Internal model for spread risk under Solvency II

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Abstract

In May 2009 the European Commission decided on new regulations regarding solvency among insurance firms, the Solvency II Directive. The directive aims to strengthen the connection between the requirement of solvency and risks for insurance firms. The directive partly consists of a market risk module, in which a credit spread risk is a sub category.

In this thesis a model for credit spread risk is implemented. The model is an extended version of the Jarrow, Lando and Turnbull model [19] as proposed by Dubrana [8]. The implementation includes the calibration of a stochastic credit risk driver as well as a simulation of bond returns with the allowance of credit transitions and defaults.

The modeling will be made with the requirements of the Solvency II Directive in mind. Finally, the result will be compared with the Solvency II standard formula for the spread risk sub-module.

Sammanfattning

I maj 2009 beslutade Europeiska kommissionen om nya bestämmelser gällande solvens bland försäkringsföretag, Solvens II-direktivet. Direktivet syftar till att stärka sambandet mellan kravet på solvens och risker för försäkringsföretag. Direktivet består delvis av en marknadsriskmodul där spread risk är en underkategori.

I detta arbete implementeras en modell för spread risk. Modellen är en utökad version av Jarrow, Lando och Turnbull modell [19], föreslagen av Dubrana [8]. Implementeringen innefattar kalibrering av en stokastisk riskdrivare samt simulering av en obligationsportföljs avkastning där övergångar mellan kreditbetyg och inställda betalningar är tillåtna.

Modellen kommer att göras med kraven i Solvens II-direktivet i åtanke. Slutligen kommer resultatet att jämföras med Solvens II standardformel för delmodulen för spread risk.

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Contents

1 Introduction

1.1 Background

In May 2009 the European Commission decided on new regulations regarding solvency among insurance firms, the Solvency II Directive. The directive provides a framework for the regulations that all members of the European Union will introduce to their legislation. Solvency II aims to strengthen the connection between the requirement of solvency and risks for insurance firms. It is also working to provide a stronger protection for consumers. The directive is furthermore a step for creating a common regulation for European insurance companies. The Solvency II Directive is based on three pillars [7]:

1. Quantitative requirements

2. Qualitative requirements

3. Disclosure and transparency requirements

This thesis will focus on the first pillar which sets out frameworks of requirements for calculations of technical provisions and Solvency Capital Requirement, using either a standard model provided by the regulators (see section 2.6) or a (partial) internal model developed by the insurance firm or a third party. The framework partly consists of a market risk module, which is defined as the risk of loss or of adverse change in the financial situation resulting, directly or indirectly, from fluctuations in the level and in the volatility of market prices of assets, liabilities and financial instruments. These risks can be divided into sub-categories; interest rate, equity, property, spread, currency and concentration [13]. The focus of this thesis is the spread risk, which is defined as risks that arise from changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure [7].

In the topic of credit risk modeling, a lot has been produced. These models revolve around two main types, structural and reduced form models. The differences between these models will be discussed and evaluated for the purpose of risk management. The implemented model is a Markov model for the term structure of credit spreads. The model is an extended version of the Jarrow, Lando and Turnbull model [19] as proposed by Dubrana [8]. The implementation includes the calibration of a stochastic credit risk driver as well as a simulation of bond returns with the allowance of credit transitions and defaults. The modeling will be made with the Solvency II Directive in mind. Finally,

the result will be compared with the Solvency II standard formula for the spread risk
sub-module.

2 Theoretical Background

This section will present the necessary mathematical theory needed to understand the credit model of choice as well as the implementation of it. Furthermore the definition of the Solvency Capital Requirement (SCR) will also be presented along with the standard formula for the spread risk sub-module.

2.1 Markov Chain Theory

The following definitions and theorems can be found in [10].

In a variety of different fields, one wishes to describe random events through time. These events through time can be described by stochastic processes. One family of such processes are called Markov processes (continuous) or Markov chains (discrete). These are characterized by their so called loss of memory, where the prediction of future outcomes are dependent only on the current state. This property is desired in the fields of predictive modeling or forecasting and hence it is useful for risk management purposes. In this section some useful definitions for both continuous and discrete Markov chains are presented, starting with the discrete time case.

Discrete-time Markov Chain

For a *discrete-time* Markov chain each stage n corresponds to a certain given *time point* with a constant *time step* between each point. Let the time step between two time points be equal to t. Then $p_{ij}(t)$ denotes the probability of moving from state i to state j over a time t. The transition matrix over a time step t is denoted as $P(t)$.

Definition 2.1: (Discrete Markov chain)

A stochastic process $\{X_n; n \geq 0\}$ is a Markov chain if

$$Pr(X_{n+1} = i_{n+1} | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = Pr(X_{n+1} = i_{n+1} | X_n = i_n)$$

for all stages n and all states $i_0, i_1, \ldots, i_{n+1}$. This definition means that a Markov chain is memory less in the sense that the distribution one time step ahead is dependent on the current state of the chain but not on previous states. The definition of states and state space can be found below.

Definition 2.2: (State space)

The finite or countable set S forms the *state space* of the Markov Chain, which is the set of all possible outcomes of $\{X_n; n \geq 0\}$. Each possible outcome $i_n \in S$ is called a *state*.

Definition 2.3: (Time Homogeneity)

The term *time-homogeneous Markov chain* implies that

$$Pr(X_{n+1} = a | X_n = b) = Pr(X_n = a | X_{n-1} = b)$$

Thus, the transition probability is independent of the stage n. Note however that a time-homogeneous Markov chain is *not* independent of the length between stages. In a time setting where the stages are time points, this would mean that the Markov chain is independent over time, but not independent of time step length. Naturally, the shorter time step, the less probable it is that the stochastic process has moved during that time.

Definition 2.4: (Transition probability)

The transition probabilities are defined as follows:

$$p_{ij} = Pr(X_1 = j | X_0 = i) \quad \text{and} \quad p_{ij}^{(n)} = Pr(X_n = j | X_0 = i)$$

corresponding to the single-step transition probability and the nth-step transition probability respectively. More specifically, p_{ij} is the probability of making a transition from state i to state j.

Definition 2.5: (Transition matrix)

For a finite state space S, we define the transition matrix P over K states as

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{pmatrix}$$

where the entries p_{ij} are transition probabilities as in Definition 2.4.

Theorem 2.1: (Properties of the transition matrix)
a) $\sum_{j=1}^{K} p_{ij} = 1$ for $i =, 1, 2 \ldots, K$
b) $p_{ij} \geq 0$ $\forall i, j = 1, 2, \ldots, K$

The claim in a) follows from Definition 2.4 since the sum of the probabilities of either staying in state j or moving to any other state j in the state space must be equal to one. The claim in b) is obvious since p_{ij} is a probability and therefore non-negative.

Continuous-time Markov Chain

A continuous-time Markov chain is defined by a finite or countable state space S (just like in the discrete time case) and a generator matrix Λ. For $i \neq j$, the elements λ_{ij} are non-negative and describe the rate of the process transitions from state i to state j. The elements λ_{ii} are chosen such that each row of the transition rate matrix sums to zero. Below are some definitions of the continuous time case, which is mostly natural modifications of the discrete time case presented above.

Definition 2.6: Continuous-time Markov Chain

A continuous stochastic process $X(t)$ is a (discrete) Markov process if

$$P(X(t_{n+1}) = i_{n+1} | X(t_0) = i_0, X(t_1) = i_1, \ldots, X(t_n) = i_n) = P(X(t_{n+1}) = i_{n+1} | X(t_n) = i_n)$$

for all n, all $0 \leq t_0 < t_1 < \cdots < t_{n+1}$ and all states.

Definition 2.7: (Generator matrix)

Let $X(t)$ denote a continuous-time Markov chain, as defined in Definition 2.6 above. Let $P(t)$ be the transition matrix in continuous time, S be the state space as in Definition 2.2 and Λ be defined as

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1K} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{K1} & \lambda_{K2} & \cdots & \lambda_{KK} \end{pmatrix}$$

where the element λ_{ij} denote the rate at which the process transitions from state i to state j.

Theorem 2.2: (Properties of the generator matrix)

The generator matrix Λ satisfies the following properties:

1. $0 \leq -\lambda_{ii} \leq \infty$
2. $\lambda_{ij} \geq 0$ for all $i \neq j$
3. $\sum_j \lambda_{ij} = 0$ for all i

Theorem 2.3: (Generator and transition matrix relation)

Consider a time step h. It follows from the definition of intensities that

$$\lambda_{ij} = \lim_{h \to 0+} \frac{p_{ij}(h) - 0}{h} \quad \text{for } i \neq j$$

$$\lambda_{ii} = \lim_{h \to 0+} \frac{p_{ij}(h) - 1}{h}$$

or in matrix form:

$$\Lambda = \lim_{h \to 0+} \frac{P(h) - I}{h}$$

where I is the identity matrix.

From Chapman-Kolmogorov,

$$P(t + h) = P(t)P(h) = P(h)P(t)$$

$$\Longleftrightarrow$$

$$P(t + h) - P(t) = P(t)(P(h) - I) = (P(h) - I)P(t)$$

and letting $h \to 0_+$ we get

$$P'(t) = P(t)\Lambda = \Lambda P(t)$$

which are called the *Kolmogorov forward* and *backward* equations. The equations are first order differential equations and have the unique solution

$$P(t) = e^{t\Lambda} \tag{1}$$

Theorem 2.4

Equation (1) can be rewritten as

$$P(t) = e^{t\Lambda} = I + t\Lambda + \frac{t\Lambda^2}{2!} + \frac{t\Lambda^3}{3!}\cdots$$

Furthermore, let

$$S = \max\{(a-1)^2 + b^2; a + bi \text{ is an eigenvalue of P}, a, b \in R\}$$

Let P be a time-homogeneous Markov transition matrix, i.e. an N $\times$ N real matrix with non-negative entries and with row-sums 1 and suppose that S < 1. Then the series

$$\tilde{\Lambda} = (P - I) + \frac{(P-I)^2}{2!} + \frac{(P-I)^3}{3!} + ..., \tag{2}$$

converges geometrically quickly and gives rise to an N $\times$ N matrix $\tilde{\Lambda}$ having row-sums 0, such that $e^{t\tilde{\Lambda}} = P(t)$ exactly.

Theorem 2.5

Suppose the diagonal entries of a transition matrix P are all greater than $1/2$ (i.e., $p_{ii} > 0.5$ for all i). Then S < 1, i.e. the convergence of the series in Equation (2) is guaranteed.

Remark.

In the case where P is diagonalizable with all eigenvalues real and positive, summing the series in (2) (if it converges) is equivalent to first diagonalizing P, then replacing the diagonal entries by their logarithms, and then converting back to the original basis.

2.2 Stochastic Differential Equation

The following definition can be found in [4].

Let $M(n, d)$ denote the class of $n \times d$ matrices, and consider as given the following objects

- A d-dimensional (column-vector) Wiener process W.

- A (column-vector valued) function μ: $R_+ \times R^n \to R^n$.

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- A function σ: $R_+ \times R^n \to M(n, d)$.

- A real (column) vector $x_0 \in R^n$.

We want to investigate whether there exists a stochastic process X which satisfies the *stochastic differential equation* (SDE)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$
$$X_0 = x_0.$$

To be more precise we want to find a process X satisfying the integral equation

$$X_t = x_0 + \int_0^t (s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \quad \text{for all } t \geq 0.$$

The standard method for proving the existence of a solution to the SDE above is to construct an iteration scheme of Cauchy–Picard type. The idea is to define a sequence of processes $X^0, X^1, X^2, \ldots$ according to the recursive definition

$$X_t^0 \equiv x_0$$
$$X_t^{n+1} = x_0 + \int_0^t \mu(s, X_s^n)ds + \int_0^t \sigma(s, X_s^n)dW_s$$

Having done this one expects that the sequence $\{X^n\}_{n=1}^{\infty}$ will converge to some limiting process X, and that this X is a solution to the SDE. Indeed, the iteration scheme works under mild conditions on μ and σ, see [24].

2.3 The Euler-Maruyama Scheme

The following definition can be found in [28].

Consider the stochastic differential equation

$$dX(t) = a(X(t))dt + b(X(t))dW(t)$$
$$X(0) = x_0$$

where $W(t)$ is a Wiener process and suppose we want to solve this SDE on some interval $[0, T]$. The Euler-Maruyama approximation is defined as follows

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- Divide the interval $[0, T]$ into N equal sub-intervals of width $\Delta t > 0$
  $$0 = \tau_0 < \tau_1 < \cdots < \tau_N = T \text{ and } \Delta t = T/N$$

- Define $Y_0 = x_0$

- Recursively define Y_n for $1 \leq n \leq N$ by

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n$$

where the random variables $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$ are independent and identically distributed normal random variables with $E[\Delta W_n] = 0$ and $Var[\Delta W_n] = \Delta t$

2.4   The Levenberg-Marquardt Algorithm

This section is based on [21], [22] and [23].

A least-squares problem is a problem where a function $f(x)$, that is a sum of squares, is minimized

$$\min_x f(x) = \|F(x)\|_2^2 = \sum_i F_i^2(x)$$

This type of problems occur when fitting model functions to data, i.e., nonlinear parameter estimation. They are also useful where you want the output, $\hat{y}(x)$, to follow some continuous model trajectory, $y(t)$, for some vector x and scalar t. This problem can be expressed as

$$\min_{x \in R^n} \int_t^T (\hat{y}(x,t) - y(t))^2 dt$$

where $\hat{y}(x, t)$ and $y(t)$ are scalar functions.

When the integral above is discretized it can be formulated as a least-squares (LS) problem:

$$\min_{x \in R^n} f(x) = \sum_{i=1}^m (\hat{y}(x, t_i) - y(t_i))^2$$

here, $\hat{y}(x, t)$ and $y(t)$ include the weights of the quadrature scheme. Note that in this problem the vector $F(x)$ is

$$F(x) = \begin{bmatrix} \hat{y}(x, t_1) - y(t_1) \\ \hat{y}(x, t_2) - y(t_2) \\ \vdots \\ \hat{y}(x, t_m) - y(t_m) \end{bmatrix}$$

Let $J(x)$ denote the $m \times n$ Jacobian matrix of $F(x)$. Furthermore let the gradient vector of $f(x)$ be denoted as $G(x)$, the Hessian matrix of $f(x)$ as $H(x)$, and the Hessian matrix of each $F_i(x)$ as $H_i(x)$, then you have

$$G(x) = 2J(s)^T F(x)$$

$$H(x) = 2J(x)^T J(x) + 2Q(x)$$

where

$$Q(x) = \sum_{i=1}^{m} F_i(x) \cdot H_i(x)$$

As the residual $\| F(x) \|$ tends to zero, so does the $Q(x)$ matrix. When the residual of the solution is small, a very effective method is to use the Levenberg-Marquardt method for an optimization. In the Levenberg-Marquardt, a search direction, d_k, is obtained at each major iteration, k, that is a solution of the linear set of equations:

$$\left(J(x_k)^T J(x_k) + \lambda_k I \right) d_k = -J(x_k)^T F(x_k)$$

where the scalar λ_k controls both the magnitude and direction of d_k. The search direction d_k can then be used in a line search strategy that ensures that the function $f(x)$ decreases at each iteration.

2.5 Arbitrage-free Cash Flows

As stated in [11], any bond may be viewed as a package of multiple cash flows — coupon or face — with each cash flow viewed as a zero-coupon instrument maturing on the date it will be received. Thus, rather than using a single discount rate one should use multiple discount rates, discounting each cash flow at its own rate. Each cash flow is then separately discounted at the same rate as a zero-coupon bond corresponding to the coupon or face date, and of equivalent credit worthiness.

2.6 Spread risk sub-module for bond in Solvency II

The regulation [7] defines the following simplified standard formula for the calculation of the solvency capital requirement (SCR) for spread risk on bonds

$$SCR_{bonds} = MV^{bonds}\left(\sum_i \%MV_i^{bonds} \cdot dur_i \cdot b_i + \%MV^{bondsNR} \cdot min[dur_{NR} \cdot 0.03, 1]\right)$$

where

- MV^{bonds} denotes the value of the bonds.

- $\%MV_i^{bonds}$ denotes the proportion of the bonds with credit quality step i, where a credit assessment is available for those assets.

- $\%MV^{bondsNR}$ denotes the proportion of the bonds for which no credit assessment is available.

- dur_i and dur_{NR} denote the modified duration denominated in years of the bonds, where a credit assessment is available and where no credit assessment is available respectively.

- b_i denotes a function of the credit quality step i of the bonds with credit quality step i, as defined in the table below.

Credit quality step i	0	1	2	3	4	5	6
b_i	0.9 %	1.1 %	1.4 %	2.5 %	4.5 %	7.5 %	7.5 %

Remark

In the regulation, the spread risk sub-module includes all form of securitization, credit derivatives as well as loans, but this thesis will only be focusing on requirement for bonds.

2.7 Value-at-Risk

The following definition is found in [17].

The Value at Risk at the 5% level is defined as the lowest value the portfolio will achieve 5% of the time. The definition is as follows

$$VaR_p(X) = \min\{m : P(mR_0 + X < 0) \leq p\}$$

where R_0 is the percentage return of a risk free asset.

This denotes the value-at-risk at level $p \in (0, 1)$ of a portfolio with value X.

In statistical terms $VaR_p(X)$ is the $(1 - p)$-quantile of X. The u-quantile of a random variable X with distribution function F_X is defined as

$$F_X^{-1}(u) = \min\{m : F_X(m) \geq u\}$$

where F_X^{-1} is the inverse of F_X if F_X is strictly increasing. If F_X is both continuous and increasing, then F_X^{-1} is the unique value m such that $F_X(m) = u$. From this it follows that
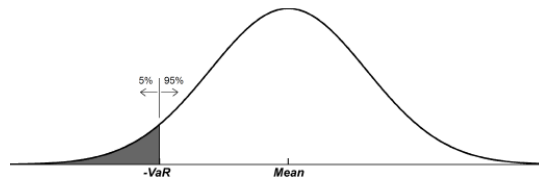
$$VaR_p(X) = F_X(1 - p)$$

Figure 1: Illustration of the Value at Risk at the 5% level

2.8 Definition of the Solvency Capital Requirement in Solvency II

Christiansen and Niemeyer [6] discuss the fundamental definition of the SCR taking into account the regulatory requirements and find the following two definitions

- Article 101 of the directive [7] requires that the SCR "shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5 % over a one-year period."

- Remark 64 of the directive [7] says that "the Solvency Capital Requirement should be determined as the economic capital to be held by insurance ... undertakings in order to ensure ... that those undertakings will still be in a position with a probability of at least 99.5 %, to meet their obligations to policy holders and beneficiaries over the following 12 months."

They present some possible interpretations of the two definitions and discuss their equivalence and consistency, which will be left out of this thesis. The one used in this thesis is

$$SCR(s) = VaR_{0.995}\big(N(0) - v(0,1)N(1)\big)$$

where $v(0,1)$ equals the discount factor that corresponds to the risk free interest rate and $N(t)$ denotes the value of the portfolio at time t.

2.9 The Inverse Transform Method

The following definition can be found in [25].

Let $F(x)$, $x \in R$, denote any cumulative distribution function (cdf) (continuous or not). Let $F^1(y)$, $y \in [0,1]$ denote the inverse function of $F(x)$. Define $X = F^{-1}(U)$, where U has the continuous uniform distribution over the interval $[0,1]$. Then X is distributed as F, that is, $P(X) = F(x)$, $x \in R$.

3 Credit Risk Modeling

3.1 Introduction

There are two types of models available for modeling credit risk; structural ones and reduced form ones. As [18] states, a common view of these types of classes is that they are disjoint or disconnected of each other, and consequently there is discussion in the academic literature which of these models that is the best in terms of hedging performance and default prediction. The key difference between these classes is often stated as the difference in the assumptions required. [18] argues that the difference lies in the information available rather than the "required" assumption. Structural models assume that the information available is the firm's asset value and liabilities. In contrast, reduced form models assume that the information is that observed by the market. Given this insight [18] states that structural models can be transformed into reduced form models by changing the informational assumptions and making them less refined. The model class preferred depends on the purpose of the model. If the purpose is to judge the own firms default risk for capital considerations, then [18] argues that structural models are the best. On the other hand, if the purpose of the model is to price or hedge, i.e. risk management, then the reduced form perspective is preferred.

3.2 Model Framework

Given that the purpose of this thesis lies in the field of risk management the model of choice is a reduced form one, as considered preferred in accordance with [18]. Furthermore, it requires less detailed information about the firms assets and liabilities and is more consistent with the market. This is a characteristic which is desired when modeling market risks in Solvency II. The model presented below is found in [8] and is based on the framework presented in [19] and [3]. The model is a Markov model for the term structure of credit spreads through time which incorporates the credit rating as an indicator of the probability of default. Other characteristics include seniority debt in terms of recovery rates (i.e. fraction of face value payed if defaulted), and possibility of combining with any term structure for default free debt. It also uses historical transition probabilities for different credit ratings for valuation.

Consider a friction less economy over some finite time period. The traded securities in the economy are a default-free money market account, default-free zero coupon bonds of

all maturities, and risky zero coupon bonds of all maturities. Assume that there exists a unique equivalent martingale measure $\tilde{P}$, making the default-free and risky zero-coupon bond prices martingales, which is the equivalent to the market of these bonds being complete and free of arbitrage [19]. Let $p(t, T)$ denote the price at time t of a default-free zero coupon bond maturing at time T. The money market account accumulates returns at the spot rate $r(t)$ (with any desired term structure) and is denoted as

$$B_t = \exp\{\int_0^t r(s)ds\}$$

Under the assumption that the market is complete and free of arbitrage the default-free zero coupon bond prices is defined as the expected, discounted value at time T

$$p(t, T) = E_t^{\tilde{P}}\left[\frac{B(t)}{B(T)}|\mathcal{F}_t\right] = E_t^{\tilde{P}}\left[\exp\{-\int_t^T r(t)dt\}|\mathcal{F}_t\right]$$

Let $v(t, T)$ be the price at time t of a risky zero coupon bond maturing at time T. In case of bankruptcy before time T, the holder of bond receives a fraction, δ, of the promised face value. The fraction, or recovery rate, is determined by the priority of the bond debt compared to other debts of the bond issuer. [19] assumes the recovery rate to be an external constant independent of the spot rate $r(t)$, an assumption that can easily be relaxed. At the same time, studies have shown that δ indeed varies dependent on time and rating [9] [2]. This is further discussed later on.

Given this assumption, if t^* is the time of default, then the price $v(t, T)$ is defined as

$$v(t, T) = E_t^{\tilde{P}}\left[\frac{B(t)}{B(T)}(\delta 1_{(t^* \leq T)} + 1_{(t^* > T)})|\mathcal{F}_t\right] = E_t^{\tilde{P}}\left[\frac{B(t)}{B(T)}|\mathcal{F}_t\right]E_t^{\tilde{P}}\left[\delta 1_{(t^* \leq T)} + 1_{(t^* > T)}|\mathcal{F}_t\right]$$

$$= p(t, T)(\delta + (1 - \delta)\tilde{P}_t(t^* > T)) \tag{3}$$

As can be seen from Equation (3) above, the risky zero coupon bond's price is the default-free zero coupon bonds price times the expected payoff at time T. This definition was first presented by [20] and was then further developed by [19] by modeling the distribution of default as a time-homogeneous Markov chain over a finite state space S.

Let the state space $S = \{1, ..., K\}$ consist of different credit ratings, with state 1 being the highest, $K - 1$ the lowest and K default. The Markov chain is specified by a $K \times K$ transition matrix

$$\tilde{P} = \begin{pmatrix} \tilde{p}_{11} & \tilde{p}_{12} & \cdots & \tilde{p}_{1K} \\ \tilde{p}_{21} & \tilde{p}_{22} & \cdots & \tilde{p}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The entries $\tilde{p}_{ij}$ represent the probabilities under the equivalent martingale measure of going from state i to state j in one time step, which implies that $\tilde{p}_{ij} \geq 0$ for all $i, j, i \neq j$ and $\tilde{p}_{ii} = 1 - \sum_{\substack{j=1 \\ j \neq i}}^{K} \tilde{p}_{ij}$ for all i. [19] assumes again, for simplicity, that default (state K) is an absorbing state. This assumption can easily be relaxed as shown by [20].

Equation (3) can then be rewritten

$$p(t,T)(\delta + (1-\delta)\tilde{P}_t(t^* > T)) = p(t,T)(1 - (1-\delta)\tilde{P}_t(t^* \leq T))$$
$$= p(t,T)(1 - (1-\delta)\tilde{p}_{iK}) \tag{4}$$

In order to obtain the pseudo-probabilities under $\tilde{P}$, [19] introduce a credit risk premium $\pi(t)$. As pointed out by [8], two major drawbacks of the original JLT-model is the assumptions of a constant transition matrix and that the credit risk premium is a deterministic function of time. These assumptions will lead to deterministic spreads and are therefore not suitable for risk purposes.

The extension of the JLT-model proposed by [3] includes a stochastic risk premium in order to better fit with historical observations. First it is assumed that the probability generator matrix is diagonalizable:

$$\Lambda = \Sigma D \Sigma^{-1}$$

where D is is the diagonal matrix of eigenvalues of Λ and the column of $\Sigma \in R^{K \times K}$ the right eigenvectors of Λ. [3] mentions that the eigenvectors have been observed being

stable over time, hence they are assumed to be constant. Given this assumption, they can be calculated from historical transition matrices by using the relationship between the transition matrix and generator matrix in accordance with Theorems 2.3, 2.4 and 2.5.

The eigenvalues are on the other hand modelled as stochastic and [8] consider the following case in order to move from historical eigenvalues to risk adjusted under :

$$\tilde{D}(t) = \pi(t)D$$

where $\pi(t)$ is a stochastic process. The transition matrix can be written

$$\tilde{P}(t,T) = \Sigma E^{\tilde{P}}\big[\exp(D\int_t^T \pi(s)ds)\big]\Sigma^{-1} \tag{5}$$

In particular, the default probabilities are then given by

$$\tilde{p}_{iK}(t,T) = \sum_{j=1}^{K-1}\sigma_{ij}\tilde{\sigma}_{jK}\{E^{\tilde{P}}\big[\exp(d_j\int_t^T \pi(s)ds)\big] - 1\} \tag{6}$$

where σ_{ij} and $\tilde{\sigma}_{jK}$ are the elements of the matrices Σ and Σ^{-1} respectively.

The choice of stochastic process, $\pi(t)$, needs to be made with some considerations in mind. [8] argues that a mean reverting model allows for different economic cycles which can be observed when analyzing spread movement. As an example, when economic conditions decline issuers creditworthiness follows and consequently spreads increase. At the same time the declining economic condition will eventually recover and thus leading to increase in issuers creditworthiness followed by a decrease in the spreads. Furthermore it is crucial that the stochastic process of choice only produce positive probabilities otherwise the spreads would become negative, which contradicts the definition of spreads. The choice of the Cox-Ingersoll-Ross (CIR) model in [8] is therefore a well motivated. The CIR process has the following SDE

$$d\pi(t) = \alpha(\mu - \pi(t))dt + \sigma\sqrt{\pi(t)}dW(t) \tag{7}$$

where the parameter α, mean reversion speed, controls at which speed the reversion movement is happening. Due to the inclusion of the square root of time $\sqrt{\pi(t)}$, the model only produces positive risk premium, which guarantees positive spreads. Given this stochastic process the expectation in Equation (6) can be derived analytically in

a similar way as performed when calculating zero coupon bond prices within the CIR-process, namely

$$E[\exp(d_j \int_t^T \pi(s)ds)] = \exp(A_j(t,T) - \pi(t)B_j(t,T))$$

with

$$A_j(t,T) = \frac{2\alpha\mu}{\sigma^2} \ln \left(\frac{2v_j e^{\frac{1}{2}(\alpha+v_j)(T-t)}}{(v_j + \alpha)(e^{v_j(T-t)} - 1) + 2v_j} \right)$$

$$B_j(t,T) = \frac{-2d_j(e^{v_j(T-t)} - 1)}{(v_j + \alpha)(e^{v_j(T-t)} - 1) + 2v_j}$$

$$v_j = \sqrt{\alpha^2 - 2d_j\sigma^2}$$

4 Case Study

This chapter consists of a case study where the model presented in the previous chapter is implemented. The steps of the implementation includes estimation of the generator matrix Λ, calibration of the stochastic risk driver $\pi(t)$ and finally the simulation of bond returns with the allowance of credit rating migrations and defaults. The data used for the estimation of the generator matrix is a one year transition matrix provided by Moody's Analytics and can be found in the Appendix.

The simulation of bond returns is based on 222 bonds with an average duration around 4.3 years. The bond data was provided by Skandia. As can be seen from the table below, the portfolio mainly consists of Government bonds, AAA-rated bonds and bonds where no rating is available. The large share of bonds without a rating is an issue since the decision of which rating to assess them will heavily influence the result of the model. The regulation [7] gives bonds with no rating available the stress of $\min[dur_{NR} \cdot 0.03, 1]$ as presented in section 2.6. Comparing this stress with the table of b_i presented in the same section and taking into account that the average duration for the bonds without a rating in the portfolio is 3.21 years, they are assessed with the credit rating BBB. The same reasoning can be found in the Swedish Financial Supervisory Authority Finansinspektionen's proposed regulation for occupational pension companies [12]. The government bonds are assessed with the credit rating AAA. Both of these assessments are considered conservative. Discussions regarding their implications will be brought up in Section 6.

Rating	% of portfolio value	Average duration (years)	Number of bonds
Government	36.5	6.94	70
AAA	31.3	3.82	36
AA	0.4	1.29	2
A	1.3	3.89	7
BBB	0.9	2.76	7
BB	0.1	3.21	1
B	0.2	3.5	4
CCC	0	N/A	0
No rating	29.3	3.21	95

Table 1: Overview of the bond portfolio used for simulations

Based on the characteristics of the bond portfolio, the data series used for calibration of the parameters of the CIR-process consists of monthly historical spread movements for Handelsbankens Mortgage Bond Index (HMSMLY35). The index is a AAA-rated index with a duration of 3-5 years and was retrieved from Bloomberg. Note that in order to get the spread movements of the AAA-rated index, the yield of the index is reduced by the government bond yield with matching duration at each point in time. Figure 2 below shows the index between the dates 1995-01 and 2017-04. This time period is considered to be representative for the calibration of the π process. The choice of the index is motivated by the fact that 2/3 of the bond portfolio is AAA-rated with an average duration around 5 years. Given the fact that the stochastic driver is calibrated to AAA spreads, one can expect that the spreads for lower ratings will be less consistent with observed market spreads. This will be discussed later on.
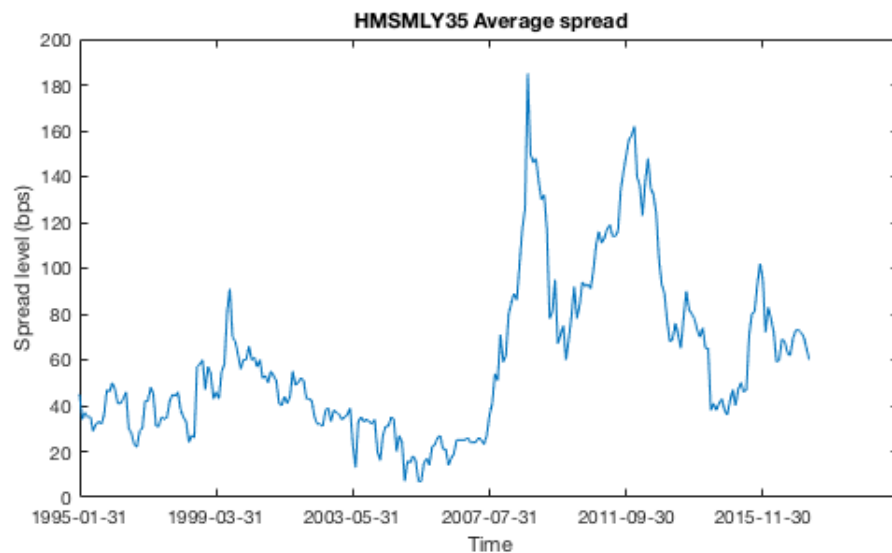

Figure 2: The calibration index plotted between Jan. 1995 and Apr. 2017

Table 2 below summarizes some distributional properties of the index.

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Statistic	HMSMLY35
Mean	0.0059
Standard Deviation	0.0035
Skewness	1.0933
Kurtosis	3.6774
Max (bps)	0.0185
Min	0.0007

Table 2: Statistics of the calibration index

The definitions for the third and fourth moment, i.e. the skewness and the kurtosis, used in thesis are:

$$Skewness = \frac{E[x - \mu]^3}{\sigma^3}$$

$$Kurtosis = \frac{E[x - \mu]^4}{\sigma^4}$$

where μ and σ are the mean and the standard deviation.

4.1 Calibration

The first step in the calibration of the CIR process to a spread level is to estimate the generator matrix. This is done by combining Theorems 2.4, 2.5 and 2.6. The estimate of the one year transition matrix provided by Moody's Analytics is used to obtain an approximation of the generator matrix Λ, which enables calculations of the transition probabilities $P(t,T)$ for any t and T, i.e. $T - t$ year transition matrix. Since P is diagonalizable with all eigenvalues real and positive ($S < 1$), the generator matrix is calculated as

$$\Lambda = S \ln(D) S^{-1}$$

where D is is the diagonal matrix of eigenvalues of P and the column of $S \in R^{K \times K}$ the right eigenvectors of P. The estimate of the 1-year transition matrix P provided by Moody's along with the resulting generator matrix can be found in the Appendix. Note that the entries of the transition matrix are defined as long term probabilities and thus the long term properties will be translated to the resulting generator matrix, making it

the long term generator matrix.

Once the estimate of the generator matrix is obtained, the calibration of the CIR process $\pi(t)$ to historical spreads can be performed. For this purpose the Levenberg-Marquardt algorithm (see section 2.4) is used. The distribution of the calibration index is considered to be a realistic distribution for possible spread levels of a AAA rated bond with maturity of 4 years. In order to produce realistic scenarios for the spread movements, the CIR-process is calibrated on monthly time steps over the course of one year. The distribution (based on 10 000 scenarios) of the of spread level after one year will be calibrated to the distribution of the historical index. The targets that the CIR-process will be calibrated to are the mean, the standard deviation and the skewness of the historical index as presented in Table 2. The initial value at time zero, $\pi(0)$, is also included in the calibration, this value was the level of the spread at the end of April 2017. With these targets the problem vector of the optimization problem is defined as

$$F(x) = \begin{bmatrix} E[s(1)] - 0.0059 \\ std(s(1)) - 0.0035 \\ skew(s(1)) - 1.0933 \\ s(0) - 0.0060 \end{bmatrix}$$

where $s(t)$ denotes the spread level at time t, of a AAA rated bond with a maturity of 4 years. A modified version of the Euler-Mayurama scheme discrete-time approximation of the CIR process is used here. The modification consists of taking the absolute value of the right hand side of Equation (7) in order to guarantee non-negativity. There are many other methods for discretization of diffusion processes including the ones presented by [14], [1] and others. The scheme used is specified as follows

$$\pi_t = \|\pi_{t-1} + \alpha(\mu - \pi_{t-1})\Delta t + \sigma\sqrt{\Delta t \cdot \pi_{t-1}}\Delta W_t\| \tag{8}$$

The recovery rate δ is also calibrated to the spread level. The benefits of calibrating this parameter includes additional degrees of freedom along with a recovery rate that is consistent with market expectations.

4.1.1 Fitted values

Rating	Target	Target value	Calibrated value
AAA	Mean	0.0059	0.0060
	Standard deviation	0.0035	0.0035
	Skewness	1.0933	1.0933
	Initial spread level	0.0060	0.0060

Table 3: Comparison between the targeted values and the calibrated values

As can be observed by Table 3 above, the targeted values are all met which implies that the calibrated parameters of the π process along with recovery rate δ should produce realistic spread movements for AAA-rated bonds. Below are the calibrated parameters of the π process and the calibrated recovery rate δ.

Parameter	Calibrated value
α	0.0592
μ	2.5112
σ	1.0816
$\pi(0)$	7.9823
δ	0.6423

Table 4: Calibrated π parameters and recovery rate

As can be observed in the table above, the initial value $\pi(0)$ is higher than the mean reverting level μ which will force the spreads to revert to a lower spread level at a fairly high pace. At the same time the relatively low α parameter in combination with the high volatility parameter σ will allow for desired fluctuations of the spreads. The plots below show the initial spread curves for different ratings, a simulation of spread movements and the spread level of a AAA rated bond for different recovery rates, as produced by the calibrated parameters in Table 4.
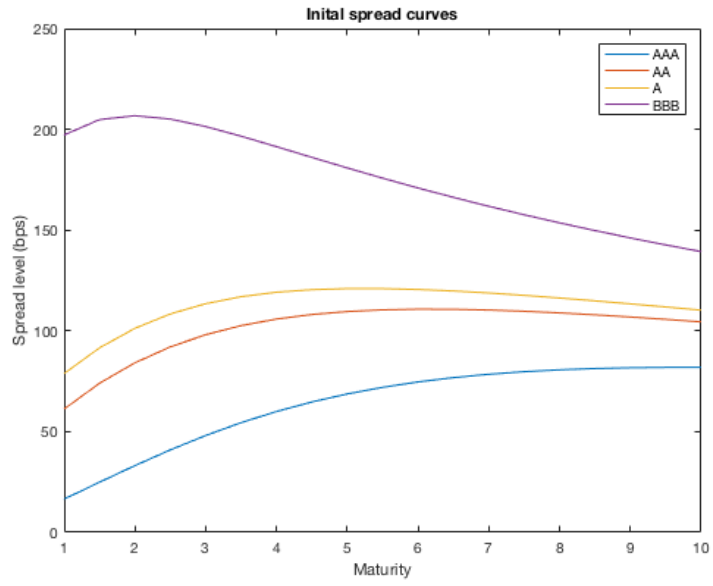
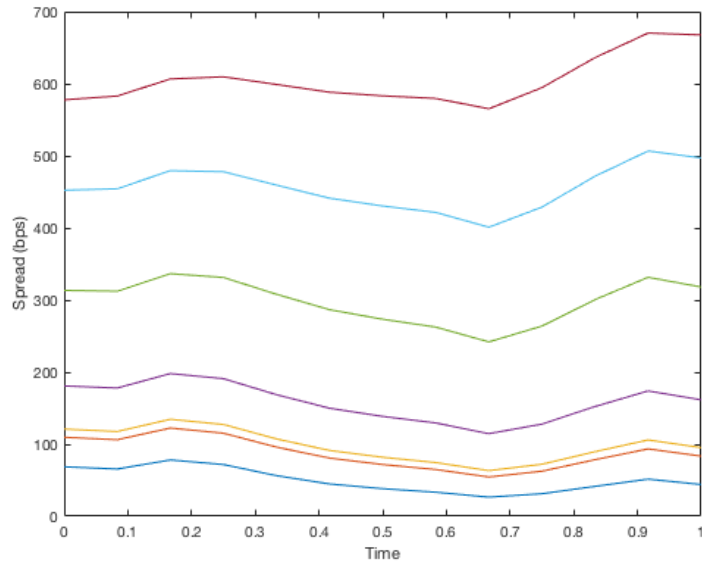Figure 3: Initial spread curve for different ratings produced by the model

Figure 4: A simulation of spread movements with rating AAA being at the bottom of the graph and CCC being at the top
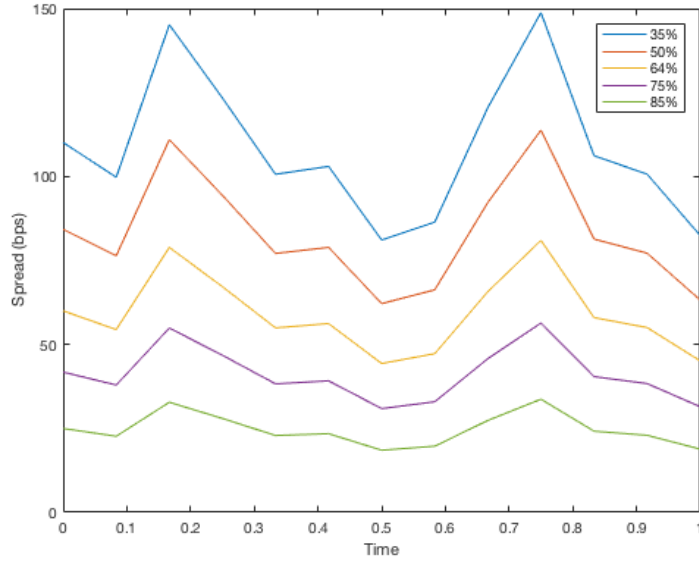
Figure 5: Spread level for different values of the recovery rate δ for rating AAA

As can be observed in figure 5, the spread level is highly sensitive towards the assumed constant recovery rate. A low value of this parameter results in a higher spread and vice versa. This along with a downward sloping spread curve for BBB in figure 3 suggests that the recovery rate should be rating and/or time dependent, given that this behaviour is not consistent with that observed in the market. A downward sloping spread curve implies that bonds with lower maturities in that rating is considered more risky than bonds with higher maturities. However, given that all of the targets in the calibration were met, the calibrated value of the recovery rate will be used and assumed to be constant for all bonds of all ratings and maturities. This rather strong assumption, and its implications, will be discussed later on.

Figure 4 shows a simulation of the spread movements for all ratings. As can be observed all ratings are exposed to the same credit risk driver π, hence they are all subject to the same movements. This guarantees that the spreads of lower ratings to higher compared to higher ratings, which is required given the properties of the transition matrix.

4.2 Simulation

In this section the method used for the simulation of credit migrations and bond portfolio returns is presented. The simulation will be based on the calibrated $\pi(t)$ process, as presented in the previous section.

4.2.1 Credit Transitions

The simulation of credit transitions is constructed by using the one month (1/12-year) transition matrix in each of the scenarios, i.e. the probabilities of migration from one rating to another in one month. The transition matrix is obtained by using Equation (5) along with the realization of the $\pi(t)$ process for each scenario as presented in Equation (8). The simulation of migration between ratings is done by using the discrete inverse transform method, see section 2.9. The steps of this method is outlined as follows.

- Draw a number U from a $U(0,1)$ distribution.

- The state at time $t+1$ for a bond with rating i at time t is j if $\sum_{k=0}^{j-1} \tilde{p}_{ik} < U \leq \sum_{k=0}^{j} \tilde{p}_{ik}$, where $\tilde{p}_{ik}$ denote the entries of the one month transition matrix $\tilde{P}(1/12)$ as defined in Equation (5).

These steps are performed for each issuer independently at each time step.

4.2.2 Bond Portfolio

For simulating the bond portfolio returns, the theory presented in section 2.5 is used. As mentioned one can view the cash flow of a coupon bond as a package of multiple zero coupon bonds, each with the maturity of the coupon payments. With this approach the value of a coupon bond is simply the sum of the package of zero coupon bonds at each point in time.

In order to obtain the risk free bond prices $p(t,T)$ as required in Equation (4), a short rate model is needed. The model used is the CIR model with the following SDE

$$dr(t) = a(b - r(t))dt + \sigma_r \sqrt{r(t)}dW(t)$$

As familiar, the CIR process does not produce negative rates but this is at the same time a desired feature considering the current low interest rate climate. In order to obtain this feature, the rate at each time step is displaced by a given constant, γ.

As presented in [4], the price at time t of a zero coupon bond with maturity T in the CIR model is given by

$$p(t,T) = A(t,T)e^{-(r(t)-\gamma)B(t,T)}$$

where

$$B(t,T) = \frac{2(e^{v(T-t)} - 1)}{2v + (a+v)(e^{v(T-t)} - 1)}$$

$$A(t,T) = \left(\frac{2ve^{(a+v)(T-t)/2}}{2v + (a+v)(e^{v(T-t)} - 1)} \right)^{2ab/\sigma_r^2}$$

with

$$v = \sqrt{a^2 + 2\sigma_r^2}$$

Here the standard Euler-Maruyama scheme, as presented in section 2.3 is used for the discretization of the short rate. The parameters of the short rate along with the initial yield curves can be found in Table 5 and figure 6 respectively. This set of parameters was obtained through discussions with Skandia and with respect to the observed yield curve of Swedish government bonds at the end of April 2017.

Parameter	Value
a	0.07
b	0.042
σ_r	0.15
$r(0)$	0.01
γ	0.017

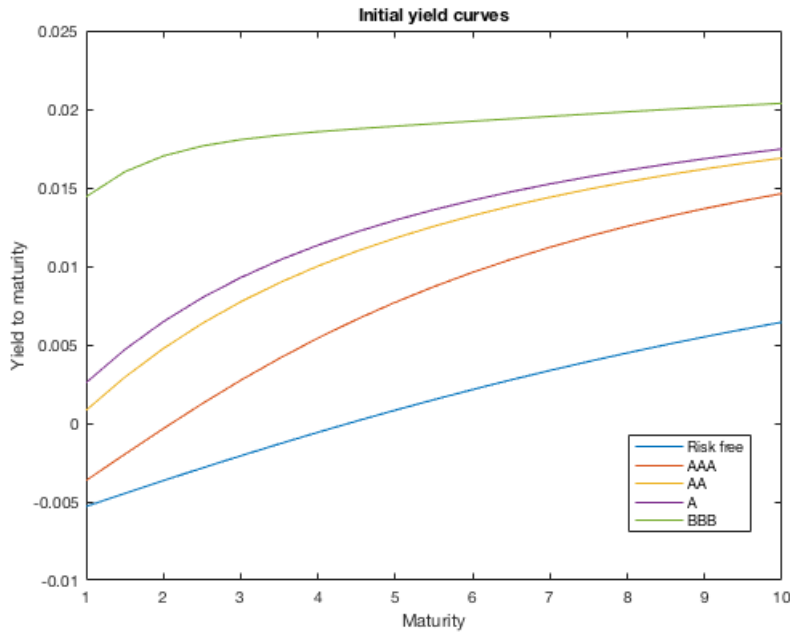Table 5: The parameters used for the short rate model

Figure 6: Initial yield curves for different ratings

The investment strategy implemented is a buy-and-hold strategy, outlined as follows. At time zero, the initial value is calculated by using the initial short rate. At each following time step t a revaluation is performed by using the simulated rating at the corresponding time step along with the short rate $r(t)$, face value, coupon and time to maturity (for coupon and/or bond). In the case when a coupon is received and/or bond has matured, it is not reinvested. In the case of a default of a bond at time t, the recovery rate δ times the face value is received which, again, is not reinvested. This way of modeling allows the bond returns to be driven by the spread model rather than the short rate model. However the model will produce some noise by the use of the short rate. A sensitivity analysis of the model will be made were a constant short rate (or equivalently, a constant yield curve) is used. The one year return of bond i is then calculated as

$$r_i = \frac{v_i(1, T)}{v_i(0, T)} - 1$$

where $v_i(1, T)$ and $v_i(0, T)$ denotes the value of bond i after one year and at time zero respectively. The portfolio return r_p is then obtained by

$$r_p = \sum_{i=1}^{n} w_i \cdot r_i$$

where w_i is the portfolio weight of bond i. This is done for each scenario, creating a one year return distribution for the portfolio from which we can calculate the SCR, as presented in section 2.7. In accordance with the regulation [7], the SCR is calculated at a 99.5% confidence level and measured over a one year horizon. The initial value $N(0)$ is set to 1 and $N(1)$ is calculated as $(1 + r_p)$ for each scenario.

4.3 Results

The below figure shows the distribution of the yearly portfolio returns. The short rate has been simulated on monthly time steps in accordance with the previous section. As one can observe, the combination of low interest rates (with a high mean reversion level of 4.2 %), a third of the portfolio value given the rating BBB produces a fairly wide downside. Given the definition of SCR presented in section 2.7 and with the discount factor equal the price of a zero coupon bond of maturity 1 under the short rate model presented in the previous section, the SCR of the implemented model is 12.82 % of the initial portfolio value at time zero.
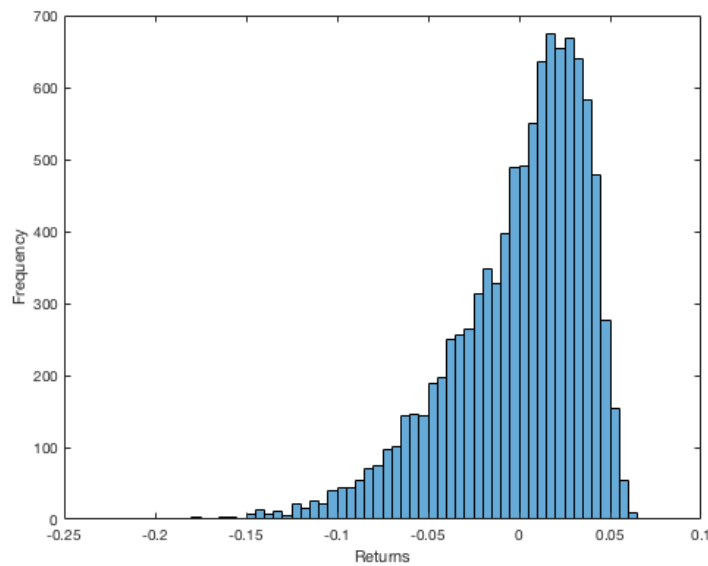

Figure 7: Histogram of the yearly portfolio returns

In order to filter some of the noise produced by the short rate model (noise in the sense that the returns driven by the spread movements is less detectable), a constant short rate is used for the same scenarios of spreads movements, transition and defaults as in the above dynamic version. In this case the initial yield curve, as presented in figure 6, is used. The resulting distribution of returns can be found below. This gives, as expected, a distribution with a thinner left tail compared to the distribution in figure 7. The SCR in this case is 7.17 %, given the same assumption regarding the discount factor as in the dynamic case.
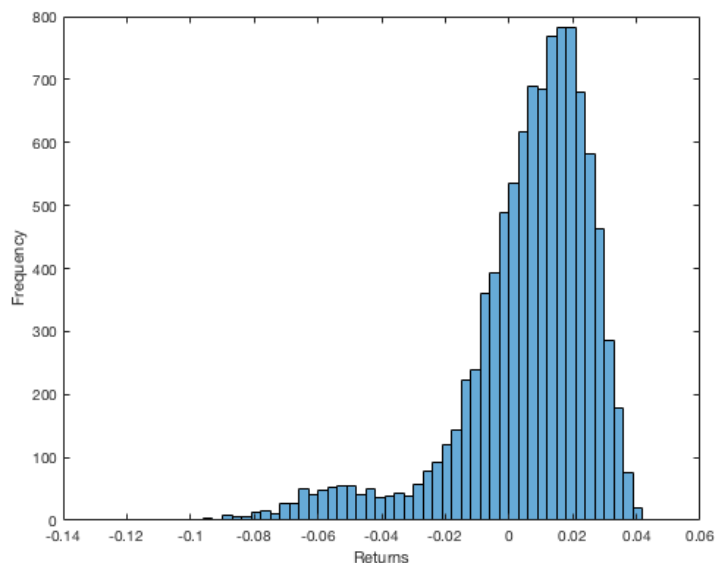

Figure 8: Histogram of the yearly portfolio returns

As be can be seen from the figure above, the distribution of portfolio returns in this case shows tendencies of being bimodal. The arise of this extra "hump" can be explained by extreme scenarios in which many issuers have defaulted. In the case of default of an issuer the return of the issued bond is

$$r = \frac{F\delta}{v(0,T)} - 1$$

where F is the face value of the bond, δ the recovery rate and $v(0,T)$ the initial value of the bond at time zero. For low values of δ the return will be smaller, moving the extra "hump" to the left. By the same reasoning, a high value of δ will result in higher returns and thus the "hump" will be centered further to the right. Hence the recovery

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rate can be said to determine the "distance" between the two modals, and consequently the quantile of the distribution.

5 Comparison with the Standard Formula

A comparison of the SCR produced by the model and the SCR of the standard formula can be seen in Table 6 below. Worth mentioning is that the standard formula does not model transitions or defaults. It only considers a parallel shift of the yield curve. This explains why the SCR of the implemented model is a lot higher. Another area of difference between the standard formula and the implemented model is that the model views government bonds as AAA-rated while the standard formula do not require any capital for these bonds. A more reasonable approach in the comparison would be to adjust for these differences by only simulating the spread movements without the allowance of transition or defaults and ignoring the returns on the government bonds. Given this approach the SCR of the model, based on the same simulation of the stochastic credit risk driver π, would be 3.68 %.

	Standard Formula	Model	Model without Gov.bonds
SCR	3.09 %	12.82 %	3.68 %

Table 6: Comparison between the SCR of the standard formula and the implemented model

Since the SCR of the Solvency II directive is defined at the 99.5 %-level, i.e. on a 1 out of 200 years scenario, an appropriate comparison would be to compare the 99.5 %-quantiles of the spreads produced in the simulation to the parallel shift proposed by the standard formula. The quantiles of the spreads are calculated based on the average duration of each rating class of the bond portfolio, which was presented in table 1. The parallel shift of the standard formula (SF) is then replicated by taking the difference between the initial spread level and the quantile. The results of this comparison can be found in table below.

b_i	0	1	2	3	4	5	6	No rating
SF	0.9 %	1.1 %	1.4	2.5 %	4.5 %	7.5 %	7.5 %	3 %
Model	1.17 %	1.24 %	1.42 %	1.77 %	1.79 %	1.42 %	0.97 %*	1.49 %*

Table 7: Comparison between the parallel shifts of the standard formula and the implemented model

Note that the quantile of credit quality step 6 was calculated by using a maturity of 4 years and the the no rating shift was calculated as the spreads of BBB-rated bonds but with the duration of 3.21 (as assumed in the bond portfolio modeling). As can be observed the model produces parallel shifts comparable to those of the standard formula between ratings AAA and A. For lower ratings the model fails to produce realistic shifts since the credit risk driver is calibrated to a AAA rated bond. This is simply an illustration of the difficulties regarding the calibration of the π process, since calibrating to lower ratings in order to match the spread movements for those would be at the cost of a poor match for higher ratings. However, for the bond portfolio used in the case study, one could argue that the model produces realistic spread quantiles when compared with the standard formula.

6 Discussion & Conclusions

This thesis has implemented and evaluated, on behalf on Skandia, a potential model for spread risk under the Solvency II Directive. The model is the extended version of the JLT-framework with a stochastic credit risk driver, as proposed by [8]. The major difference between the standard formula and the proposed model is that the latter includes default risk when simulating the bond returns, whereas the standard formula simply assumes a parallel shift of the yield curve for each credit rating. Naturally this results in a higher SCR (12.82 %) than the standard formula (3.05 %).

Another difference between the two, is that the implemented model relies on a short rate model. As shown the modeling of this effects the outcome of the model heavily, since using the model with a constant short rate (or yield curve) reduced the requirement by roughly 40 %. To make a comparison between the proposed model and the standard formula more fair, a third approach was considered. This approach, presented in chapter 5, was performed by ignoring the returns of government bonds and without the allowance of transitions and defaults. This resulted in a SCR of 3.68 %. Notably in the first two approaches is that while the standard formula assumes a zero stress for government bonds, the implementation of the model assumes that these bonds are AAA-rated which is considered a conservative assumption. The choice of this assumption was made through discussions with Skandia, and is motivated by the fact that these bonds indeed are exposed to spread risk. A more refined way to handle these bonds, in the framework of the proposed model, would be to use a sovereign transition matrix to obtain the default probabilities of government bonds of different ratings. Natural questions are then how to calibrate the credit risk driver, to which spreads and given which base. Other ways to handle government bonds have been proposed by [26], [5] and [27] among others, but given the framework outlined in this thesis the best approach would be to make use of the sovereign transition matrices available and calibrate a sovereign credit risk driver for these spreads.

For the case study the π process was calibrated to historical movements of AAA-rated spreads with a maturity of 3-5 years (based on the spread between the HMSMLY35 average yield and the Swedish government bond yield of matching maturity). This was motivated by the fact that, given the above mentioned assumption regarding government bonds, around two thirds of the given bond portfolio value was allocated in this rating

class with an average maturity of 5 years. This calibration made the implemented model produce less market consistent spread movements for lower credit ratings, as expected. Possible remedies for this can be obtained by the use of the recovery rate δ. The recovery rate was used in the calibration of the credit risk driver, in order to produce a better fit, and was then assumed to be constant for all bonds, for all ratings. However, as shown through a sensitivity analysis, the sensitivity of the spread levels towards the recovery rate is high. This holds true especially for speculative grade bonds. In order to obtain reasonable spread movements across all ratings, the recovery rate should be delicately handled. Literature has specified different variables explaining the recovery rate, including seniority, debt ratio and macroeconomic climate [2] [15].

One of the main strengths of the implemented model is its flexibility, making it adaptable to different preferences. This includes the above mentioned recovery rate, as well as dependency towards other market risks within the Solvency II framework. The simulation of transitions was made by the use of the inverse transformation method and was assumed to be independent between issuers. This assumption could be developed into being dependent with other risk factors within the market risk sub-module. The standard formula assumes 0.75 correlation between spread risk and equity risk which is based on historical stressed scenarios. However as stated by [16], this correlation is based on a momentarily observation of daily spreads levels and equity returns over a short time period, which can be questioned given the yearly measurement of the Solvency II directive. Limitations of the model include the above mentioned limited degrees of freedom in the calibration of the stochastic credit risk driver as well as the reliance of an accurate estimate of the generator matrix, making the performance of the model heavily dependent of the transition matrix used for this estimation. This is however a well documented area, and due to the scope of this thesis it was left out.

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Appendix

The following transition matrix has been used in the case study in section 4. It is based on observed transitions and defaults between 1990 and 2016 and was provided by Moody's Analytics.

$$P(1) = \begin{pmatrix} 0.9420 & 0.0542 & 0.0020 & 0.0004 & 0.0004 & 0.0004 & 0.0004 & 0.0001 \\ 0.0103 & 0.9010 & 0.0669 & 0.0143 & 0.0039 & 0.0014 & 0.0012 & 0.0010 \\ 0.0096 & 0.0280 & 0.9198 & 0.0345 & 0.0039 & 0.0014 & 0.0012 & 0.0016 \\ 0.0087 & 0.0094 & 0.0545 & 0.8779 & 0.0408 & 0.0016 & 0.0012 & 0.0059 \\ 0.0010 & 0.0092 & 0.0109 & 0.0582 & 0.8406 & 0.0601 & 0.0012 & 0.0189 \\ 0.0002 & 0.0083 & 0.0099 & 0.0106 & 0.0692 & 0.8163 & 0.0313 & 0.0542 \\ 0.0000 & 0.0008 & 0.0009 & 0.0028 & 0.0692 & 0.0902 & 0.7046 & 0.1314 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The estimate of the corresponding probability generator matrix is

$$\Lambda = \begin{pmatrix} -0.0598 & 0.0559 & 0.0021 & 0.0004 & 0.0004 & 0.0004 & 0.0004 & 0.0001 \\ 0.0108 & -0.1042 & 0.0705 & 0.0151 & 0.0041 & 0.0014 & 0.0013 & 0.0011 \\ 0.0101 & 0.0292 & -0.0836 & 0.0360 & 0.0040 & 0.0014 & 0.0013 & 0.0017 \\ 0.0093 & 0.0100 & 0.0581 & -0.1302 & 0.0435 & 0.0017 & 0.0013 & 0.0062 \\ 0.0011 & 0.0100 & 0.0119 & 0.0634 & -0.1737 & 0.0655 & 0.0013 & 0.0206 \\ 0.0002 & 0.0092 & 0.0110 & 0.0117 & 0.0765 & -0.2029 & 0.0346 & 0.0599 \\ 0.0000 & 0.0009 & 0.0011 & 0.0034 & 0.0820 & 0.1070 & -0.3502 & 0.1558 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$