proposal BCBS has transitioned from VaR to ES and in light of this we choose to use ES as our preferred risk measure as well.

Later on we will compute ES from a simulated samples and hence we need an expression for an empirical estimate of it. If we denote the floor function $\lfloor . \rfloor$, Hult et al. (2012) shows

[1] Sometimes called Economic Value-at-Risk (EVaR) for EVE and Earnings-at-Risk (EaR) for NII.

that if we have a sample of n losses of X and order them such that $L_{1,n} \geq L_{2,n} \geq \cdots \geq L_{n,n}$ an empirical estimator of ES_α is

$$\widehat{\text{ES}}_\alpha(X) = \frac{1}{\alpha}\left(\sum_{i=1}^{\lfloor n\alpha \rfloor} \frac{L_{i,n}}{n} + \left(\alpha - \frac{\lfloor n\alpha \rfloor}{n}\right) L_{\lfloor n\alpha \rfloor+1,n}\right). \tag{3.5.4}$$

3.6 Models combining NII and EVE

As was mentioned in chapter 1 there does not exist a lot of research investigating the relationship between NII and EVE. The model that is closest to ours is presented in Memmel (2014) where NII and EVE, similarly to our model, is studied for a rolling portfolio of coupon bonds. The validity of this approach is based on Memmel (2008), where the NII sensitvity of German cooperative and savings banks is approximated as the interest income generated from a portfolio of rolling par-coupon bonds. The used method requires the following assumptions

- The portfolio's maturity structure remains constant over time. This means that as soon as a bond (bought or issued) matures the same amount is reinvested in a bond with the same time to maturity as the maturing bond initially had.
- Both the replacement process and the coupon payments are continuous, with the same fraction of the portfolio maturing at every point in time. Meaning that if the portfolio invests in bonds with an initial time to maturity of M years, $\frac{1}{M}$ of the portfolio matures every year. Alternatively expressed we reinvest $\frac{1}{M}dt$ of the portfolio over a small period of time dt.
- All bonds are non-defaultable and issued at par.

In the empirical setting in Memmel (2008) this is approximated using a monthly discretization. Each bank is then approximated as a weighted sum of several rolling portfolios with different maturities. Of course, this is a simplification of a real bank's asset and liability structure, disregarding features such as defaults and the usage of derivatives. Nevertheless, Memmel (2008) finds that the method works fairly well as an approximation of German banks' NII. In Memmel (2014) this model is expanded to also include an EVE measure. To get analytically tractable expressions an additional assumption is made, the zero-coupon curve is assumed to have been historically constant and thus all par-coupon bonds bought before today, denoted as $t = 0$, have yielded the same coupon. Using the same notation as equation 3.1.5 this means that

$$c(t, t + M) = c(0, M), \quad \forall t \leq 0.$$

We denote $S(M)$ the strategy that consists of investing in a rolling portfolio of continuously yielding, risk free, par-coupon bonds with maturity equal to M years. EVE is defined as the present value of the run-off portfolio. To find an expression for EVE we notice that during a small time period $[t, t + dt]$ where $t < M$ the portfolio strategy $S(M)$ pays its holder

$$(1 - t/M)c(0, M)dt + 1/M dt. \tag{3.6.1}$$

The first part coming from the coupons paid by bonds that have yet to mature at t and the second part coming from the bonds maturing at t[1]. EVE of the strategy $S(M)$ can hence be computed by discounting all cash flows from the portfolio and we have that

$$\text{EVE}(M) = \int_0^M \left[(1 - t/M)c(0, M) + 1/M\right] p(0, t)dt. \tag{3.6.2}$$

Memmel (2014) defines the base case for NII as the coupon payments the portfolio would have yielded during the first year provided the zero-coupon curve did not change during the first year. Hence we have that

$$\text{NII}(M) = \int_0^1 c(0, M)dt = c(0, M). \tag{3.6.3}$$

To investigate how NII and EVE of different portfolio strategies changes with different zero-coupon curve shifts, Memmel (2014) assumes that the zero-coupon curve can be modeled using the Nelson-Siegel model. This means that the continuously compounded zero-coupon rate between 0 and t is written as

$$R(0, t) = \beta_0 + \beta_1 \frac{1 - \exp(-\lambda t)}{t\lambda} + \beta_2 \left(\frac{1 - \exp(-\lambda t)}{t\lambda} - \exp(-\lambda t)\right), \tag{3.6.4}$$

and thus

$$p(0, t) = e^{-R(0,t)t}. \tag{3.6.5}$$

Assuming that λ is a constant, zero-coupon curve changes can be viewed as a function of $(\beta_0, \beta_1, \beta_2)$ and we can proceed by deriving expressions for $\frac{\partial \text{EVE}}{\partial \beta_i}(M)$ and $\frac{\partial \text{NII}}{\partial \beta_i}(M)$. For EVE the resulting derivatives are easily derived from equation 3.6.2 as

$$\frac{\partial \text{EVE}}{\partial \beta_i}(M) = \int_0^M \left[(1 - t/M)c(0, M) + 1/M\right] \frac{\partial p(0, t)}{\partial \beta_i} dt. \tag{3.6.6}$$

Since the only part that is sensitive to a change in β_i is the ZCB price (in this setting it can be thought of as a discount factor). For NII the assumption is made that after the initial zero-coupon curve shift it remains constant until at least the end of year 1. We note that the change in NII occurs due to the bonds that are renewed during year 1 and thus

$$\frac{\partial \text{NII}}{\partial \beta_i}(M) = \begin{cases} \left(\int_0^M t/M dt + \int_M^1 dt\right) \frac{\partial c_\delta(0,M)}{\partial \beta_i}, & M \leq 1 \\ \int_0^1 t/M dt \frac{\partial c_\delta(0,M)}{\partial \beta_i}, & M > 1. \end{cases} \tag{3.6.7}$$

Where we have used δ to denote that the change in coupon only affects coupons paid by bonds that have been bought after the zero-coupon curve has shifted. Notice that integrands represent how much that has been rolled into bonds with the "new" coupon. Before proceeding it is worth emphasizing that NII is measured as realized NII and thus not measured at the same point in time as EVE.

[1] Bonds that were bought at $t - M$

Translating a zero-coupon curve shift into changes in $(\beta_0, \beta_1, \beta_2)$, we can together with equations 3.6.6 and 3.6.7 compute the approximate effect on EVE and NII as

$$\Delta NII \approx \sum_{i=0}^{2} \frac{\partial \text{NII}}{\partial \beta_i}(M)\Delta\beta_i, \tag{3.6.8}$$

and

$$\Delta EVE \approx \sum_{i=0}^{2} \frac{\partial \text{EVE}}{\partial \beta_i}(M)\Delta\beta_i. \tag{3.6.9}$$

Memmel (2014) then uses a Principal Component Analysis of historical 12-month changes of the zero-coupon curve to translate the three factors that explain most of the variation into changes in $(\beta_0, \beta_1, \beta_2)$. Using a historical simulation approach the effects of changes in the parameters are investigated and related according to changes in the β_is. As expected, it is also shown that portfolios with smaller Ms are more sensitive to changes in NII, whereas the converse holds for EVE. In contrast to Memmel (2008) combinations of several rolling portfolios is only illustrated briefly. The sole example being combinations of one long and one short portfolio both having the same size, thus studying an aggregate portfolio of a different notional value than a simple long portfolio.

Iwakuma and Hibiki (2015) sets out to study how two different measures of IRR interact, which are defined as EVE and a three-year NII measure that also includes changes in the market value of assets and liabilities. The authors utilize more advanced models of assets and liabilities that include prepayments, credit risk, time-varying deposit volumes, non-perfect correlation between the zero-coupon rate and e.g. the deposit rate. The zero-coupon curve is modeled as a Nelson-Siegel model where the β_is are assumed to follow an AR(1) model. The effects of changes in the zero-coupon curve are studied for a hypothetical balance sheet using Monte Carlo simulations. The authors argue that their NII measure is superior to EVE due to it taking into account future transactions. However, it should be added that the model is rather sensitive to parameter estimates and hence one could question if it is possible to model future business reliably, especially for periods up to three years. We also note that the model is specifically tailored for the Japanese setting and thus has a few Japan-specific characteristics. Unfortunately we have not been able to find the full report in English and the conference paper is not very detailed.

Whereas both Iwakuma and Hibiki (2015) and Memmel (2014) investigate two different risk measures simultaneously they take no position on what an optimal portfolio is. Ozdemir and Sudarsana (2016) proposes a framework to select the optimal duration of the banking book given the objective being to optimize profit and certain constraints being satisfied with respect to NII and EVE. The use of duration as the variable the bank can change is based on a survey of large global banks[1] showing that a majority had a target duration of equity that was used for IRR management. Rather than focus on modeling a bank the focus is on motivating and defining an optimization problem that the bank should solve when managing IRR. The EVE and NII constraints are defined as VaR (or ES) at a certain level being below a given threshold. The objective function that is to be optimized is a type of risk-adjusted profit. Both EVE and NII are computed as the

[1] https://www.pwc.com/gx/en/banking-capital-markets/assets/balance-sheet-management-benchmark-survey.pdf

difference between the initial value and the value after a year. This is problematic for EVE since the duration of a static portfolio will, ceteris paribus, decrease as we move forward in time, meaning that the position becomes less risky. It is worth mentioning that not a lot is said about how well defined the optimization problem is. For a banking book with complex instruments it might be computationally hard or even infeasible to solve the defined problem.

We end this section with noting that all of these studies use slightly different definitions of NII and EVE and all of them differ from the IRRBB definitions.

Chapter 4

Modeling

In this chapter we present the portfolios and the analytical expressions for EVE and NII that are used to measure IRR. However, before this a discussion regarding our model requirements is warranted. Thus we begin with a presentation of our modeling assumptions and the simplifications that are made.

4.1 Portfolio composition

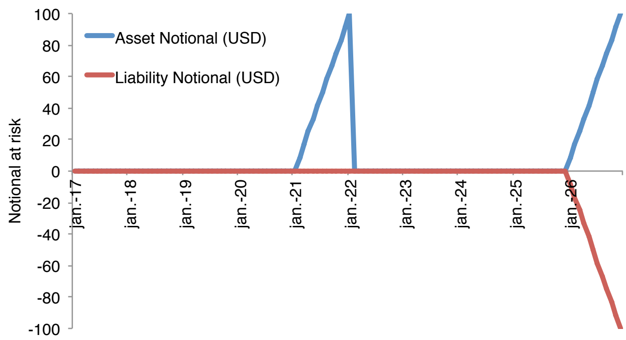
All companies are more or less sensitive to movements in interest rates due to the time value of money. Nevertheless, regulators do not actively study IRR for most industries. As was mentioned in 2.2, IRRBB requires banks to compute two IRR measures, EVE and NII. A natural requirement is then that the studied portfolio should, at least partially, react similarly as a banking book to zero-coupon curve shifts. A real bank's banking book is in general made up of a complex composition of financial instruments on both the asset and liability side of the balance sheet. Examples of complex components on the liability side are deposits with no contractual maturity and whose interest rates are seldom perfectly correlated with any market rate. On the asset side examples are different types of mortgages with included optionality. This optionality is sometimes exercised in a non-rational manner, requiring behavioral models to value them. For both deposits and mortgages the complexity is further increased by the fact that products often can have both bank- and country-specific features. As an example, mortgages in Sweden can typically only be prepaid if the borrower pays a break-fee, whereas in the U.S. fixed-rate mortgages are typically prepaid without any fees. However, the fundamental relationships between EVE, NII and the maturity of instruments should remain the same for most instruments. Building on Memmel (2008) we make the assumption that parts of a commercial bank's banking book could roughly be approximated using a portfolio strategy that consists of rolling over non-defaultable coupon bonds. The reader should note that we thus disregard basis and optionality risk. But, we note that when Iwakuma and Hibiki (2015) tries to take these effects into account they end up with a model that is sensitive to assumptions and less useful for banks not based in Japan.

Another important decision that is implied by using a strategy of rolling over coupon bonds is the distribution of maturing instruments on the time axis. For NII, positions repricing for the first time after the measurement period, set to be one (1) year in this thesis, are viewed as non-risky. With regards to NII risk, this implies that we are indifferent between a position that reprices in five years time and a position that reprices in ten years time. While this effect is inherent to the NII measure and thus not necessarily problematic it does not hold as we move forward in time. We illustrate this using a simple example assuming a bank that starts its business on the 1st of January 2017 by

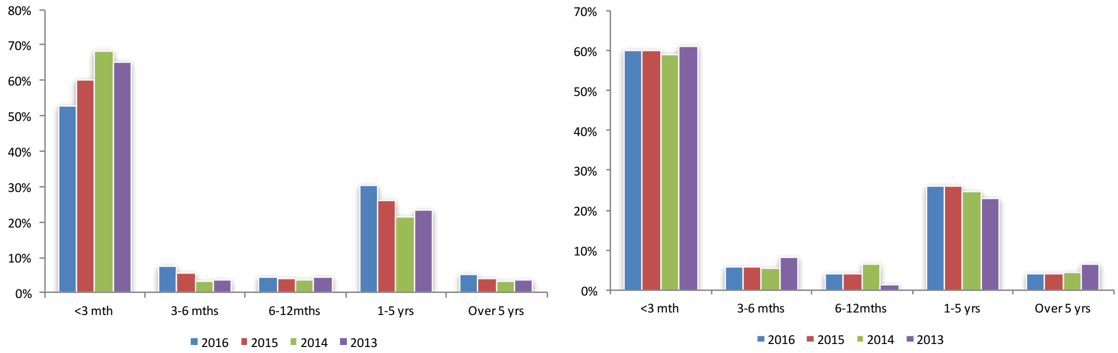
- buying a 10-year continuously paying coupon bond issued at par and,
- funding this by issuing a 5-year continuously paying coupon bond at par.

Assuming that NII does not take into account any discounting risk, the bank faces no NII risk between the 1st of January 2017 and the 31st of December 2020. However, on the 1st of January 2021 NII risk gradually increases until the 31st of December 2021 since the bank has to refinance the liabilities the following year to a (possibly) new coupon. Following the refinancing it sharply decreases to zero again on the 1st of January 2022 when the new liability has been issued and remains there until the 1st of January 2026 when it increases again assuming both the asset and liability side will be rolled over. This is illustrated in figure 4.1 where, assuming that the face value of both the asset and liability side is 100, we plot the notional amount (negative values for liabilities) that is exposed to NII risk against time.

Figure 4.1: Notional exposed to NII risk.



We see that if we construct a portfolio in this way we will not have NII sensitivity to points on the interest rate curve further out than the measurement period and we have cliff effects when moving forward in time. The problem is partially solved by using a roll-over strategy. This allows the portfolio to be sensitive to points on the zero-coupon curve further out than the measurement period, e.g. the portfolio could contain 10-year bonds that mature during the first year. We should note that the assumption does not work well if the maturity composition of the balance sheet is frequently altered. However, for a commercial bank the assumption that the bank does not drastically change its exposure to different parts of the zero-coupon curve is not that far-fetched. This can be seen in figures 4.2, where the distribution of Svenska Handelsbanken AB's interest rate adjustment periods between 2013 and 2016 is shown. In figure 4.2a, interest rate sensitive asset notionals are grouped by time to next rate repricing date. The amounts have been normalized by total asset notionals for each respective year. Figure 4.2b shows the same information for the liabilities. Off-balance sheet notionals have been included in assets.



(a) Assets and off-balance sheet items

(b) Liabilities

Figure 4.2: Interest rate adjustment periods for Svenska Handelsbanken AB, from Annual reports 2013-2016

4.2 Run-off, static or dynamic portfolios

After having chosen a portfolio model we now have to decide if the risk should be measured for a run-off, static or dynamic portfolio. Remember from section 2.1 that a run-off portfolio is a portfolio where new and renewed business is ignored. A static portfolio ignores new business and instead assumes that a maturing component in the portfolio is rolled over into a component with the same maturity as the maturing component initially had. Whereas a dynamic measure takes both new and renewed business into account, making assumptions about factors such as how volumes vary depending on the zero-coupon rate (Bessis, 2011). Typically EVE is measured on a run-off portfolio, whereas NII models using all three types of portfolios are used in practice (BCBS, 2016b).

Using non-consistent portfolio types for each measure becomes problematic if we, simultaneously, want to say something about how a zero-coupon curve shift affects both risk measures. Memmel (2008) measures EVE at $t = 0$ for a run-off portfolio. For NII, a static portfolio is assumed with the additional assumption that the zero-coupon curve remains the same for the first year following the initial shift. A zero-coupon curve where the forward zero-coupon curve is equal to today's zero-coupon curve is only true if the zero-coupon curve is flat. Hence, if the zero-coupon curve is not assumed to be flat, risk is not measured against the same curve for NII and EVE. We illustrate this using a simple example. Assuming that today is $t = 0$ and that forward-starting zero-coupon bonds are priced with the forward rate as defined in equation 3.1.1, the forward price for a 6-month ZCB with maturity at $t = 1$ is $p(0, 1)/p(0, 1/2)$. We can then compare the following strategies

(i) Buy $1/p(0, 1/2)$ 6-month ZCBs, the price of which is 1 today.

(ii) Buy $1/p(0, 1/2)$ 6-month ZCBs today and enter into $1/p(0, 1)$ forward contracts for 6-month ZCBs maturing at $t = 1$. This gives us $1/p(0, 1)$ at $t = 1$, the price of which is 1 today.

In EVE risk these two strategies are equivalent and (ii) is only introduced since we also need to compute NII. We can view the interest income as the amount over 1 (the initial value of the portfolio) we get at $t = 1/2$ and $t = 1$. Thus the NII we could expect during the first year is

$$\text{NII} = \left(\frac{1}{p(0,1/2)} - 1\right) + \left(\frac{1}{p(0,1)} - 1\right).$$

Memmel (2008)'s assumption is that $p(0,1/2) = p(1/2,1)$ with certainty. With a strategy of buying $1/p(0,1/2)$ 6-month ZCBs today and then rolling them into 6-month ZCBs in 6 months we get

$$\text{NII} = \left(\frac{1}{p(0,1/2)} - 1\right) + \left(\frac{1}{p(0,1/2)^2} - 1\right).$$

It is then easy to see that these two NIIs are the same if and only if $p(0,1/2)^2 = p(0,1)$ or equivalently stated $R(0,1/2) = R(0,1)$. If the initial zero-coupon curve is not parallel (or if the curve is parallel but the shift non-parallel) the resulting risks are not consistent with each other. This inconsistency is problematic if the purpose is to compare how EVE and NII reacts to a zero-coupon curve shift at the same point in time.

To avoid this problem we choose to measure both EVE and NII at the same time. Where NII is viewed as the expected NII the coming year and computed using the forward rates implied by the zero-coupon curve. In line with IRRBB, we also choose to measure discounted NII and use a static portfolio where it is assumed that assets can be reinvested according to the forward rate. A consequence of this is that NII would converge to EVE if the measurement period was extended toward infinity, which is also shown in appendix C. This means that in contrast to IRRBB we also include the sensitivity of known cash flows to present value changes in the NII measure. While not usually included in the NII measure this highlights the fact that there is short-term present value risk in locked-in cash flows.

To summarize the assumptions defining our model for measuring risk simultaneously in NII and EVE are

- A static portfolio for NII.
- A run-off portfolio for EVE.
- A rolling investment strategy to keep the portfolio's maturity composition constant over time and reducing the ability to "hide" NII risk in instruments maturing beyond year 1.
- Letting NII be sensitive to forward rates, thus allowing us to measure NII and EVE risk against the same zero-coupon curve when the zero-coupon curve is not flat.
- Discounting both known and expected cash flows for NII.

4.3 Mathematical formulation

The building block of the model is a non-defaultable bond paying a continuous coupon. At time u, the forward-coupon for this type of M-year bond, paying its first coupon at time t and with a forward price of 1 is as previously stated in equation 3.1.5 is

$$c(u,t,t+M) = \frac{p(u,t) - p(u,t+M)}{\int_t^{t+M} p(u,s)ds}.$$

Using these types of bonds we can construct the portfolio strategy that we will denote as $S(M)$. This strategy can be described as

- Buying M-year maturity continuously paying coupon bonds that are issued at par and with face value equal to 1.
- Evenly distribute when bonds mature on the time axis. This means having a fraction of $1/M$ of the portfolio expiring each year[1].

Risk is computed as the 1-month change in EVE and NII and we denote $\text{EVE}(t, M)$ as EVE of the $S(M)$ strategy at t and similarly for $\text{NII}(t, M)$. The change in EVE and NII during 1 month is calculated as

$$\begin{aligned}\Delta\text{EVE}(t,M) &= \frac{\text{EVE}(t+\frac{1}{12},M)-\text{EVE}(t,M)}{\text{EVE}(t,M)}\\ \Delta\text{NII}(t,M) &= \frac{\text{NII}(t+\frac{1}{12},M)-\text{NII}(t,M)}{\text{EVE}(t,M)},\end{aligned} \tag{4.3.1}$$

where the changes have been normalized by $\text{EVE}(t, M)$ to account for the fact that the original portfolio's value will vary between different $S(M)$ and we want to compare initial strategies of equal value.

Depending on the assumptions we make about the roll-over and the previous term structure we end up with slightly different versions. These are described and formulated below. To ease notation the discrete examples are only for $M = k/12,\ k = 1, 2, \ldots$.

Discrete roll-over and actual historical coupons (D-H)

A natural starting point is making the assumption that bonds mature and are renewed on a discrete basis, here chosen to be once a month. As mentioned in section 4.2, EVE is computed for a run-off balance sheet. Hence we need to discount all the payments the current bonds in our portfolio make up to their maturity. Assuming that the most recent bond was bought just before t the payments can be divided into two parts, coupons and face value payments from maturing bonds. The discounted value of the face value payments is simply

$$\sum_{k=1}^{12M}\frac{1}{12M}p(t,t+k/12), \tag{4.3.2}$$

since an even fraction of the portfolio matures every month. The coupon payments are slightly more complicated and can be expressed as

$$\int_0^M \underbrace{\left(1-\frac{\lfloor 12s\rfloor}{12M}\right)\left(\sum_{k=0}^{12M-1-\lfloor 12s\rfloor}\frac{c(t-\frac{k}{12},t-\frac{k}{12}+M)}{12M-\lfloor 12s\rfloor}\right)}_{\bar{c}_{D1}(s;t,M)} p(t,t+s)ds. \tag{4.3.3}$$

[1] Assuming that the portfolio has a face value of USD 1 this means that if $M = 5$ we have to reinvest USD 0.2 during the first year and if $M = 0.5$ the whole portfolio will have been reinvested twice during the first year.

Remember that even though bonds are rolled over discretely coupons are paid continuously and hence we have to integrate. The first term represents the fraction of bonds that have not matured at s. The second term represents the size of the coupon that is paid during that interval, which is just the average coupon of the bonds that have yet to mature. Summing up equations 4.3.2 and 4.3.3 gives us $\text{EVE}(t, M)$ as

$$\text{EVE}(t, M) = \sum_{k=1}^{12M} \frac{1}{12M} p(t, t + \frac{k}{12}) + \int_0^M \bar{c}_{D1}(s; t, M) p(t, t + s) ds. \tag{4.3.4}$$

In section 4.2 we settled on using a static portfolio for NII and as per IRRBB measuring the risk of the discounted cash flows. For NII, the expressions will depend on if M is less than or greater than 1. The simplest part of NII is the part coming from coupons of bonds bought up until t, the discounted value of which we have already expressed in equation 4.3.3. A slight alteration has to be made since we only measure NII up to $t + 1$, giving us

$$\int_0^{\min(1,M)} \bar{c}_{D1}(s; t, M) p(t, t + s) ds, \tag{4.3.5}$$

where the upper limit is $\min(1, M)$ since if $M < 1$ all bonds bought up to t will have matured at $t + M$. Secondly, we have the part of NII that stems from bonds not yet bought but whose expected contribution to NII we measure as the coupons implied by the forward curve. The contribution from this part is

$$\int_0^{\min(1,M)} \underbrace{\frac{\lfloor 12s \rfloor}{12M} \left(\sum_{k=1}^{\lfloor 12s \rfloor} \frac{c(t, t + \frac{k}{12}, t + \frac{k}{12} + M)}{\lfloor 12s \rfloor} \right)}_{\bar{c}_{D2}(s;t,M)} p(t, t + s) ds. \tag{4.3.6}$$

Similarly to equation 4.3.3 the first part is the fraction of the portfolio invested in bonds bought after t at s and the second part is the average of these coupons. In the case of $M < 1$ we have to add a term for coupons paid on $[M, 1]$ when we only have bonds bought after t, for which we get

$$\int_M^1 \underbrace{\left(\sum_{k=\lfloor 12s \rfloor - 12M + 1}^{\lfloor 12s \rfloor} \frac{c(t, t + \frac{k}{12}, t + \frac{k}{12} + M)}{12M} \right)}_{\bar{c}_{D3}(s;t,M)} p(t, t + s) ds. \tag{4.3.7}$$

Summing up equations 4.3.5, 4.3.6 and 4.3.7 gives us

$$\text{NII}(t, M) = \int_0^{\min(1,M)} \left(\bar{c}_{D1}(s; t, M) + \bar{c}_{D2}(s; t, M) \right) p(t, t + s) ds + \int_{\min(1,M)}^1 \bar{c}_{D3}(s; t, M) p(t, t + s) ds. \tag{4.3.8}$$

Discrete roll-over and constant historical coupons (D-C)

When trying to study how EVE and NII relate to each other, using historical coupons could introduce effects that we are not interested in studying. As an example we could imagine that we compute $\Delta\text{NII}(t, M)$ and $\Delta\text{EVE}(t, M)$ using equations 4.3.4 and 4.3.8 for $M \in \{1/12, 2/12\}$ just after a large parallel shift in the zero-coupon curve. For $M = 2/12$, $\text{EVE}(t + \frac{1}{12}, M)$ and $\text{NII}(t + \frac{1}{12}, M)$ would inherently drift up or down depending on if the shift resulted in a gain or loss. However, for $M = 1/12$ there is no inherent effect since all bonds have been bought after the shift. Memmel (2008) makes the assumption that all bonds bought up to t yield the same coupon to make it easier to study the effects that zero-coupon curve shifts have. We note that this approach is not perfect either since results will be biased by the shape of the current zero-coupon curve. If we make the same assumption we need to redefine the function $\bar{c}_{D1}(s; t, M)$, which is used in equations 4.3.3 and 4.3.5, setting

$$\bar{c}_{D1}(s; t, M) = c(t, t + M),$$

i.e. the par-coupon at t. Note that besides this change the expressions for EVE and NII remain the same. However, we measure EVE risk as the difference between $\text{EVE}(t + 1/12, M)$ and $\text{EVE}(t, M)$ and similarly for NII. We want to take into account the bonds that are rolled over between t and $t + 1/12$. Hence we have to make a slight alteration for the function at $t + 1/12$ to get

$$\bar{c}_{D1}(s; t + 1/12, M) = \left(1 - \frac{\lfloor 12s \rfloor}{12M}\right) \left(\sum_{k=0}^{12M-1-\lfloor 12s \rfloor} \frac{c(t - \frac{k-1}{12}, t - \frac{k-1}{12} + M)}{12M - \lfloor 12s \rfloor} \right), \quad (4.3.9)$$

$$c(t - \frac{k-1}{12}, t - \frac{k-1}{12} + M) = c(t, t + M), \ \forall k \neq 0. \quad (4.3.10)$$

Continuous roll-over and historical coupons (C-H)

In section 4.1 we discussed the cliff effects that we would see if we did not use a roll-over strategy. It is possible that some of these effects could still exist when bonds are replaced discretely. To remove the possibility of this effect we could make the (slightly unrealistic) assumption that bonds are replaced continuously, i.e. over a small interval $[t, t + dt]$, a fraction of dt/M matures and is replaced with new bonds. The expressions for EVE and NII are derived in a similar manner to the discrete case. For the face value part of EVE we have that

$$\int_0^M \frac{1}{M} p(t, t + s) ds.$$

Before showing the coupon part we remark that we only need to redefine the functions $\bar{c}_{D1}$, $\bar{c}_{D2}$ and $\bar{c}_{D3}$ in the continuous roll-over case to arrive at expressions for NII as well. These can be written as

$$\begin{aligned}
\bar{c}_{C1}(s; t, M) &= \left(1 - \frac{s}{M}\right) \int_{t+s-M}^{t} \frac{c(u, u + M)}{M - s} du, \\
\bar{c}_{C2}(s; t, M) &= \frac{s}{M} \int_{t}^{t+s} \frac{c(t, u, u + M)}{s} du, \\
\bar{c}_{C3}(s; t, M) &= \int_{t+s-M}^{t+s} \frac{c(t, u, u + M)}{M} du.
\end{aligned} \quad (4.3.11)$$

Where the expressions before the integrals are the fractions that are invested in the bonds and the integrals are the averages of the coupons these bonds are paying. Hence, we can express NII and EVE as

$$\text{EVE}(t, M) = \int_0^M \frac{1}{M} p(t, t+s) ds + \int_0^M \bar{c}_{C1}(s; t, M) p(t, t+s) ds, \tag{4.3.12}$$

and

$$\begin{aligned}\text{NII}(t, M) = \int_0^{\min(1,M)} &\left(\bar{c}_{C1}(s; t, M) + \bar{c}_{C2}(s; t, M)\right) p(t, t+s) ds \\ &+ \int_{\min(1,M)}^1 \bar{c}_{C3}(s; t, M) p(t, t+s) ds.\end{aligned} \tag{4.3.13}$$

Continuous roll-over and constant historical coupons (C-C)

Lastly, we could make both the assumption of continuous roll-overs and assuming that all bonds bought up to t were done at the same coupon. Similarly to the discrete case the only alteration that is needed is a change of $\bar{c}_{C1}$, which is then redefined as

$$\bar{c}_{C1}(s; t, M) = c(t, t+M).$$

As in the discrete case we note that at $t + 1/12$ the function is different than at t. Hence, we set

$$\bar{c}_{C1}(s; t + 1/12, M) = \left(1 - \frac{s}{M}\right) \int_{t+1/12+s-M}^{t+1/12} \frac{c(u, u+M)}{M - s} du, \tag{4.3.14}$$

$$c(u, u+M) = c(t, t+M), \ \forall u \leq t. \tag{4.3.15}$$

Notes on implementation

We end with some comments regarding the implementation of these in MATLAB. It is of course not possible to exactly compute $\bar{c}_{C1}$ in equation 4.3.11 since we do not have continuous observations of the historical zero-coupon curve. Instead, we approximate the historic coupons using equation 4.3.3. For both continuous models we have a similar problem for the coupons coming from bonds bought between t and $t + 1/12$. We approximate this by using a weekly discretization (4 obs). Most of these integrals are rather nasty and are integrated numerically. It is also the case that ZCBs will be log-normally distributed when using the Vasicek models presented in chapter 3. Unfortunately, the distribution of a sum of log-normally distributed random variables is not known. This leads to us not being able to derive analytical expressions for Expected Shortfall for EVE and NII. Instead, we implement a (naïve) Monte Carlo simulation to estimate their values.

Chapter 5

Data

This chapter presents the historical data that is used and the bootstrapping method that is later utilized to transform this into observations of an approximated historical zero-coupon curve. It ends with a description of the used simulation scheme.

5.1 Description of the data

As was mentioned in section 3.1 a continuous zero-coupon curve is a theoretical concept in the sense that bonds of all maturities do not trade on the market. Another problem is that it is far from given that a (risk-free) zero-coupon curve should be constructed using government bonds, e.g. IRRBB suggests the usage of a secured interest rate swap curve instead. However, in this thesis we choose to use a yield curve constructed by the U.S. Treasury (UST) using UST issued bonds[2]. Daily, the UST uses a "quasi-cubic hermite spline function" to construct the Constant-Maturity Treasury (CMT) curve using the yields of on-the-run Treasury securities (ODM, 2009). This means that the UST uses close-of-business yields of "the most recently auctioned 4-, 13-, 26-, and 52-week bills, plus the most recently auctioned 2-, 3-, 5-, 7-, and 10-year notes and the most recently auctioned 30-year bond, plus the composite rate in the 20-year maturity range." (ODM, 2009). The yields are called CMT yields because even if there currently does not exist a Treasury bond with an outstanding maturity of, say exactly 5 years, this yield is computed. The CMTs published by the UST are the 1-, 3-, 6-month, 1-, 2-, 3-, 5-, 7-, 10-, 20- and 30-year yields. As in Bolder (2001) we use monthly data during a 10-year period, in our case the last business day every month between April 2007 and April 2017. This gives us a time series of 121 observations of each CMT rate, some of the data expressed in percentages are shown in table 5.1.

5.2 Transformation of the data

Unfortunately the CMT yields are not expressed as continuously compounded zero-coupon rates and thus must be transformed into such rates. To do this we use the algorithm from

[2] The data can be accessed here.

Table 5.1: CMT yield data from the U.S. Department of Treasury (in %).

Date	1 Mo	3 Mo	6 Mo	1 Yr	...	20 Yr	30 Yr
4/28/17	0.68	0.80	0.99	1.07	...	2.67	2.96
3/31/17	0.74	0.76	0.91	1.03	...	2.76	3.02
2/28/17	0.40	0.53	0.69	0.88	...	2.70	2.97
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5/31/07	4.78	4.73	4.96	4.95	...	5.10	5.01
4/30/07	4.80	4.91	5.03	4.89	...	4.88	4.81

Whaley (2007) described below.

Step 1

The yields for maturities less than one year, i.e. the 1-, 3- and 6-month maturities, correspond to securities with no coupons. For these the CMT yields are expressed as a simple interest rate meaning that we can convert them to the continuously compounded zero-coupon rate at t using the following formula

$$R(t, t+t_i) = \frac{\log(1 + Y(t, t+t_i)t_i)}{t_i}, \quad t_i \in \{1/12, 3/12, 6/12\}, \tag{5.2.1}$$

with $Y(t, t+t_i)$ denoting the t_i-CMT yield at t.

Step 2

For the other CMT yields the process is slightly more complex, this is due to the CMT yields being expressed as the semi-annually compounded par coupon. Using equation 3.1.2 and the definition of the zero-coupon rate we can write this relation as

$$\begin{aligned} 1 &= p(t, t+t_j) + \sum_{k=1}^{2t_j} \frac{Y(t, t+t_j)}{2} p(t, t+k/2) \\ &= e^{-R(t,t+t_j)t_j} + \frac{Y(t, t+t_j)}{2} \sum_{k=1}^{2t_j} e^{-R(t,t+k/2)k/2}. \end{aligned} \tag{5.2.2}$$

The problem when using this recursive algorithm is that we do not have CMT yields for every 6 months between 1 and 30 years. As an example we have both the 1- and 2-year CMT yield but not the 1.5-year yield. One could think of several methods for interpolating between CMT yields, in this thesis we choose the simplest method and linearly interpolate. This allows us to "bootstrap" the zero-coupon curve using equation 5.2.2. An explicit expression for the zero-coupon rate at t_i is then

$$R(t, t+t_j) = -\log\left(\frac{1 - \frac{Y(t,t+t_j)}{2}\sum_{k=1}^{2t_j-1} e^{-R(t,t+k/2)k/2}}{1 + \frac{Y(t,t+t_j)}{2}}\right)/t_j, \tag{5.2.3}$$

which we can use to derive the zero-coupon curve up to the final CMT maturity (30 years). Some of the bootstrapped zero coupon rates are shown, in percentage, in table 5.2.

Table 5.2: Zero-coupon rates bootstrapped from CMT yields (in %).

Date	1 Mo	3 Mo	6 Mo	1 Yr	...	20 Yr	30 Yr
4/28/17	0.68	0.80	0.99	1.07	...	2.76	3.15
3/31/17	0.74	0.76	0.91	1.03	...	2.85	3.20
2/28/17	0.40	0.53	0.69	0.88	...	2.78	3.15
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5/31/07	4.77	4.70	4.90	4.89	...	5.13	4.93
4/30/07	4.79	4.88	4.97	4.83	...	4.94	4.77

5.3 Simulations

As previously noted in 4.3 we are unable to derive analytic solutions for the risk in different strategies $S(M)$. Therefore, we choose to assess risk of portfolio strategies using Monte Carlo simulation.

The models evaluated by this method are

- Model(C-C) with 1-factor Vasicek, 8000 scenarios
- Model(C-C) with 2-factor Vasicek, 8000 scenarios
- Model(C-C) with 3-factor Vasicek, 8000 scenarios
- Model(C-H) with 3-factor Vasicek, 2000 scenarios
- Model(D-C) with 3-factor Vasicek, 2000 scenarios

A smaller number of scenarios have been used for the discrete and historical models. This is due to the computations becoming time-consuming for the discrete case. We have estimated approximate confidence intervals for EVE and NII risk of each Monte Carlo simulation. The 95%-confidence intervals for Model 3F-C-C with 8000 scenarios and Model 3F-D-C with 2000 scenarios can be found in Appendix D. The confidence intervals of all models seem to suggest that for our purposes, the sample sizes chosen produce sufficient approximations of EVE and NII risk in different $S(M)$ strategies.

Simulation algorithm

The simulation algorithm has been implemented in the programming language MATLAB[1]. For reproducibility purposes MATLAB's random number generator is set to 'Mersenne-Twister' with seed 2 for all model simulations. A description in pseudo-code of the simulation algorithm is provided below.

[1] MATLAB code can be provided upon request

Algorithm 1 EVE and NII risk simulation

Precondition: n scenarios, $dt = \frac{1}{12}$, $step = \frac{dt}{4}$ $M \in \{0.25, 0.5, \ldots, 10\}$ and parameters for the short-rate model

function SIMULATE 1-MONTH EVE & NII OUTCOMES
– Calculate historical zero-coupon rates and perform a least square estimate
of current zero-coupon rates($t = 0$) to get y_t state vector

simulate paths for the short rate for each scenario
for $i \leftarrow 1$ to n **do**
simulate paths for the short rate 1 month ahead by a 4 time step discretization
$y_{i,1} \leftarrow y_t + \kappa(\theta - y_t)step + \sigma\sqrt{step}\mathcal{N}(0,1)$
$y_{i,2} \leftarrow y_{i,1} + \kappa(\theta - y_{i,1})step + \sigma\sqrt{step}\mathcal{N}(0,1)$
$y_{i,3} \leftarrow y_{i,2} + \kappa(\theta - y_{i,2})step + \sigma\sqrt{step}\mathcal{N}(0,1)$
$y_{i,4} \leftarrow y_{i,3} + \kappa(\theta - y_{i,3})step + \sigma\sqrt{step}\mathcal{N}(0,1)$
for $M_i \leftarrow M_i$ in M **do**
– Calculate the historical par-coupons for strategy M_i
– Calculate the $\text{NII}(t, M_i)$ and $\text{EVE}(t, M_i)$ of strategy $S(M_i)$
by calling one of the $\text{NII}(t, M_i)$ and $\text{EVE}(t, M_i)$ functions defined in section 4.3
and provide the historical par-coupons, y_t state vector and M_i

Now calculate the NII and EVE outcomes of strategy $S(M)$
after 1 month for each scenario
for $i \leftarrow 1$ to n **do**
– Calculate the additional par-coupons from the simulated y_is
and extend the historical par-coupon vector
– Calculate $\text{NII}(t + dt, M_i)$ and $\text{EVE}(t + dt, M_i)$ of strategy $S(M_i)$
by calling one of the NII and EVE functions defined in section 4.3
and provide the extended historical par-coupon vector,
$y_t, y_{i,4}$ state vectors and M_i
– Calculate the normalized differences
$\Delta\text{EVE}(t, M_i) = \text{EVE}(t + dt, M_i)/\text{EVE}(t, M_i) - 1$,
$\Delta\text{NII}(t, M_i) = (\text{NII}(t + dt, M_i) - \text{NII}(t, M_i))\,/\text{EVE}(t, M_i)$.
– Calculate $\widehat{ES}_\alpha(\Delta\text{EVE}(t, M_i))$ and $\widehat{ES}_\alpha(\Delta\text{NII}(t, M_i))$
using equation 3.5.4

Chapter 6

Simulations and discussion

In this chapter we will present and discuss the results from the simulations of the models presented in chapter 4. In chapter 4 we used a two letter code to categorize the model. The first letter denotes if bonds were rolled over continuously and the second denotes if the zero-coupon curve was assumed to have been constant up to t. In this chapter we expand on this adding an XF before the two letters, with $\mathrm{X} \in \{1, 2, 3\}$, representing the number of factors used in the Vasicek model. As an example, 3F-C-H denotes the model where the zero-coupon curve is simulated using a **3-F**actor Vasicek model, with **C**ontinuous roll-overs and assuming the use of non-constant **H**istorical zero-coupon curves. The models with their respective notation and assumptions are shown in table 6.1. To ease notation we will drop the t from $\mathrm{NII}(t, M)$ and $\mathrm{EVE}(t, M)$ since we only measure risk at one point in time. We also remark that M is measured in years, the present value, i.e. the initial EVE, is normalized to \$1 for all M and ES is measured with $\alpha = 0.05$.

Model name	Vasicek factors	Continuous roll over	Continuous coupons	Historical ZC
3F-D-C	3	No	Yes	No
3F-C-H	3	Yes	Yes	Yes
3F-C-C	3	Yes	Yes	No
2F-C-C	2	Yes	Yes	No
1F-C-C	1	Yes	Yes	No

Table 6.1: Model assumptions

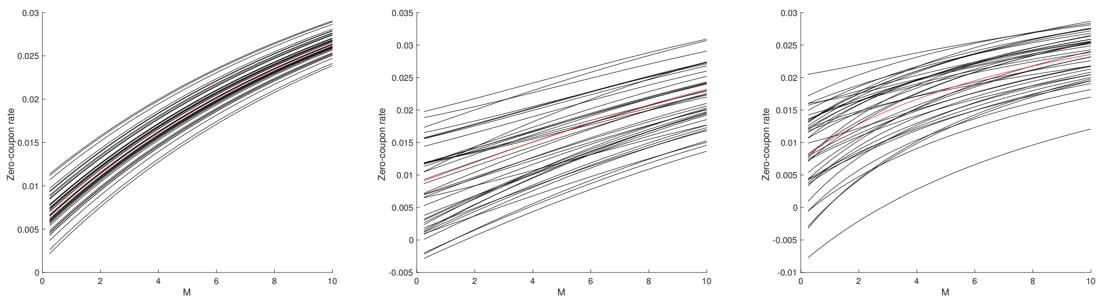
6.1 Zero-coupon curve simulations

Before investigating the risk of the different portfolio strategies it is useful to observe and discuss the underlying zero-coupon curve simulations. As was outlined in section 2.2, IRRBB prescribes using three types of shifts for EVE: parallel, change of steepness and change of curvature. However, for NII only parallel shifts are prescribed. We relate this to section 3.2 where we showed how a single-factor model will only generate parallel shifts, something which is not the case for the multi-factor models. Roughly we can say that 2-factor models could change the level and steepness of the curve, whereas 3-factor models could also change the curvature. To show that this holds when simulating we estimate the parameters using the Kalman filter algorithm from section 3.4 and the data described in section 5.1. Estimated parameters for the three models are shown in table 6.2, a sample of zero-coupon curve simulations from the three models are shown in figure 6.1 and some empirical statistics in table 6.3. As expected the range and types of shifts

Model	Factors	θ_i	κ_i	σ_i	λ_i
1F Vasicek	1	0.01891	0.13379	0.00687	-0.65014
2F Vasicek	2	0.00947	0.26227	0.01629	0.11665
		0.00401	0.01876	0.02043	-0.28524
3F Vasicek	3	0.00013	0.28288	0.02082	-0.50992
		0.00016	0.71816	0.01721	0.79681
		0.014374	0.03216	0.01326	-0.15369

Table 6.2: Estimated parameters

differ for the single-, 2- and 3-factor model. In figures 6.1 we plot every two-hundredth simulated zero-coupon curve, with the red curve being the original one. In figure 6.1a we notice that the single-factor model produces parallel shifts. The 2-factor model simulations in 6.1b have the parallel shifts of the 1-factor model as well as shifts with changed slopes, i.e. a steepening or flattening of the zero-coupon curve. For the 3-factor model, the simulations in 6.1c shows that we can achieve the 2-factor model's shifts as well as an increased variation of slope for different maturities, i.e. we can achieve shifts in the zero-coupon curve's curvature. Notice that the range of zero-coupon curve shifts differs between the three models. This might be due to the accuracy of our Kalman filter's parameter estimation as previously discussed in section 3.4. It is outside the scope of this thesis to perform a robustness check of our estimated parameters and for our analysis we are instead interested in investigating how increasing the number of types of shifts affects the risk levels for EVE and NII.



(a) 1-factor Vasicek (b) 2-factor Vasicek (c) 3-factor Vasicek

Figure 6.1: Every two-hundredth simulated zero-coupon curve.

Table 6.3: Empirical statistics for $R(t + 1/12, t + 1/12 + M)$ under the different Vasicek models, in %.

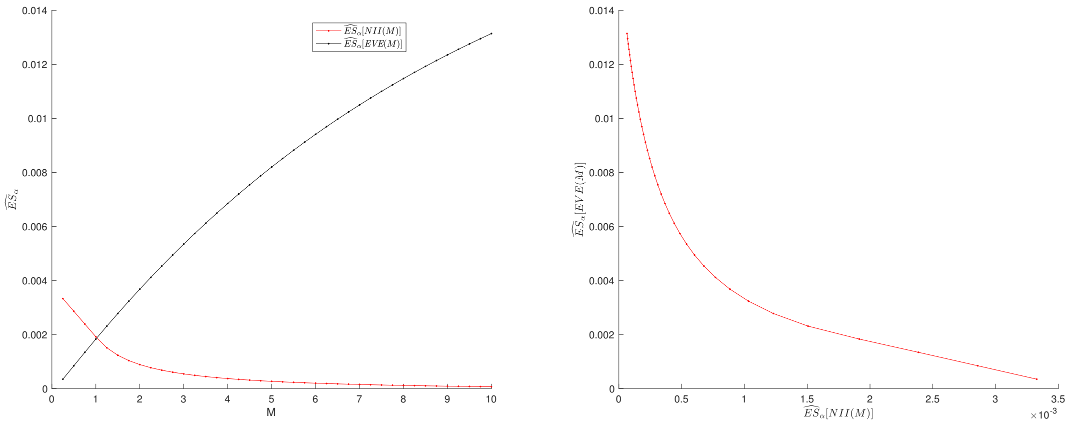
Vasicek 1F			
	M=3/12	M=5	M=10
5% Quantile	0.3593	1.6069	2.4676
Mean	0.6831	1.8469	2.6491
95% Quantile	1.0012	2.0828	2.8275
Vasicek 2F			
	M=3/12	M=5	M=10
5% Quantile	-0.3078	0.6331	1.3916
Mean	0.9012	1.6514	2.3158
95% Quantile	2.113	2.6671	3.2291
Vasicek 3F			
	M=3/12	M=5	M=10
5% Quantile	-0.5143	1.0167	1.7212
Mean	0.8552	1.8163	2.354
95% Quantile	2.2246	2.646	2.9997

6.2 Risk under the different Vasicek models

Having simulated the zero-coupon curves we can investigate if increasing the number of factors also increases the range of $\text{ES}_{0.05}(\text{NII}(M))$ and $\text{ES}_{0.05}(\text{EVE}(M))$ for the XF-C-C models. Remember that these models assumed continuous rolling over of bonds and a constant historical zero-coupon curve. We begin with the 1F-C-C model, in figure 6.2a we plot the estimated $\text{ES}_{0.05}(\text{NII}(M))$ and $\text{ES}_{0.05}(\text{EVE}(M))$ against M. As expected, a strategy $S(M)$ will have lower risk in NII than $S(N)$ if $M > N$ and conversely for EVE, $S(M)$ will be riskier than $S(N)$. For NII this is understood by remembering that as M increases a smaller fraction of the portfolio matures during the following year and in the case of $M \geq 1$, only $1/M$ matures and is reinvested. This effect could be partially offset by the discounting risk of the first year's fixed coupons, but we note that for large M this effect is small. For EVE, risk increases with M, the intuitive explanation being that that the sensitivity of cash flows to a zero-coupon curve shift increases with the cash flows time to maturity and the strategy $S(M)$'s average cash flow maturity date is increasing in M. This could not be said with certainty if the zero-coupon curve is downwards sloping but as remarked previously the single-factor model will only produce parallel shifts and since the original curve, as seen in figure 6.1a, is strictly increasing it holds. Figure 6.2b depicts the same risks shown in figure 6.2a but for each $S(M)$'s risk pair

$$(\widehat{\text{ES}}_\alpha[NII(M)], \widehat{\text{ES}}_\alpha[EVE(M)]).$$

This relationship is of importance since it illustrates the trade-off in EVE and NII risk that an investor will face when deciding on which strategy $S(M)$ to pursue. An investor unwilling to take on more than a certain amount of NII risk can see that this implies that a certain amount of EVE risk will have to be accepted. More importantly, the investor with a pre-specified preference of EVE risk and NII risk can see if the combination is attainable. Of course, the investor need not settle on a single $S(M)$ but could instead invest fractions summing to one in several strategies instead, which is a discussion we will revisit in section 6.4. Before proceeding we remark that if we set the initial short rate to $\bar{\theta}+0.01$ the zero-coupon curves will be downward sloping instead. Doing this and redoing the simulations gives us roughly the same risks as in figure 6.2 but as expected decreases the EVE risk for larger Ms. The resulting plots can be seen in Appendix A where we also provide results for a simple stress test of the model's parameters θ, κ, σ and λ.



(a) $\widehat{ES}_\alpha$ for different maturities M

(b) Risk pairs $(\widehat{ES}_\alpha[NII(M)], \widehat{ES}_\alpha[EVE(M)])$

Figure 6.2: Expected Shortfall risk for different maturity strategies, model 1F-C-C, $\alpha = 0.05$

Having investigated the single-factor model we proceed to the multi-factor cases. In figure 6.4 we compare the risks of all three models. EVE- and NII-risk for different $S(M)$ can be seen in figure 6.4a and their respective risk pairs in figure 6.4b. Irrespective of a single-, 2- or 3-factor Vasicek model EVE and NII risk for different $S(M)$-strategies and their risk pairs have a similar shape. However the level of risk produced by the three short-rate models differs significantly, with the single-factor model producing the lowest risk. The 2- and 3-factor models have similar risk levels in NII and EVE for $S(M)$-strategies with shorter maturities M, while EVE risk for the 2-factor model exceeds the 3-factor model for longer maturities. An explanation for this is our parameter estimates. Using equation 3.3.4 we can compute the theoretical one-month conditional variance for $R(t, t+M)$ as a function of M. This is shown in figure 6.3 where we can see that using our estimated parameters the 3-factor model will produce a more volatile zero-coupon curve than the 2-factor model for M less than roughly 1.5 years, whereas the converse holds for M roughly greater than 1.5 years. We also note that this is in line with the estimates in table 6.3.

Regardless of the different risk levels it is encouraging that the shape of the three factor models are similar since some results should hold independently of the factor model, e.g. EVE risk should generally increase with M and the converse should generally hold for NII. However, some additional comments are needed regarding how the correlation between different portfolios changes with the number of factors used. This since it will effect

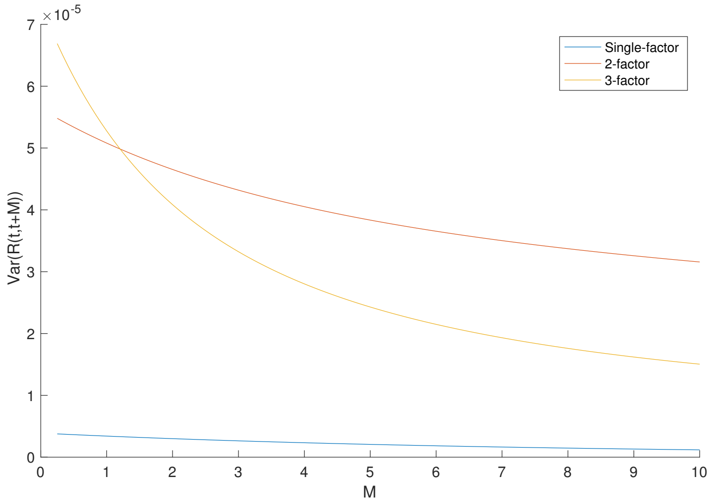
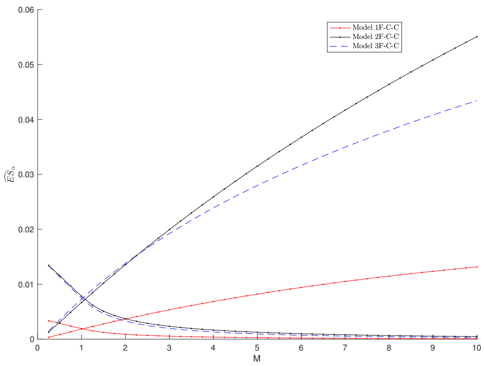
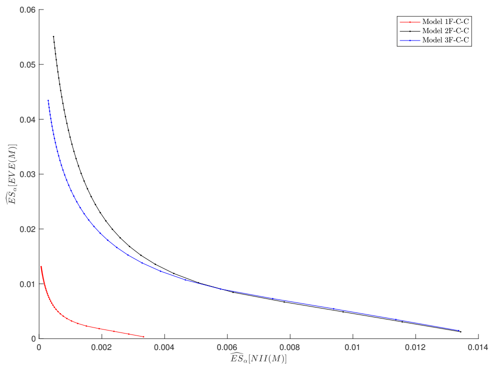


Figure 6.3: Theoretical one-month variance, $Var^{\mathbb{P}}(R(t, t+M)|\mathcal{F}_{t-1/12})$



(a) $\widehat{ES}_\alpha$ for different maturities M

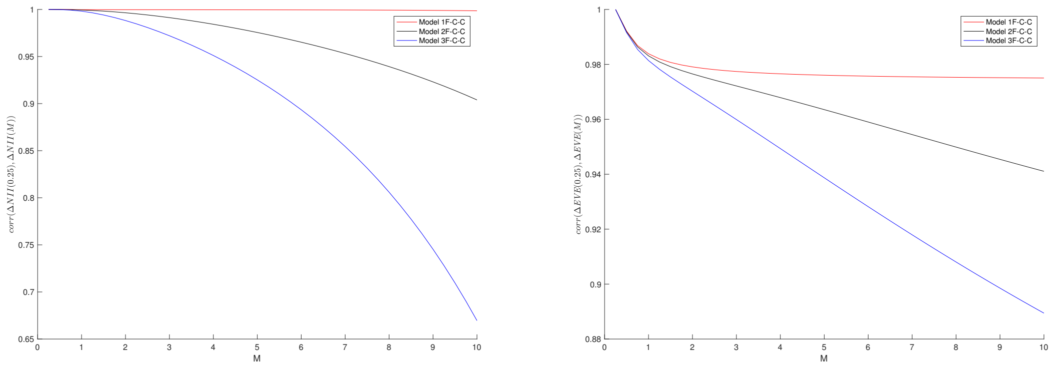


(b) Risk pairs $(\widehat{ES}_\alpha[NII(M)], \widehat{ES}_\alpha[EVE(M)])$

Figure 6.4: Expected Shortfall risk 1-,2- and 3-factor Vasicek, $\alpha = 0.05$

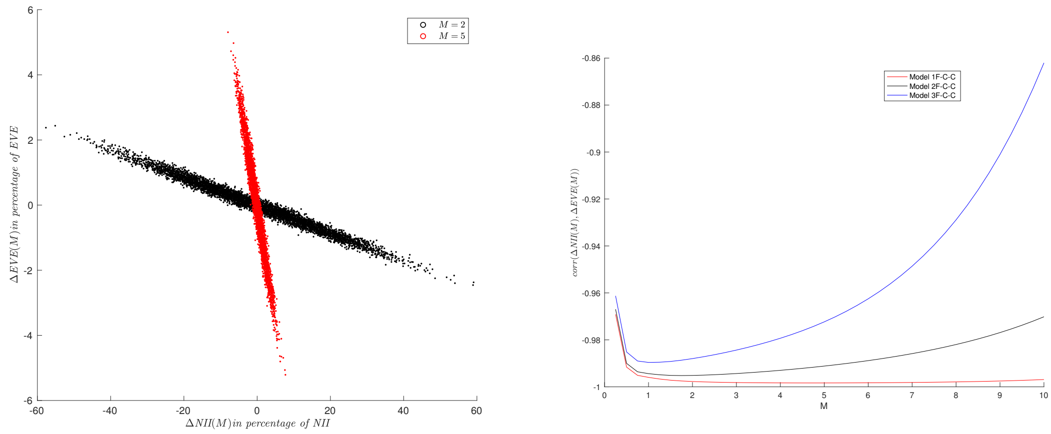
the possible EVE risk and NII risk combinations that are possible if we combine several $S(M)$-strategies. As previously stated in equation 3.5.3, ES is a convex risk measure. In other words, the risk of an investment in several $S(M)$ strategies will have less or equal risk than the sum of the $S(M)$-strategies' individual risks. This diversification benefit is likely to be smaller when $S(M)$-portfolios are highly correlated. In figure 6.5 estimated correlations between the shortest $S(M)$-strategy, $S(3/12)$, and longer $S(M)$-strategies can be seen for EVE and NII risk. The single-factor and 2-factor models show very high correlations while correlations for the 3-factor model are slightly lower. The results are similar for other choices of $S(M)$. Whilst high correlations for individual outcomes does not necessarily imply high correlations for the worst outcomes (in our case beyond the 5% quantile), the worst outcomes are still likely to be highly correlated. The correlation of outcomes beyond the 5% quantile for an $S(M)$ portfolio and the corresponding outcomes of other $S(M)$ portfolios can be found in Appendix E. These outcomes show slightly lower correlation.

Another interesting result for all three models is the strong negative correlation between EVE and NII risk as seen in figure 6.6. Notice in figure 6.6a that the individual outcomes of percentage changes in EVE tend to be negative when percentage changes in NII are positive. Estimated EVE and NII correlations for different strategies $S(M)$ are shown in figure 6.6 for the three different factor models. The single-factor model has the highest negative correlation, followed by the 2-factor model and 3-factor model. As can be seen in figure 6.5a, estimated NII correlations are close to 1 between any $S(M)$ for the single-factor model. If a short portfolio was introduced we could find a very good hedge in NII risk for a long strategy $S(M)$ and short $S(N)$ for any M, N we have considered, which in a real world setting is less plausible.



(a) $\widehat{corr}(\Delta NII(0.25), \Delta NII(M))$ (b) $\widehat{corr}(\Delta EVE(0.25), \Delta EVE(M))$

Figure 6.5: Correlations of different M-strategies' simulated outcomes for NII and EVE



(a) EVE & NII changes in 2-year- & 5-year-strategy for each simulation, model 3F-C-C

(b) $\widehat{corr}(\Delta NII(M), \Delta EVE(M))$

Figure 6.6: Correlation between EVE and NII of different M-strategies' simulated outcomes

As previously mentioned, in IRRBB the Basel Committee suggests using shifts corresponding to a single-factor model and 3-factor model for measuring NII sensitivity and

EVE sensitivity respectively. In our setting it does not make any sense to use one model for NII and another for EVE. However, the exercise above is done since it allows us to essentially study how introducing one more type of shift affects the risk profile. An otherwise complicated task since we lack closed-form expressions for the ES risks. In the proceeding sections we will only use the 3-factor model since it covers the scenarios from the single- and 2-factor model (albeit for our parameters estimates the 2-factor model seems to produce more extreme values).

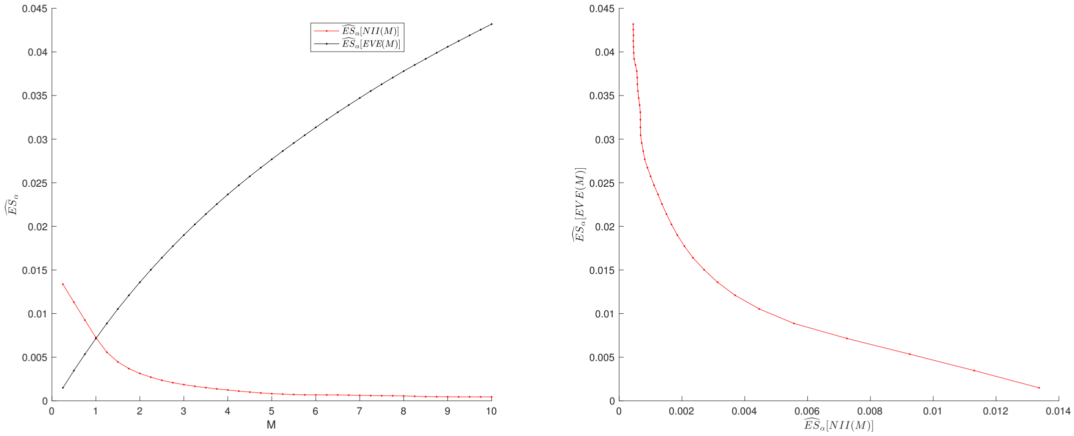
6.3 Altering assumptions

In this section we will investigate how altering the assumption of a constant historical zero-coupon curve and the assumptions of being able to roll over bonds continuously affects EVE and NII risk.

Non-constant historical zero-coupon curve

Remember that the simulations presented in the previous section were with the model assumption that the historical par-coupon rate had been constant or equivalently stated the zero-coupon curve had been constant. In section 4.3 we discussed the reasoning behind making this assumption when comparing EVE risk and NII risk. However, in a real setting this is of course not true and the use for our model would lessen if this had a huge impact on the simulated risks. To study this we compare model 3F-C-H with model 3F-C-C, i.e. the two portfolio models using continuous rolling of bonds, differing on the assumption regarding the historical zero-coupon curve, and with zero-coupon curves simulated from the 3-factor Vasicek model.

EVE and NII risk plots can be seen in figure 6.7. In figure 6.7a, for M greater than approximately 5 years, NII risk is not evenly decreasing in M. This is easier to spot in the left part of figure 6.7b where NII risk "inconsistently" decreases for increasing EVE risk. A possible explanation for this can be found if we study the historical evolution of zero-coupon rates. In figure 6.8 we notice that from our historical zero-coupon 5- and 7-year rates there was a sharp decline approximately 6 years ago. From the portfolio's perspective, when M increases past this point the portfolio will suddenly include coupon bonds with much higher coupon rates that will affect the portfolio's risk in the present.



(a) $\widehat{ES}_\alpha$ for different maturities M (b) Risk pairs $(\widehat{ES}_\alpha[NII(M)], \widehat{ES}_\alpha[EVE(M)])$

Figure 6.7: Expected Shortfall risk for different maturity strategies, $\alpha = 0.05$, model 3F-C-H

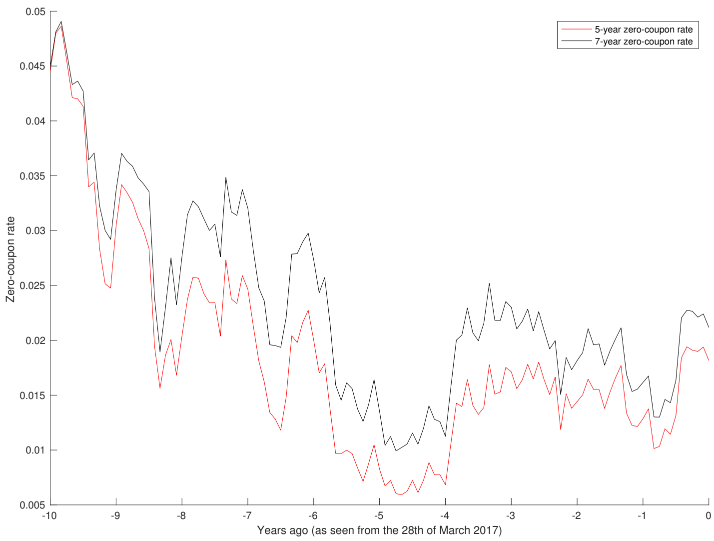
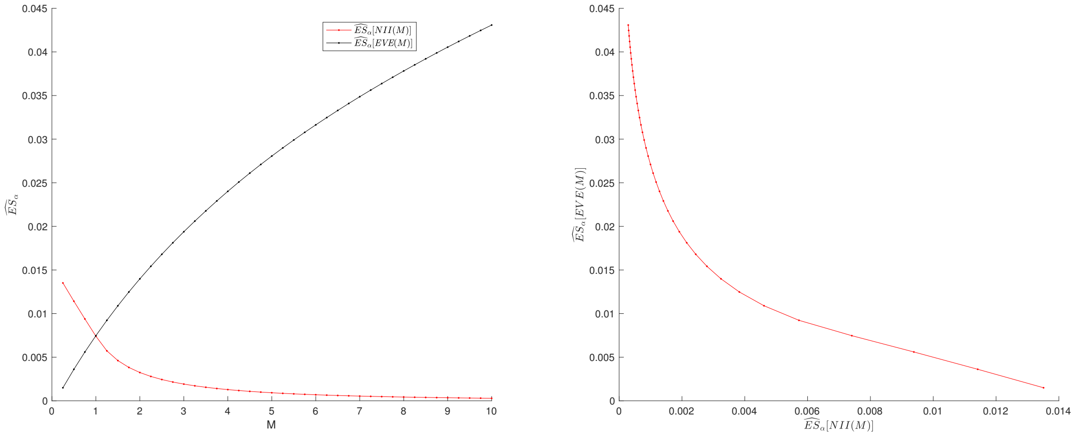


Figure 6.8: Historical 5 and 7-year zero-coupon rates

Discrete roll-over

The other unrealistic assumption that is made is that we can roll over bonds continuously. To see if altering this assumption affects the simulated NII and EVE risk we compare the 3F-C-C model with the 3F-D-C model, i.e. comparing continuous roll-over with discrete roll-over and in both cases assuming a constant historical zero-coupon curve. The resulting risk pairs can be seen in figure 6.9.

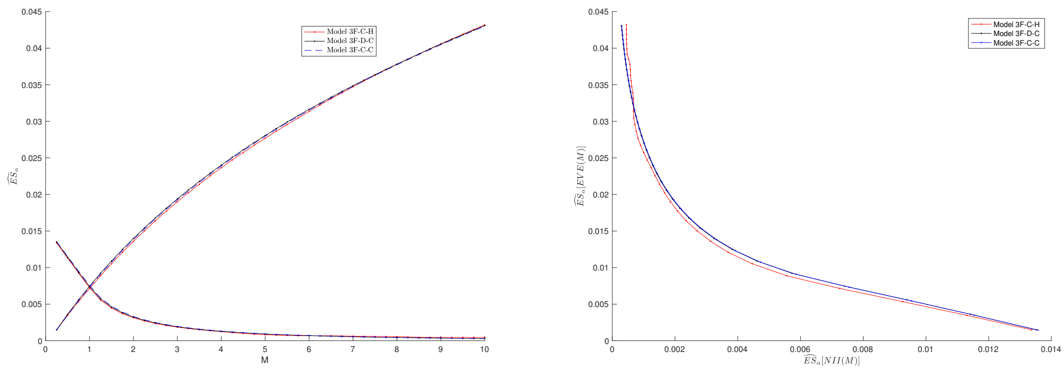
A comparison between the three models we have discussed above, i.e. models 3F-C-C, 3F-



(a) $\widehat{ES}_\alpha$ for different maturities M (b) Risk pairs $(\widehat{ES}_\alpha[NII(M)], \widehat{ES}_\alpha[EVE(M)])$

Figure 6.9: Expected Shortfall risk for model 3F-D-C and different maturity strategies, $\alpha = 0.05$

C-H and 3F-D-C, can be seen in figure 6.10. There we can see that the risk relations and risk levels are essentially equal for the discrete roll-over and continuous roll-over models. It is barely noticeable but if we look closely we could see that the EVE risk is slightly higher for 3F-D-C compared to 3F-C-C. The explanation for this is that the duration will be slightly higher when using discrete roll-over since the average time to maturity will be greater than when using continuous roll-over. However, the difference is negligible. The reader interested in seeing the resulting risk profile for 3F-D-H can find it in appendix B. To conclude the portfolio model comparison, we have seen that neither assumptions regarding continuous investments nor assumptions regarding historical par-coupon bond investments have had a large effect on the risk. To strip out the small effects that are inherent to using discrete rolling and a non-constant historical zero-coupon curve we only study the 3F-C-C model in the proceeding sections.



(a) $\widehat{ES}_\alpha$ for different maturities M (b) Risk pairs $(\widehat{ES}_\alpha[NII(M)], \widehat{ES}_\alpha[EVE(M)])$

Figure 6.10: Expected Shortfall risk for models 3F-D-C, 3F-C-H and 3F-C-C, $\alpha = 0.05$

6.4 Long M-portfolio combinations

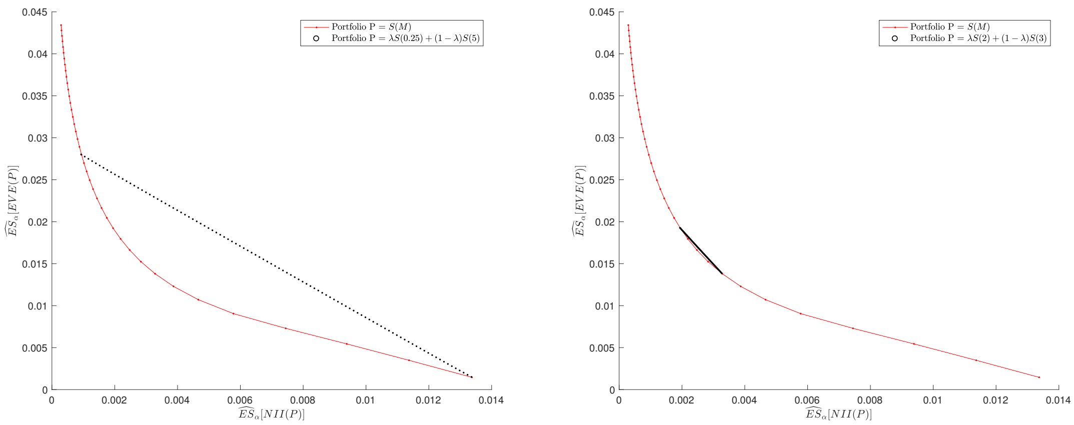
So far we have considered the investment strategy of investing in one rolling bond portfolio $P = S(U)$ with $U \in \{0.25, 0.5, \dots, 9.75, 10\}$. A natural extension is to investigate the risk pairs that are possible if the investor has the possibility to divide his investment into multiple investment strategies of type $S(M)$. We consider the portfolios (of portfolios)

$$P(\lambda, M, N) = \lambda S(M) + (1-\lambda)S(N), \tag{6.4.1}$$

where $M, N \in \{0.25, 0.5, \dots, 9.75, 10\}$ and $\lambda \in \{0, 0.01, 0.02, \dots, 0.99, 1\}$. To ease notation we denote $P(\lambda, M, N)$'s risk pair

$$(\widehat{ES}_\alpha[\text{NII}(P(\lambda, M, N))], \widehat{ES}_\alpha[\text{EVE}(P(\lambda, M, N)]) =$$
$$(\widehat{ES}_\alpha[\lambda\text{NII}(S(M)) + (1-\lambda)\text{NII}(S(N))], (\widehat{ES}_\alpha[\lambda\text{EVE}(S(M)) + (1-\lambda)\text{EVE}(S(N))]) \tag{6.4.2}$$

In figure 6.11 two examples are given, figure 6.11a considers portfolio combinations where $M = 0.25$ and $N = 5$. Figure 6.11b considers the portfolio combinations where $M = 2$ and $N = 3$. As can be seen, both combined portfolios' risk pairs $P(\lambda, M, N)$ are worse or equal to, for all $\lambda \in \{0.01, 0.02, \dots, 0.99\}$, a risk pair achievable with a simple portfolio $P = S(U)$ for some U.



(a) Portfolio $\lambda S(0.25) + (1-\lambda)S(5)$

(b) Portfolio $\lambda S(2) + (1-\lambda)S(3)$

Figure 6.11: Risk pairs $(\widehat{ES}_\alpha[\text{NII}(P(\lambda, M, N))], \widehat{ES}_\alpha[\text{EVE}(P(\lambda, M, N)])$ for two different strategies. $\lambda \in \{0, 0.01, 0.02, \dots, 1\}$

Since risk is measured as the coherent risk measure Expected Shortfall, by the risk measure's convexity property it holds that

$$ES_\alpha[\text{NII}(P(\lambda, M, N))] \leq \lambda ES_\alpha[\text{NII}(M)] + (1-\lambda)ES_\alpha[\text{NII}(N)] \tag{6.4.3}$$

and

$$ES_\alpha[\text{EVE}(P(\lambda, M, N))] \leq \lambda ES_\alpha[\text{EVE}(M)] + (1-\lambda)ES_\alpha[\text{EVE}(N)]. \tag{6.4.4}$$

In words, risk of the portfolio will always be less or equal to the linear combination of the two separate risk pairs $S(M)$ and $S(N)$. However, as seen in figure 6.5 EVE risk is highly correlated between different $S(M)$ and $S(N)$ and albeit lower, the correlation is also high for NII risk. This indicates that there will not be a large reduction in risk for a combined portfolio such as $P(\lambda, M, N)$ since the worst outcomes (losses beyond the 5% quantile) for each investment are likely to occur from the same scenario.

In figure 6.12 risk pairs for all $P(\lambda, M, N)$ are shown, with $M, N \in \{0.25, 0.5, \ldots, 9.75, 10\}$ and $\lambda \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$. We note that almost all our risk pairs from portfolios $P(\lambda, M, N)$ seem to be worse or equal compared to the risk pairs produced by a simple portfolio, $S(U)$. The simple $S(U)$ portfolio looks like an attractive investment from a combined EVE and NII risk perspective. The few portfolio combinations that are slightly better than the simple portfolios are in the rightmost part of figure 6.12 and thus corresponding to small Us. If we study this closer it can be seen that this only holds for $U \leq 1$. A graphical explanation for this is that the S(U) risk pair curve is approximately linear for $U \leq 1$ and thus the convexity of ES guarantees that combinations will be at least as good as the risk pair curve.

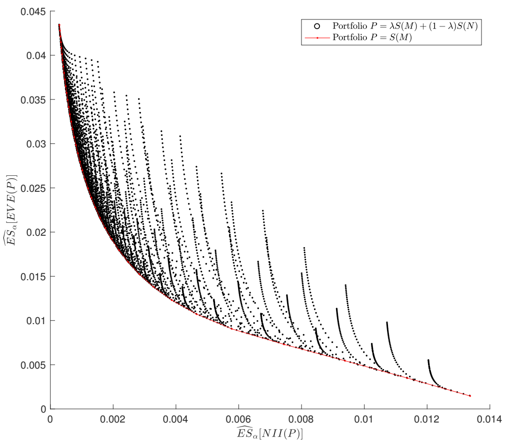


Figure 6.12: Risk pairs for all portfolios $\lambda S(M) + (1-\lambda)S(N)$, $\lambda \in 0, 0.1, \ldots, 1$, $M, N \in 0.25, 0.5, \ldots, 10$

6.5 Long and short M-portfolio combinations

Up to this point we have considered the trade-off problem in EVE risk and NII risk an investor faces when deciding how to invest his assets in various portfolio strategies $S(M)$ and combinations thereof. We will now introduce the possibility for the investor to borrow money by short-selling one or combinations of portfolio strategies $S(M)$. The investor will thus have the following amounts to be considered for his investment strategy

- USD 1 investment
- USD 1 additional investment
- USD 1 borrowed

We assume that the assets are invested in portfolio strategy $P(\lambda, M, N)$

$$\begin{aligned} &2P(\lambda, M, N) = 2(\lambda_1 S(M) + (1 - \lambda_1) S(N)), \\ &M, N \in \{0.25, 0.5, \ldots, 9.75, 10\}, \\ &\lambda \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}, \end{aligned} \tag{6.5.1}$$

and that this is funded by short-selling USD 1 of another portfolio combination. Resulting in aggregate portfolios of the type

$$2P(\lambda_{asset}, M_{asset}, N_{asset}) - P(\lambda_{liability}, M_{liability}, N_{liability}). \tag{6.5.2}$$

Below we will design three different strategies of this kind. Strategy 1 in figure 6.13 shows the risk pairs for

$$M_{asset}, N_{asset} \in \{7.5, 7.75 \ldots, 9.75, 10\},$$

and

$$M_{liability}, N_{liability} \in \{0.25, 0.5\},$$

for all $\lambda_{asset}, \lambda_{liability} \in \{0, 0.1 \ldots, 0.9, 1\}$. Strategy 1 is thus a strategy to borrow short and invest long. As can be seen the risk pairs produced are greater than matching assets and liabilities represented by simple portfolios $S(U)$.

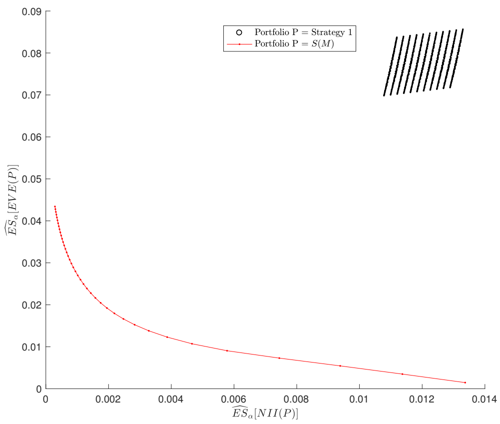


Figure 6.13: Risk pairs for strategy 1

For strategy 2 we design a strategy with the objective of reducing NII risk. For the long-short portfolio considered, low NII risk can be attained by multiple strategies. We could as an example choose the longest investment horizon $M = 10$ for all long and short investments, this would reduce our portfolio to a simple $S(10)$ portfolio and as our results have showed this portfolio yields the lowest NII risk of a single strategy $S(M)$. However, this leaves us exposed to the highest EVE risk of the simple portfolios. We would like to choose another approach for our long short portfolio strategy. As previously noted in section 6.2 there are high correlations in the worst NII outcomes between simple $S(U)$ portfolios. Utilizing this fact we would like to design a strategy where the long portfolio $2P(\lambda, M, N)$, which is twice the size of the short portfolio, is invested in $S(M)$ and $S(N)$ portfolios of approximately half the NII risk as the short portfolio. With an aggregate risk in the long portfolios of approximately the same size as the short portfolio and high correlation between the long and short portfolios' worst outcomes a decent hedge in NII risk should be achieved. Following this approach, Strategy 2 in figure 6.14a is a strategy where both asset and liabilities are at short maturities with assets slightly longer at

$$M_{asset}, N_{asset} \in \{1, 1.25, 1.5\},$$

and liabilities at

$$M_{liability}, N_{liability} \in \{0.5, 0.75, 1\}.$$

With Strategy 2 we notice that low NII risk corresponding to the risk of an $S(U)$ investment of $U \geq 1.5$ can be achieved without any investment in $S(U)$ of $U \geq 1.5$, which implies that we have diversification effects.

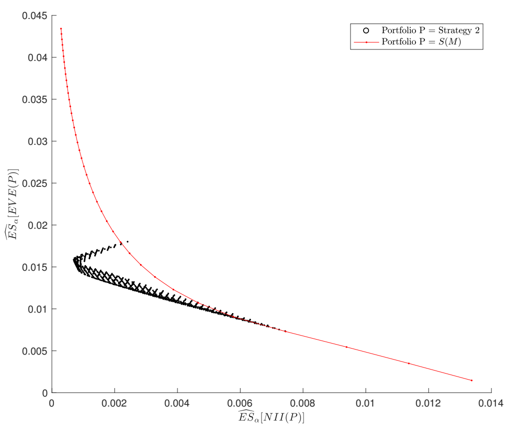
We could design a similar strategy with the goal of reducing EVE risk, namely our Strategy 3. By following the same argument as for strategy 2, we choose a long portfolio with approximately half the EVE risk of our short portfolio.

$$M_{asset}, N_{asset} \in \{2, 2.25, 2.5, 2.75, 3\},$$

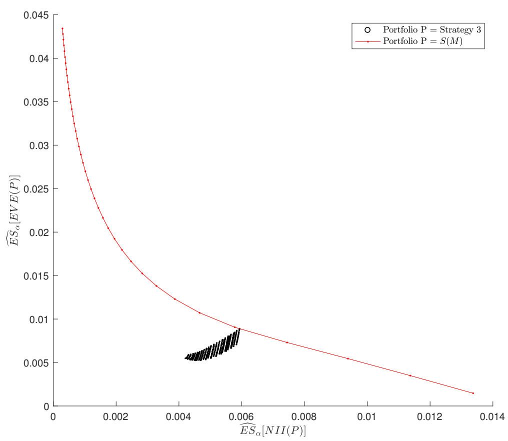
and liabilities at

$$M_{liability}, N_{liability} \in \{5.75, 6, 6.25\}.$$

Risk pairs for Strategy 3 can be seen in figure 6.14b. Strategy 2 and 3 illustrate that less risky risk pairs can be achieved than matching assets and liabilities to the largest extent possible. With Strategy 2 producing low NII risk and Strategy 3 producing low EVE risk.



(a) Risk pairs for strategy 2



(b) Risk pairs for strategy 3

Figure 6.14: Risk pairs $(\widehat{ES}_\alpha[\text{NII}(P)], \widehat{ES}_\alpha[\text{EVE}(P)])$ for two different strategies

6.6 Banking book applicability

We have now considered the trade-off problem in EVE and NII risk an investor faces when deciding how to invest his assets in various portfolio strategies $S(M)$ and combinations thereof. We have also considered the trade-off problem when borrowing and investing additional funds. Remember the balance sheet equation, assets are equal to equity plus liabilities. The investment problem discussed in section 6.4 could be thought of as the investment problem facing a stylized bank that has matched its liability portfolio with a chunk of its asset portfolio and then has the possibility to freely invest the remaining assets (corresponding to the equity liability). If we instead assume that the bank perfectly matches most but not all of its liabilities with its assets, the investment decision could be formulated as the investment problem discussed in 6.5, i.e. making a decision about

- USD 1 in assets corresponding to the equity liability,
- USD 1 in additional assets that were previously matched to a liability,
- USD 1 in liabilities that were previously matched to an asset.

This is illustrated in figure 6.15, where balance sheet #1 illustrates the previous case when liabilities were perfectly matched with assets and balance sheet #2 illustrates the new problem. For balance sheet #1, our results in 6.4 indicated that an investment in a single strategy $S(M)$ yields approximately equal or better risk pairs than a combination of $S(M)$ strategies.

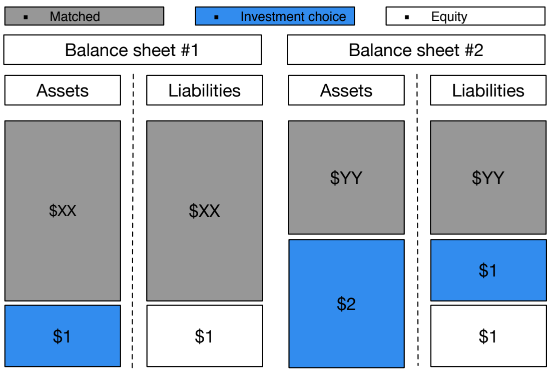


Figure 6.15: Liabilities perfectly matched with assets and liabilities not perfectly matched with assets.

For balance sheet #2 we notice that in our model less risky risk pairs can be achieved by Strategy 2 and 3 from 6.5 than alternatively matching assets and liabilities to the largest extent possible, i.e. balance sheet #1 in figure 6.15. Meanwhile, Strategy 1 in 6.5 produces more risky risk pairs for balance sheet #2 than a balance sheet #1 strategy. Thus the choice of strategy for balance sheet #2 will decide if the risk pairs achieved are better or worse than balance sheet #1.

Chapter 7

Conclusion

The purpose of this thesis was to propose a model that can be used to consistently study how varying the maturity structure of a portfolio affects both NII and EVE risk. Only a small amount of publicly available research exists in this area and due to various definitions of NII risk there is no uniform method for measuring this risk. By taking into consideration the upcoming regulatory framework IRRBB we have proposed a model of rolling investments in par-coupon bonds that we deem consistently captures how the maturity structure affects NII and EVE risk. Since we are not able to derive analytical expressions of the model's risk we have employed a Monte Carlo simulation. With the simulation procedure, risk of different maturity strategies is examined. The plausibility of some of the model's theoretical assumptions have also been investigated. The theoretical assumption of continuous par-coupon bond investments seems to be a good approximation of a discrete time investment model. The theoretical assumption of a constant historical zero-coupon curve is also found to be a good approximation of a model where actual historical zero-coupon curves are included. We conclude that a constant historical zero-coupon curve is arguably a more suitable choice for determining the risk trade-off in different strategies.

Simulations are performed with the Vasicek short-rate model. The number of state variables governing the short rate does not influence the risk relations of different strategies to a large extent. However, the different Vasicek models employed gives rise to varying levels of risk and by this we conclude that a lot of effort should be put into the short-rate model's parameter estimates. Different zero-coupon curve shapes seem to have a limited effect on the risk trade-off when varying the maturity structure. We suggest using the 3-factor Vasicek model to be more in line with the prescribed stress scenarios of IRRBB.

In regards to the risk trade-off problem with only an investment considered, the best strategy is almost always found to be a single $S(M)$ portfolio, regardless of the EVE and NII risks preferred. This means that for investment strategies consisting of long investments in two different $S(M)$-strategies, the achieved risk pairs are worse or approximately equal to the risk pairs that are achieved with a single $S(M)$ investment.

When the risk trade-off problem concerns borrowing and investing the additional funds, a large range of risk pairs can be achieved. The risk pairs of a single investment in $S(M)$ can be achieved by matching the short portfolio $S(M)$ with a corresponding long strategy. An actively risk-taking strategy such as borrowing short and lending long will produce much greater risks than a strategy of matching borrowed and invested funds. We have also shown that better risk pairs than a matched asset and liability strategy can be achieved by taking into consideration the risk levels and correlations of different $S(M)$.

We end by noting that, as the industry commented on the IRRBB proposal, banking books contain many heterogeneous instruments, which are not easily modeled. Thus when trying to study how a generic banking book reacts to certain effects simplifications have to be made. In this thesis we chose to study IRR effects for a rather simple portfolio, which enabled us to focus solely on the maturity structure. In section 6.6 we showed how one could translate our portfolio approach to a banking book setting. However, it is reasonable to assume that this approach does not fully capture effects that might exist when more complicated instruments are included. This means that our simulation results should not be interpreted as showing universally true results, but should instead be viewed as a starting point for determining a desirable maturity structure in the banking book with respect to NII and EVE risk.

7.1 Further research

We propose two different paths for further research into the relationship between NII and EVE risk. The first being to further study the risk profile generated by the model studied in this thesis and the second being to extend this model to more accurately describe a complete banking book.

To further study our model we identify several options. Firstly, in this thesis short-rate paths were simulated from the Vasicek model. The Vasicek model, belonging to the class of ATS models, provided us with simpler expressions and made the computations more tractable. The Vasicek model is arguably a simpler short-rate model and although multiple types of zero-coupon curve shifts can be achieved with the 3-factor version it would be of interest to see how the resulting risk profile changes using alternative models. Secondly, we have evaluated risk over a period of one month and the effects of varying this assumption could be investigated. For this problem a possible approach would be to see how national regulators motivate the interest rates shifts that are to be used in their implementations of IRRBB. Thirdly, a more thorough investigation of how combining different portfolio strategies affect the risk profile could be conducted. In this thesis usage of a single rolling portfolio is found to be a good approximation for efficient risk pairs if only long positions are allowed. However, this is not the case when introducing the possibility to short sell similar portfolio strategies. An optimization approach could be attempted to find the most efficient risk pairs of long-short strategies. Furthermore, we limited ourselves in this thesis to evaluating if combinations of portfolios could produce more, in the Pareto sense, efficient risk profiles than in the single portfolio case. We did not try to say anything about which of these efficient portfolios would be the most preferable. There are several ways of tackling this problem, one could be to take into account the expected return of different strategies, another could be to use utility functions to value EVE and NII risk.

Regarding possible extensions of the model, the two natural extensions would be to include the two main types of IRR not studied in this thesis: basis risk and optionality risk. Basis risk could be included in our model by assuming that asset and liability portfolios are sensitive to non-perfectly correlated interest rates of the same currency. This would lead to two portfolio strategies with the same maturity structure not necessarily cancelling each other out. The model could also be extended to include multiple currencies, which introduces basis risk but unfortunately also foreign exchange risk. Another extension could be to include items such as non-maturing deposits with optionality risk and which limits banks' possibility to freely choose a maturity structure. Our suggestion in this area is to use the Pillar 1 approach from IRRBB or the resulting implementations from national regulators due to the heterogeneity of instruments in the banking book and BCBS's movements towards the use of more standardized models. We end by noting that Alessandri and Drehmann (2010) have argued that IRR should be studied in tandem with credit risk. A final extension to our model could therefore be to incorporate credit risk by including credit spreads and modeling default risk.

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Appendix A

Model sensitivity to parameter estimates

In this appendix we illustrate how the resulting risk profile changes depending on if the zero-coupon curve is upward or downward sloping. We will also show how the risk profile is affected by upward and downward shocks of 20% to the parameters θ, κ, σ and λ.

If we in the single-factor Vasicek model set the initial short rate, $r(t)$, to be greater than $\bar{\theta}$, the resulting zero-coupon curves will be downward sloping. We set $r(t) = \bar{\theta} + 0.01$ and simulate. The resulting simulated zero-coupon curves and risks can be seen in figures A.1 and A.2, where we compare them to the upward sloping risks. It should be noted that our altered initial short rate is significantly higher than the original short rate which should impact the results as well.

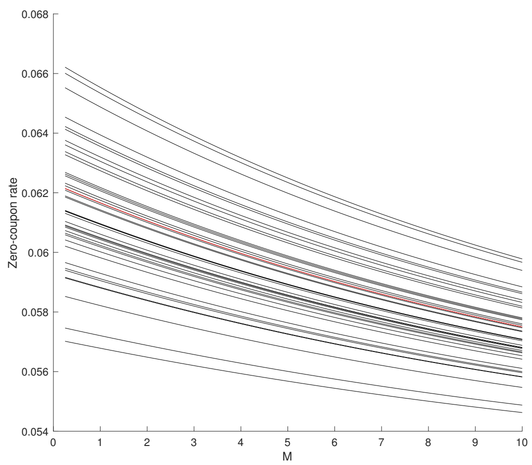
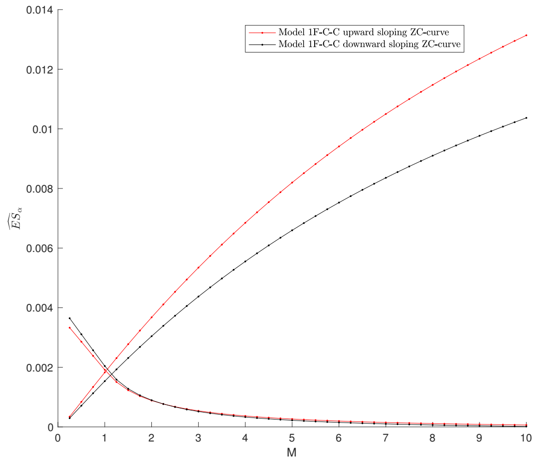
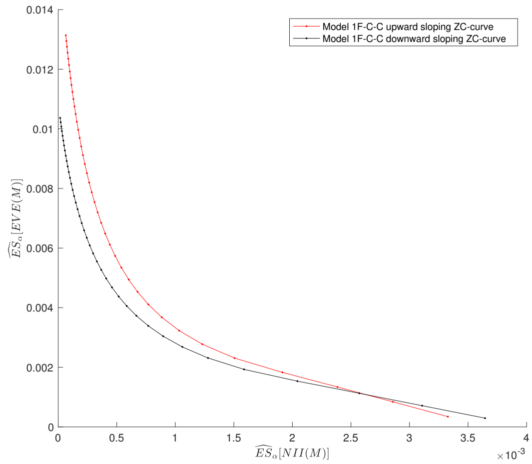


Figure A.1: Simulated downward sloping single-factor Vasicek zero-coupon curves

The single-factor model's parameters are stressed by a 20% increase and decrease in each parameter. The results for θ and κ can be seen in figure A.3. The results for σ and λ can be seen in figure A.4. We notice that the model's risks are most sensitive to changes in σ and κ.

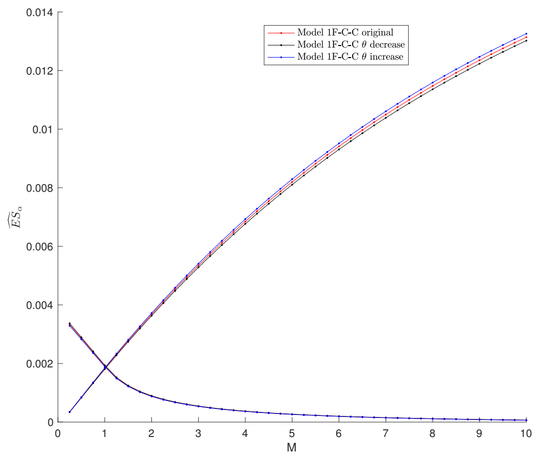


(a) $\widehat{ES}_\alpha$ for different maturities M

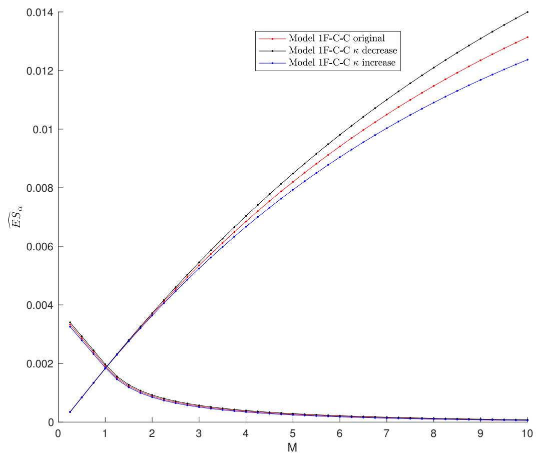


(b) Risk pairs $(\widehat{ES}_\alpha[NII(M)], \widehat{ES}_\alpha[EVE(M)])$

Figure A.2: Expected Shortfall risk for upward and downward sloping zero-coupon curve in the single-factor Vasicek model

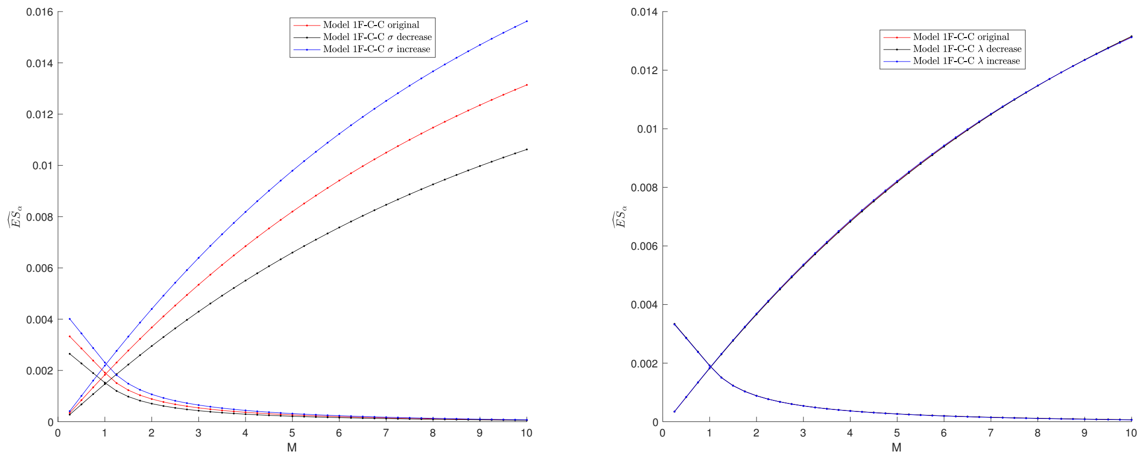


(a) Stressed θ



(b) Stressed κ

Figure A.3: $\widehat{ES}_\alpha$ for different maturities M in the single-factor Vasicek model, 20% increase and decrease of θ and κ



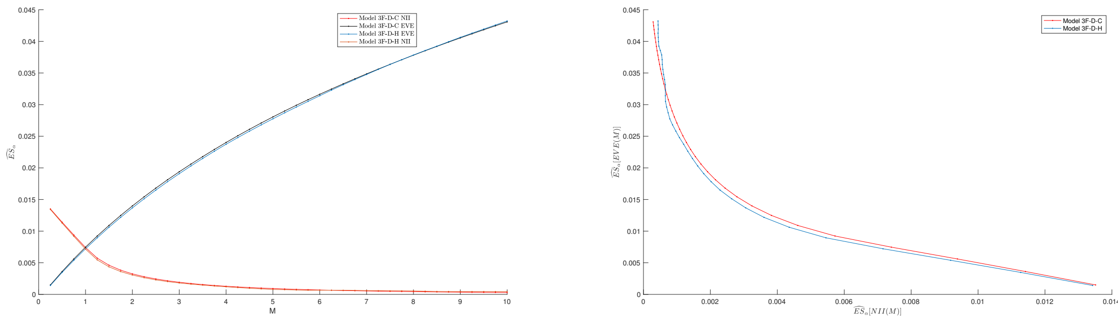
(a) Stressed σ (b) Stressed λ

Figure A.4: $\widehat{ES}_\alpha$ for different maturities M in the single-factor Vasicek model, 20% increase and decrease of σ and λ

Appendix B

3F-D-C and 3F-D-H comparison

In this appendix we compare the resulting risk-profiles from 3F-D-C and 3F-D-H, i.e. altering the assumption regarding the historical zero-coupon curve when using discrete rolling and the 3-factor Vasicek model.



(a) $\widehat{ES}_\alpha$ for different maturities M

(b) Risk pairs $(\widehat{ES}_\alpha[NII(M)], \widehat{ES}_\alpha[EVE(M)])$

Figure B.1: Expected Shortfall comparison between 3F-D-C and 3F-D-H

Appendix C

NII with the measurement period extended toward infinity

In this appendix we show that using our definitions of NII and EVE, NII converges to EVE if the measurement period is extended toward ∞. We will denote this NII measure as $\text{NII}_\infty(t, M)$. To find the expression for NII as the measurement period is extended toward ∞ we change the limits of integration in equation 4.3.13 such that

$$\text{NII}_\infty(t, M) = \int_0^M \left(\bar{c}_{C1}(s; t, M) + \bar{c}_{C2}(s; t, M)\right) p(t, t+s) ds + \int_M^\infty \bar{c}_{C3}(s; t, M) p(t, t+s) ds, \quad \text{(C.0.1)}$$

with $\bar{c}_{C1}$, $\bar{c}_{C2}$ and $\bar{c}_{C3}$ defined as in equation 4.3.12. The first integral represents all coupon payments we receive before all bonds bought before t mature. Hence, the second integral represents all the coupon payments received after these have matured. Without loss of generality we set $t = 0$ to ease notation and see that

$$M(\text{EVE}(0, M) - \text{NII}_\infty(0, M)), \quad \text{(C.0.2)}$$

can be written as

$$\int_0^M p(0, s) ds - \int_0^M \int_0^s c(0, u, u+M) p(0, s) du ds - \int_M^\infty \int_{s-M}^s c(0, u, u+M) p(0, s) du ds. \quad \text{(C.0.3)}$$

We now try to rewrite $p(0, s)$ to show that this expression is equal to zero. To do this we use the definition of the forward par-coupon from equation 3.1.5 iteratively to show that

$$\begin{aligned} p(0, s) &= p(0, s+M) + \int_s^{s+M} c(0, s, s+M) p(0, u) du \\ &= p(0, s+2m) + \int_s^{s+M} c(0, s, s+M) p(0, u) du + \int_{s+M}^{s+2M} c(0, s+M, s+2M) p(0, u) du \\ &= p(0, s+nM) + \sum_{k=1}^{n} c(0, s+(k-1)M, s+kM) \int_{s+(k-1)M}^{s+kM} p(0, u) du \end{aligned} \quad \text{(C.0.4)}$$

We now note that if we integrate one of the terms from the sum between 0 and M with respect to s and set $\bar{s} = s + M$ we get

$$\int_0^M c(0, s+(k-1)M, s+kM) \int_{s+(k-1)M}^{s+kM} p(0,u)duds = \int_{(k-1)M}^{kM} c(0,\bar{s},\bar{s}+M) \int_{\bar{s}}^{\bar{s}+M} p(0,u)dud\bar{s}. \tag{C.0.5}$$

Thus if we integrate $p(0, s)$ from 0 to M we can using equations C.0.4 and C.0.5 get

$$\int_0^M p(0,s)ds = \int_0^{nM} \int_s^{s+M} c(0,s,s+M)p(0,u)duds + \int_0^M p(0,s+nM)ds.$$

Given that the ZCB price goes toward zero as the maturity increases toward infinity we have that

$$\begin{aligned}\int_0^M p(0,s)ds &= \int_0^\infty \int_s^{s+M} c(0,s,s+M)p(0,u)duds \\ &= \int_0^M \int_0^u c(0,s,s+M)p(0,u)dsdu + \int_M^\infty \int_{u-M}^u c(0,s,s+M)p(0,u)dsdu,\end{aligned} \tag{C.0.6}$$

where in the final step we have changed the order of integration. Now we see that by plugging in this expression in equation C.0.3 we have shown that $\text{NII}_\infty(t,M) = \text{EVE}(t,M)$.

Appendix D

Confidence intervals using the non-parametric bootstrap

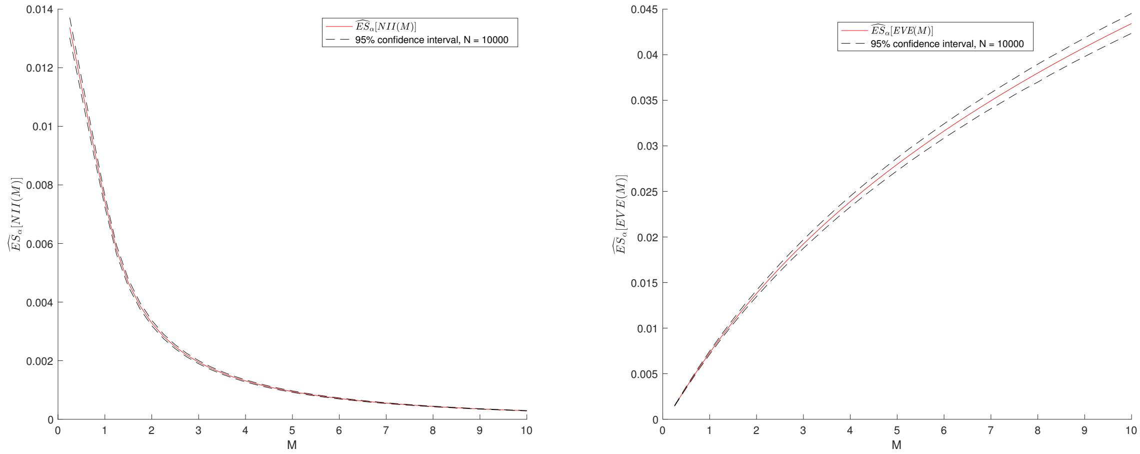
In this appendix we construct confidence intervals for our $\widehat{ES}$ estimates when our outcomes are of unknown distribution F. A useful method for constructing approximate confidence intervals for a risk measure such as $\widehat{ES}$ is the non-parametric bootstrap method. The following methodology is based on (Hult et al., 2012). For a discussion on the validity of this method, see (Hult et al., 2012).

Consider the observed sample outcomes $\{x_1, \ldots, x_n\}$ with empirical distribution F_n. To estimate the quantity $\theta(F)$ we have our empirical estimate $\hat{\theta}_{obs} = \theta(F_n)$. Since we only have one sample of outcomes, a method for producing more samples is to draw with replacement from our original sample $\{x_1, \ldots, x_n\}$. By drawing with replacement n times from our original sample we can construct a new sample $\{X_1^*, \ldots, X_n^*\}$ with empirical distribution F_n^* and quantity θ estimate $\hat{\theta}^*$. By repeating this procedure N times we can compute N estimates of $\hat{\theta}^*$, $\{\hat{\theta}_1^*, \ldots, \hat{\theta}_N^*\}$ and N residuals $R^* = \hat{\theta}_{obs} - \hat{\theta}^*$, $\{R_1^*, \ldots, R_N^*\}$. With our bootstrapped samples we can construct an approximate confidence interval with confidence level q as

$$I_{\theta,q} = (\hat{\theta}_{obs} + R^*_{[N(1+q)/2]+1,N}, \hat{\theta}_{obs} + R^*_{[N(1-q)/2]+1,N}),$$

where $R^*_{1,N} \geq R^*_{2,N} \geq \cdots \geq R^*_{N,N}$ is our ordered sample of $\{R_1^*, \ldots, R_N^*\}$.

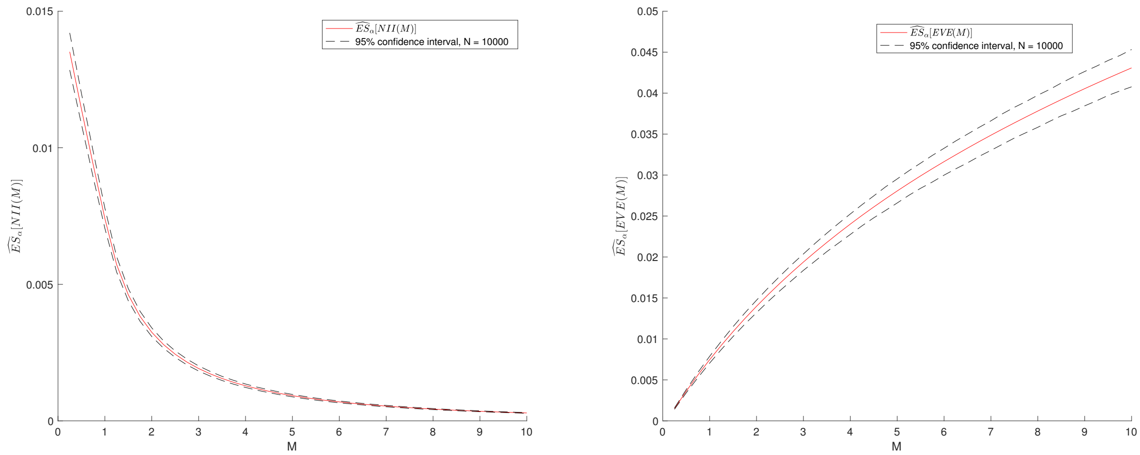
We employ the above procedure for our models $\widehat{ES}$ estimates. In figure D.1 the results can be seen for model 3F-C-C with 8000 samples, confidence intervals for $\widehat{ES}_\alpha[\text{NII}(M)]$ are found in figure D.1a and $\widehat{ES}_\alpha[\text{EVE}(M)]$ in figure D.1b. Figure D.2 shows the corresponding results for model 3F-D-C with 2000 samples. The remaining model simulations yield similar results and have been left out. $\widehat{ES}_\alpha[\text{NII}(M)]$ and $\widehat{ES}_\alpha[\text{EVE}(M)]$ have also been calculated for our samples up until the sample sizes to show convergence. These results for our models 3F-C-C and with 8000 samples and 3F-D-C with 2000 samples can be seen in figures D.3 and D.4.



(a) $\widehat{ES}_\alpha$[NII(M)] confidence intervals

(b) $\widehat{ES}_\alpha$[EVE(M)] confidence intervals

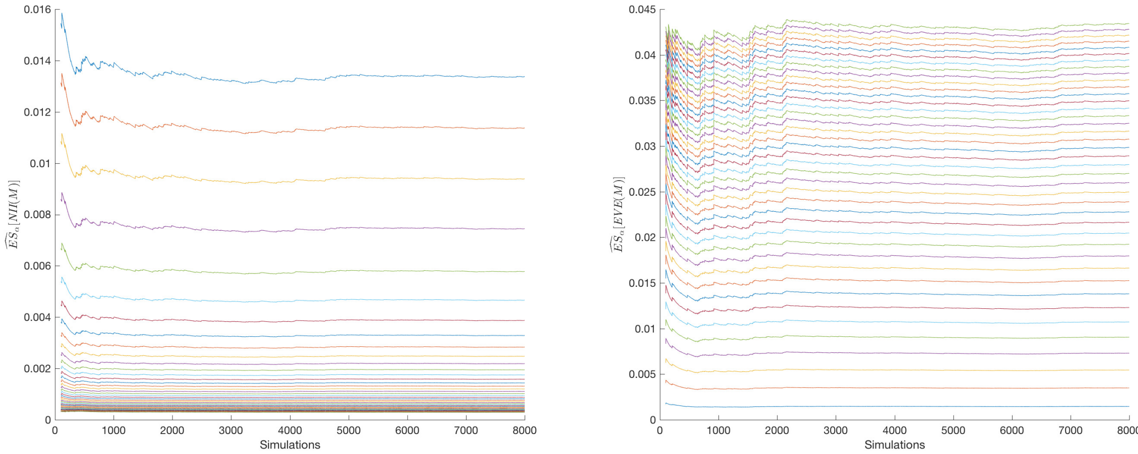
Figure D.1: $\widehat{ES}_\alpha$[NII(M)] and $\widehat{ES}_\alpha$[EVE(M)] confidence intervals for model 3F-C-C, $n = 8000$, $N = 10000$, $q = 0.95$



(a) $\widehat{ES}_\alpha$[NII(M)] confidence intervals

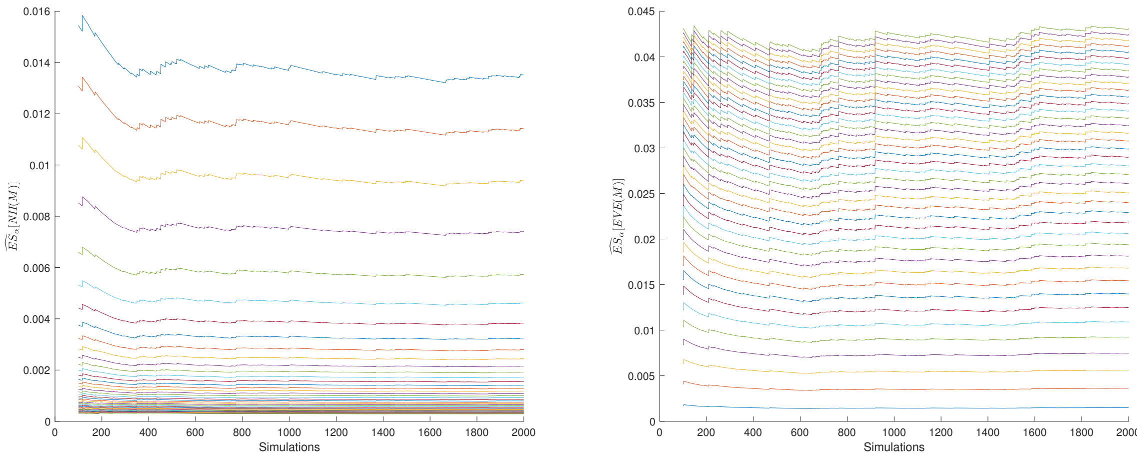
(b) $\widehat{ES}_\alpha$[EVE(M)] confidence intervals

Figure D.2: $\widehat{ES}_\alpha$[NII(M)] and $\widehat{ES}_\alpha$[EVE(M)] confidence intervals for model 3F-D-C, $n = 2000$, $N = 10000$, $q = 0.95$



(a) $\widehat{ES}_\alpha[\text{NII}(M)]$ convergence (b) $\widehat{ES}_\alpha[\text{EVE}(M)]$ convergence

Figure D.3: $\widehat{ES}_\alpha[\text{NII}(M)]$ and $\widehat{ES}_\alpha[\text{EVE}(M)]$ convergence for model 3F-C-C, up until 8000 simulated samples, $M \in \{0.25, \ldots, 10\}$



(a) $\widehat{ES}_\alpha[\text{NII}(M)]$ convergence (b) $\widehat{ES}_\alpha[\text{EVE}(M)]$ convergence

Figure D.4: $\widehat{ES}_\alpha[\text{NII}(M)]$ and $\widehat{ES}_\alpha[\text{EVE}(M)]$ convergence for model 3F-D-C, up until 2000 simulated samples, $M \in \{0.25, \ldots, 10\}$

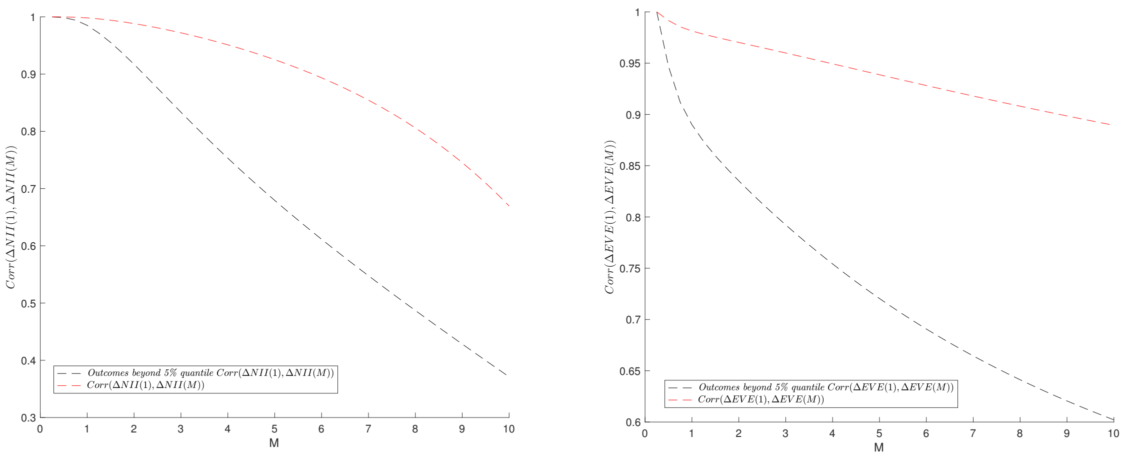
Appendix E

Correlation of worst outcomes for S(M)-portfolios

In this appendix we investigate the estimated correlations of $S(N)$ outcomes with $S(M)$ outcomes conditional on the $S(N)$ outcomes being beyond the 5%-quantile. In figure E.1a NII correlations of the outcomes beyond the 5%-quantile for $S(0.25)$ and corresponding outcomes of other $S(M)$ are shown, i.e.

$$\widehat{corr}(\Delta NII(0.25), \Delta NII(M)|\Delta NII(0.25) \geq \text{VaR}_{\alpha}(NII(0.25)))$$

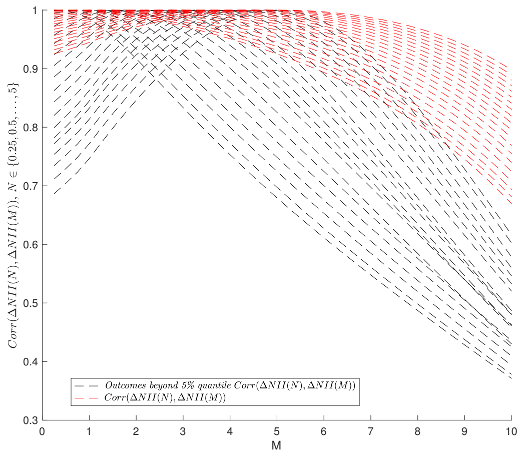
together with the correlations of all outcomes. The corresponding correlations for EVE are shown in figure E.1b. NII correlations for $0.25 \leq N \leq 5$ and $5 \leq N \leq 10$ are shown in figure E.2 and the respective EVE correlations are shown in figure E.3



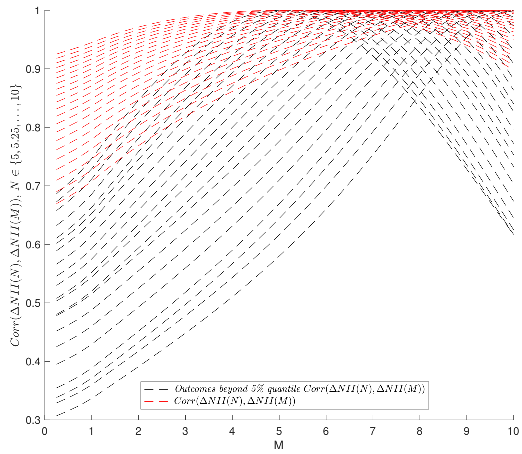
(a) NII correlations

(b) EVE correlations

Figure E.1: Correlations between worst outcomes of $S(0.25)$ and corresponding outcomes of $S(M)$

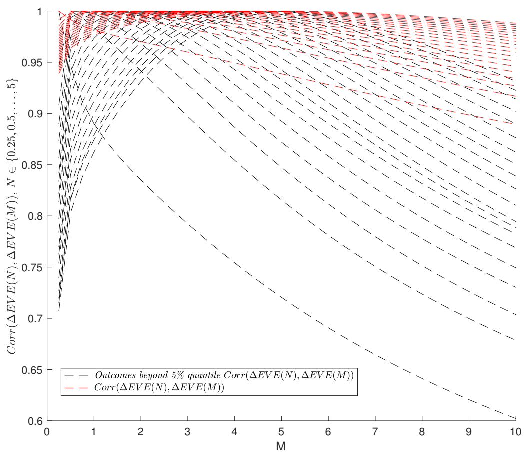


(a) $N \in \{0.25, 0.5, \ldots, 5\}$

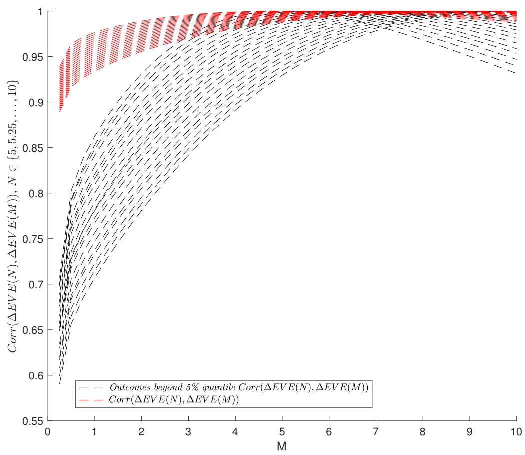


(b) $N \in \{5, 5.25, \ldots, 10\}$

Figure E.2: $\widehat{corr}(\Delta\text{NII}(N), \Delta\text{NII}(M) | \Delta\text{NII}(N) \geq \text{VaR}_\alpha(\text{NII}(N)))$



(a) $N \in \{0.25, 0.5, \ldots, 5\}$



(b) $N \in \{5, 5.25, \ldots, 10\}$

Figure E.3: $\widehat{corr}(\Delta\text{EVE}(N), \Delta\text{EVE}(M) | \Delta\text{EVE}(N) \geq \text{VaR}_\alpha(\text{EVE}(N)))$

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