Master Thesis Report

X-Value Adjustments for Interest Rate Derivatives

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Abstract

In this report, we present the X-Value Adjustments and we introduce a simulation approach to compute these adjustments. We present the steps for the calculation of the Credit Value Adjustment (CVA) on interest rate derivatives as a practical example. An important part of the report will focus on the different methods to compute the expected future exposure. In this context, we consider two methods based on Monte Carlo simulations in order to compute the expected exposure. We study also the G2++ interest rate model used for the simulations and we detail the calibration process and apply it on market data.

Sammanfattning

I den här rapporten presenterar vi definitioner och formler för X-värdes justeringar, XVA (eng. X Value Adjustment), samt en simuleringsbaserad teknik för att beräkna dessa justeringar. Som ett praktiskt exempel presenteras stegen för beräkning av CVA (eng. Credit Value Adjustment) för räntederivat. En viktig del av rapporten fokuserar på de olika metoderna för att beräkna den förväntade framtida exponeringen (eng. expected future exposure). Vi studerar två metoder baserade på Monte Carlo-simuleringar. Också G2++-modellen som används för simuleringarna presenteras, liksom detaljerna i kalibreringsprocessen och denna tillämpas sedan på marknadsdata

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Chapter 1

Introduction

Before the financial crisis of 2007-2008, the standard approaches to pricing and fair value measurement of portfolios and trading books were based on the assumption of risk-free counterparties and rates. However, the crisis highlighted the importance of counterparty risk and showed that pricing approaches should be revised. In fact, the losses due to the deterioration in the creditworthiness of a counterparty during the financial crisis exceeded the losses arising from actual defaults, according to The Basel Committee on Banking Supervision (BCBS) [5].

Therefore, the regulators gradually introduced new valuation adjustments in order to take into account the effects of credit, funding and capital costs and, as a consequence, the pricing of derivatives has become more and more complicated. These adjustments, named X Value Adjustments (XVA) , are considered today among the main Profit & Losses centers of investment banks, and they affect many areas such as modeling, pricing, risk management, regulation ....

The objective of this thesis is to study the different valuation adjustments and present a formal framework in order to compute the XVAs. As a practical case, We will outline different approaches to calculating the Credit Value Adjustment (CVA) and counterparty exposure. The main methods studied in this report are simulation-based approaches. In addition, the calculations and comparison will be performed for interest rate derivatives. Therefore, we will present in this report the interest rate model used to compute the CVA which is the two-Additive-Factor Gaussian Model (G2++).

Finally, we will show the different results of the calculations for interest rate swaps and swaptions, discuss the efficiency and accuracy of the methods and present the possible improvements.

Chapter 2

X-Value Adjustments

In this chapter, different valuation adjustments will be presented. First, we will derive the expressions of the Credit Value Adjustment (CVA) and Debit Value Adjustment (DVA) following the approach presented by Jon Gregory in [6]. Then, we will introduce the adjustments related to other risks such as funding risk.

2.1 Credit Value Adjustment (CVA)

First, We consider X a set of derivatives positions with a maximum maturity date T, and we note:

- $B(t)$, the value of the Money Market Account at time t, defined as:

$$B(t) = e^{\int_0^t r(s)ds} \tag{2.1}$$

  where r is the instantaneous spot interest rate.

- $D(t,s)$, the discount factor from time t to time s, defined as:

$$D(t,s) = \frac{B(t)}{B(s)}$$

  where $t \leq s$.

- $V^{risky}(t,T)$, the risky value of X at time t.

- $V(s,T)$, the risk-free value (which doesn't take into account default risk) at time s of the derivatives cash flows between s and T for $t < s < T$.
- $V_t(s,T)$, the discounted value of $V(s,T)$ at time t, defined as:

$$V_t(s,T) = D(t,s)V(s,T) \tag{2.2}$$

- G_t, the filtration containing all the market information up to time t
- F_t, a sub-filtration similar to G_t but does not contain events related to defaults.

The Credit Value Adjustment (CVA) is defined as the difference between the risk-free portfolio value and the true portfolio value that takes into account the possibility of counterparty default.

$$CVA(t,T) = V(t,T) - V^{risky}(t,T) \tag{2.3}$$

To introduce the counterparty risk, we suppose that we have a bilateral contract on the set of derivatives X with a counterparty *cpty* and that only the counterparty can default. We have, then, two cases to consider:

- **The Counterparty does not default before maturity** T

  In this case, the payoff is the risk-free value of the positions:

$$CF^{no\_default} = V_t(t,T)$$

- **The Counterparty defaults before maturity** T

  We define τ as the default time of the counterparty. In this case, the payoff is composed of two terms:

  - the cash flows paid before the default time τ: $V_t(t,\tau)$
  - the cash flows paid after the default time τ:
    - If the value of the positions is positive, we receive a recovery amount $R^{cpty} \times V_t(\tau,T)$ where R^{cpty} is the recovery rate of the counterparty.
    - If the value of the positions is negative, we still have to pay the value.

  Then, we have:

$$CF^{default} = V_t(t,\tau) + R^{cpty} \times \max(V_t(\tau,T),0) + \min(V_t(\tau,T),0)$$

Therefore, the risky value of X is given, under the risk-neutral measure Q, by:

$$V^{risky}(t,T) = E^Q\left[CF^{no\_default} I\{\tau > T\} + CF^{default} I\left\{\tau \leq T\right\}|G_t\right]$$

where $I\{A\}$ is the indicator function of the event A defined as:

$$I\{A\} := \begin{cases} 1 & \text{if the event A happens,} \\ 0 & \text{if the event A does not happen.} \end{cases} \tag{2.4}$$

We replace $CF^{no\_default}$ and $CF^{default}$ by their values:

$$\begin{aligned} V^{risky}(t,T) =& E^Q\left[V_t(t,T)I\{\tau > T\}|G_t\right] + \\ & E^Q\left[\left(V_t(t,\tau) + R^{cpty} \times \max(V_t(\tau,T),0) + \ min(V_t(\tau,T),0)\right) I\left\{\tau \leq T\right\}|G_t\right] \end{aligned}$$

Using the relationship $\min(x,0) = x - \max(x,0)$, we obtain:

$$\begin{aligned} V^{risky}(t,T) =& E^Q\left[V_t(t,T)I\{\tau > T\}|G_t\right] + \\ & E^Q\left[\left(V_t(t,\tau) + V_t(\tau,T) + (R^{cpty} - 1) \times \max(V_t(\tau,T),0)\right) I\left\{\tau \leq T\right\}|G_t\right] \end{aligned}$$

We know that $V_t(t,T) = V_t(t,\tau) + V_t(\tau,T)$. Then:

$$\begin{aligned} V^{risky}(t,T) =& E^Q\left[V_t(t,T)I\{\tau > T\}|G_t\right] + \\ & E^Q\left[\left(V_t(t,T) + (R^{cpty} - 1) \times \max(V_t(\tau,T),0)\right) I\left\{\tau \leq T\right\}|G_t\right] \end{aligned}$$

Using $V_t(t,T)I\{\tau > T\} + V_t(t,T)I\{\tau \leq T\} = V_t(t,T)$, we finally obtain:

$$V^{risky}(t,T) = V_t(t,T) - E^Q\left[(1 - R^{cpty}) \max(V_t(\tau,T),0)I\{\tau \leq T\}|G_t\right]$$

Considering the CVA formula in (2.2), we have:

$$CVA(t,T) = E^Q\left[(1 - R^{cpty}) \max(V_t(\tau,T),0)I\{\tau \leq T\}|G_t\right] \tag{2.5}$$

To derive the classic CVA formula, we suppose that the recovery rate R^{cpty} obtained by the surviving counterparty is deterministic. Then, we can write:

$$\begin{aligned} CVA(t,T) &= (1 - R^{cpty})E^Q\left[\max(V_t(\tau,T),0)I\{\tau \leq T\}|G_t\right] \\ &= (1 - R^{cpty})E^Q\left[E^Q\left[\max(V_t(s,T),0)I\{s \leq T\}|\{\tau = s\} \vee F_t\right]|G_t\right] \\ &= (1 - R^{cpty})E^Q\left[I\{\tau \leq T\}E^Q\left[\max(V_t(s,T),0)|\{\tau = s\} \vee F_t\right]|G_t\right] \end{aligned}$$

Introducing the risk-neutral cumulative distribution function $\Phi(t)$ of the default time τ, the previous equation can be rewritten as:

$$CVA(t,T) = LGD^{cpty} \int_t^T E^Q \left[\max(V_t(s,T),0) | \{\tau = s\} \vee F_t\right] d\Phi(s) \tag{2.6}$$

where $LGD^{cpty} = (1 - R^{cpty})$ is the Loss Given Default of the counterparty.

To simplify Equation (2.6), we assume that the default event $\{\tau = s\}$ and the exposure $V_t(s,T)$ are independent. We recall also that $V_t(s,T) = D(t,s) \times V(s,T)$.

Finally, we obtain **the classic CVA formula** :

$$CVA(t,T) = LGD^{cpty} \int_t^T E^Q \left[D(t,s) \max(V(s,T),0) | F_t\right] d\Phi(s) \tag{2.7}$$

2.2 Debit Value Adjustment (DVA)

The Debit Value Adjustment (DVA) is defined as the difference between the risk-free portfolio value and the true portfolio value that takes into account the possibility of the institution's own default. It can be seen as the CVA from the counterparty point of view.

To derive the expression for DVA, we follow the same approach as for the CVA in the previous section. However, we suppose, this time, that the institution can default and that the counterparty can not default. We find the following expression for the Debit Value Adjustment:

$$DVA(t,T) = LGD^{inst} \int_t^T E^Q \left[D(t,s) \min(V(s,T),0) | F_t\right] d\Phi^{inst}(s) \tag{2.8}$$

Where LGD^{inst} is the Loss Given Default of the institution and Φ^{inst} is the risk-neutral cumulative distribution function for the default time of the institution.

2.3 Simple expressions for CVA and DVA

2.3.1 Credit Default Swap

A Credit Default Swap (CDS) is a contract which provides protection to the buyer over the credit risk associated with an underlying asset. The buyer of the CDS makes a series of payments (CDS spread) to the seller. In exchange, he expects to receive a payoff if a credit event such as bankruptcy, restructuring and failure to pay, occurs.

2.3.2 The CVA and DVA expressions

Ruiz [9] showed that we can simplify the CVA and DVA expressions by introducing the credit spread at time t of the Credit Default Swap (CDS) of the counterparty $spread_t^{cpty}$ and of the institution $spread_t^{inst}$. Using the derivation presented in the Appendix A, we have the following formulas:

$$CVA(t,T) = \int_t^T spread_s^{cpty} EPE(s,T)ds \tag{2.9}$$

and

$$DVA(t,T) = \int_t^T spread_s^{inst} ENE(s,T)ds \tag{2.10}$$

Where

- $EPE(s,T) = E^Q[D(t,s)\max(V(s,T),0)]$ is the Expected Positive Exposure between time s and T.
- $ENE(s,T) = E^Q[D(t,s)\min(V(s,T),0)]$ is the Expected Negative Exposure between time s and T.

2.4 Other Value adjustments

In addition to the CVA and the DVA, there are other valuation adjustments accounting for hedging, collateral and funding risk.

2.4.1 Hedging Valuation Adjustment (HVA)

Hedging Valuation Adjustment (HVA) is defined as the additional funding adjustment due to the difference in cash needed from the trade that we have with a counterparty and the hedging position that we buy in the market. Following the same formulation of the

CVA and DVA expression in Equation (2.9) and (2.10), we have:

$$\begin{aligned} HVA(t,T) = & \int_t^T EPE^{hedge}(s,T) spread_s^{borrow} ds \\ & + \int_t^T ENE^{hedge}(s,T) spread_s^{lend} ds \end{aligned}$$

where $spread_s^{borrow}$ and $spread_s^{lend}$ are the spread at time s over the risk-free rate at which we can borrow and lend cash, and EPE^{hedge} (resp. ENE^{hedge}) is the expected positive (resp. negative) exposure of the extra cash needed for the hedging positions.

2.4.2 Collateral Valuation Adjustment (ColVA)

Collateral Valuation Adjustment (ColVA) is due to the funding of the net collateral, that is posted and received constantly between financial institutions. The collateral requirements are strongly dependent on the agreements and contract specifications such as margin requirements, etc.

$$\begin{aligned} ColVA(t,T) = & \int_t^T EPE^{collateral}(s,T)(spread_s^{borrow} + spread_s^{post}) dt \\ & + \int_t^T ENE^{collateral}(s,T) spread_s^{lend} ds \end{aligned}$$

where $spread_s^{post}$ is the spread at time s over the risk-free rate that we are charged on collateral posted, and $EPE^{collateral}$ (resp. $ENE^{collateral}$) is the expected positive (resp. negative) exposure of the collateral.

2.4.3 Liquidity Valuation Adjustment (LVA)

LVA is defined as an adjustment that we must add to the risk-neutral value of a portfolio of trades in order to account for the real liquidity constrains that we face in the funding and credit market. For example, collateralization requires sometimes a huge amount of cash or liquid assets, and there is also a high pressure on the liquidity risk management in the current regulations.

As a consequence, The Funding Valuation Adjustment can be defined as the sum of the three previous adjustments:

$$FVA(t,T) = HVA(t,T) + ColVA(t,T) + LVA(t,T) \tag{2.11}$$

Chapter 3

XVA Calculation: Application to CVA

3.1 XVA General Formula

We can see from the results in chapter 2 that the XVA have the similar expressions *i.e.* they can be written on the following form:

$$XVA(t,T) = \int_t^T spread(s) \times Expected\_Exposure(s,T)ds \tag{3.1}$$

where the properties of the spread and the exposure depend on which XVA we want to compute. For example, we use the exposure of the contract positions for CVA and DVA, the exposure of the hedge positions for HVA and the exposure of the collateral for ColVA. (See Chapter 2 for more details)

As a consequence of these similarities, we decide to focus, in the remaining part of the report, on the CVA calculation for interest rate derivatives as a practical example of the XVA calculations.

3.2 CVA Approximation

We recall that the CVA can be computed using the formula (starting from for $t_0 = 0$)

$$CVA(0,T) = LGD \int_0^T E^Q \left[D(0,t) \max(V(t,T),0)|F_0\right] d\Phi(t) \tag{3.2}$$

To simplify the calculation of the CVA, we introduce a time partition $t = t_0 < t_i < ... < t_n = T$. Then, The formula (3.2) can be approximated and rewritten as

$$CVA(0,T) = (1-R)\sum_{t=0}^{n-1}(\Phi(t_i) - \Phi(t_{i+1}))E^Q[D(0,t_i)\max(V(t_i,T),0)] \tag{3.3}$$

where Φ_t is the cumulative distribution function of the default time τ of the counterparty.

Chapter 4

CVA Calculation Framework

In order to compute the CVA, we use the summation formula presented in (3.2). As a consequence, we need to take into account the three following factors:

- Loss given default of the contract
- Default probability of the counterparty
- Expected exposure

Therefore, we will present in this chapter the models used to compute each one of these factors, and we will introduce the structure used to implement and integrate the CVA calculation components.

4.1 Loss Given Default

The Loss Given Default (LGD) is defined as the percentage amount that would be lost if the counterparty were to default, and is expressed as

$$LGD = 1 - R \tag{4.1}$$

were R is the recovery rate. The LGD depends on many attributes such as the sector of the entity, and the seniority (rank) of the derivatives. It is generally estimated using historical analysis on recovery rates. In this project, we suppose a constant recovery rate R of 40% which is considered by many as a market standard.

4.2 Default Probability

To compute the default probability of a counterparty, we use an intensity default model based on the presentation of Brigo and Mercurio [2].

We assume the existence of a deterministic default intensity λ_t, also known as the **hazard rate**. Supposing that the counterparty survived up to time t, the default probability of the entity in a infinitesimal interval from time t to $t + dt$ is given by:

$$P(t < \tau \leq t + dt) = \lambda_t dt \tag{4.2}$$

We suppose that the default time τ of the counterparty is driven by a Poisson process *i.e.* the cumulative distribution function of τ can be written as

$$\Phi(t) = 1 - e^{-\int_0^t \lambda_u du} \tag{4.3}$$

To compute the default probability, we only need to find the instantaneous hazard rate λ_t. One of the most popular methods to compute this rate is the use of the the Credit Triangle relation (used also in Appendix A) :

$$spread_t = (1 - R)\lambda_t \tag{4.4}$$

where $spread_t$ is the spread of the Credit Default Swap (CDS) on the counterparty, and R is the recovery rate of the same counterparty.

As a numerical experiment, we suppose a deterministic piece-wise constant hazard rate function λ_t with

$$\lambda_{t_i} = 0.01 \times i$$

for $t_i = i$ years and $i \in [0, 30]$.

The resulting default intensity and default probability are shown in Figure 4.1

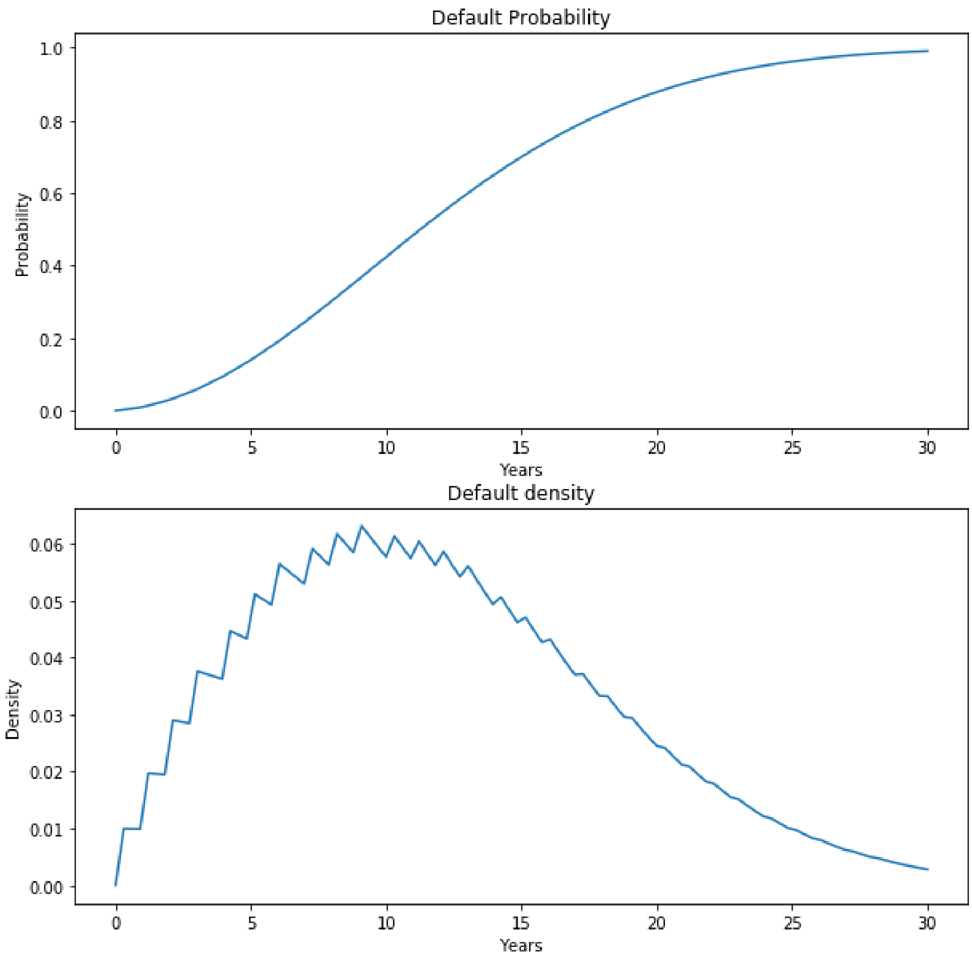

Figure 4.1: Default intensity and default probability for a deterministic piece-wise constant hazard rate function

4.3 Expected Exposure

The Calculation of the expected (positive or negative) exposure can be considered as the most important (and also the most difficult) part of the CVA Calculation and, as a consequence, it will represent the main part of the project. Therefore, we will focus in the remaining part of the report on the methods and ways to compute the expected exposure.

4.3.1 Calculation Approaches

The expected exposure can be computed using three different approaches:

Parametric approach

This objective of this approach is to approximate the future expected exposure using a number of simple parameters. An example of the parametric approaches is the Add-on method which is based on the following formula:

$$FutureExposure = CurrentExposure + \textit{Add-on}$$

The add-on can include time horizon, assets classes and volatilities parameters.

Semi-Analytical approaches

These methods are based on identifying the risk factors driving the exposure and making simple assumptions related to these factors in order to derive the distribution of the exposure.

An example of this approach is the method of approximating the exposure of an interest rate swap by a series of interest rate swaptions, introduced by Sorensen and Bollier [10] in 1994.

Simulation Approach

The Monte Carlo simulation method is the most widely used approach and also the most time-consuming method to compute the expected future exposure. It includes the following steps:

Step 1: Identifying the risk factors driving the exposure

Step 2: Choosing an appropriate time grid for the simulation

Step 3: Generate market scenarios by simulating the identified factors

Step 4: Evaluate the positions at each time point of the grid

Step 5: Aggregating and post-processing: take into account netting sets, collateral postings, thresholds ...

Comparison of the Approaches

Table 4.1 summarizes the advantages and the limitations of the three approaches. We choose to base our exposure calculations on a Monte Carlo simulation approach. In fact, This method can be difficult to implement but its genericity can prove to be very useful, especially when dealing with different types of derivatives. We will show, in chapter 6, the implementation steps for this the method and also introduce an alternative approach based on the American Monte Carlo method in order to improve the performance and the calculation speed.

Approach	Advantages	Limitations
Parametric method	• very easy to implement	• does not take into account transactions specifics such as netting and collateralization
Semi-Analytical method	• easy to implement • takes into account market data	• does not work well with complicated distributional assumptions for risk factors • not good for path-dependent derivatives
Simulation method	• generic • flexible to market data and portfolio changes • Works well under high-dimensionality of risk factors • Works well with netting and collaterals	• complex and time-consuming

Table 4.1: Comparison of the 3 approaches

4.4 Structure and Framework

in this report, we choose to focus on the implementation of a CVA calculation framework for interest rate derivatives as a practical example of the XVA calculations. The framework includes four main components which are described in this section.

4.4.1 Interest Rate Modelling

The interest rate model is implemented and calibrated in this component using market data such as swaption prices and volatilities. It is based on the G2++ Interest rate model. The generated interest rates, discount factors and curves represent the core of the derivatives pricing in the steps that follow.

4.4.2 Simulation Component

This component implements Monte Carlo simulations in order to generate multiple "states" of the market used to value the portfolio. A direct Monte Carlo approach and an alternative one based on the American Monte Carlo approach are implemented as well.

4.4.3 Valuation and Expected Exposure Calculation

In this part, the portfolio will be priced using valuation formulas and approximation based on the paths generated in the simulation component. The complexity pricing will depend on the complexity of the derivatives and the availability of closed formulas.

4.4.4 CVA Integration

This is the final component of the framework. It is responsible for the integration of the computed exposures with the recovery rates and the default probabilities which can be obtained from the credit market data (CDS spreads).

Chapter 5

The G2++ Interest Rate Model

In this chapter, we present the interest rate model used for the CVA calculations. We consider the G2++ interest-rate model where the instantaneous short-rate process is given by the sum of two correlated Gaussian factors, in addition to a deterministic function that is properly chosen so as to exactly fit the current term structure of discount factors. This chapter is inspired by the G2++ model presentation in [2].

5.1 Definition

The dynamics of the instantaneous-short-rate process under the risk-adjusted measure Q is given by

$$r(t) = x(t) + y(t) + \phi(t) \tag{5.1}$$

$$r(0) = r_0 \tag{5.2}$$

where the processes $x(t)$ and $y(t)$ satisfy for $t > 0$

$$dx(t) = -ax(t)dt + \sigma dW_1(t) \tag{5.3}$$

$$dy(t) = -by(t)dt + \nu dW_2(t) \tag{5.4}$$

where $x(0) = 0$, $y(0) = 0$, and (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ as from

$$dW_1(t)dW_2(t) = \rho dt$$

and r_0, a, b, σ, ν are positive constants and where $-1 \leq \rho \leq 1$. The function ϕ is deterministic and well defined in the time interval $[0, T]$ with T a given time horizon. In particular $\phi(0) = r_0$.

5.2 Remarks

a) **Short rate conditional law**

Integrating the formulas (5.3) and (5.4) yields,

$$\begin{aligned} r(t) =& x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} \\ &+ \sigma \int_s^t e^{-a(t-u)} dW_1(u) + \nu \int_s^t e^{-b(t-u)} dW_2(u) + \phi(t) \end{aligned} \tag{5.5}$$

This means that the instantaneous-short-rate process, conditional on F_s, is normally distributed with

$$E[r(t)|F_s] = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \phi(t) \tag{5.6}$$

$$\begin{aligned} Var[r(t)|F_s] =& \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right] + \frac{\nu^2}{2b}\left[1 - e^{-2b(t-s)}\right] \\ &+ 2\rho\frac{\sigma\nu}{a+b}\left(1 - e^{-(a+b)(t-s)}\right) \end{aligned} \tag{5.7}$$

b) **Conditional law of the integral**

We consider the following integral

$$I(t,T) = \int_t^T [x(u) + y(u)]du \tag{5.8}$$

It can be shown that $I(t,T)$, conditional to F_t, is normally distributed with

$$M(t,T) = E[I(t,T)|F_t] = \frac{1 - e^{-a(T-t)}}{a} x(t) + \frac{1 - e^{-b(T-t)}}{b} y(t) \tag{5.9}$$

and

$$\begin{aligned} V(t,T) =& Var[I(t,T)|F_t] \\ =& \frac{\sigma^2}{a^2}\left[T - t + \frac{2e^{-a(T-t)}}{a} - + \frac{e^{-2a(T-t)}}{2a} - \frac{3}{2a}\right] \\ &+ \frac{\nu^2}{b^2}\left[T - t + \frac{2e^{-b(T-t)}}{b} - + \frac{e^{-2b(T-t)}}{2b} - \frac{3}{2b}\right] \\ &+ 2\rho\frac{\sigma\nu}{ab}\left[T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a+b}\right] \end{aligned} \tag{5.10}$$

c) **Fitting the spot curve**

The model fits the currently-observed term structure of discount factors if and only if, for each T,

$$\phi(T) = f^M(0,T) + \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2 + \frac{\nu^2}{2b^2}(1 - e^{-bT})^2 + \rho\frac{\sigma\nu}{ab}(1 - e^{-aT})(1 - e^{-bT}) \tag{5.11}$$

where $f^M(0,T)$ is the observed forward rate at time T.

5.3 Derivatives pricing

5.3.1 Zero-coupon bond

Under the no arbitrage condition, the pricing formula at time t of a zero-coupon bond maturing at T is given by:

$$P(t,T) = E^Q\left[\exp\left(-\int_t^T r(u)du\right)|F_t\right] \tag{5.12}$$

This equation can be rewritten, using Remark b), as:

$$P(t,T) = \exp\left[-\int_t^T \Phi(u)du - M(t,T) + \frac{1}{2}V(t,T)\right] \tag{5.13}$$

Where the functions M and V are defined in (5.9) and (5.10).

Using Equation (5.11), we obtain the following pricing formula for the zero-coupon bond:

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left[-M(t,T) - \frac{1}{2}[V(0,T) - V(0,t) - V(t,T)]\right] \tag{5.14}$$

Where $P(0,T)$ and $P(0,t)$ are taken from the currently-observed market spot curve.

We define the following functions:

$$A(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left[\frac{1}{2}[-V(0,T) + V(0,t) + V(t,T)]\right] \tag{5.15}$$

and

$$B(z,t,T) = \frac{1 - e{-}z(T-t)}{z} \tag{5.16}$$

Using these functions, we can express the zero-coupon price as a relatively simple function of the G2++ factors $x(t)$ and $y(t)$:

$$P(t,T) = A(t,T)\exp\left[-B(a,t,T)x(t) - B(b,t,T)y(t)\right] \tag{5.17}$$

5.4 Option on a zero-coupon bond

The price of a European call option with strike K and maturity T, written on a zero-coupon bond with with face value N and maturity S at time $t \in [0, T]$, is given by:

$$\begin{aligned} ZBC(t,T,S,N,K) =& NP(t,S)F\left[\frac{\ln\frac{NP(t,S)}{KP(t,T)}}{\Sigma(t,T,S)} + \frac{1}{2}\Sigma(t,T,S)\right] \\ & - KP(t,T)F\left[\frac{\ln\frac{NP(t,S)}{KP(t,T)}}{\Sigma(t,T,S)} - \frac{1}{2}\Sigma(t,T,S)\right] \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} \Sigma(t,T,S)^2 =& \frac{\sigma^2}{2a^3}[1-e^{-a(S-T)}]^2[1-e^{-2a(T-t)}] \\ & + \frac{\nu^2}{2b^3}[1-e^{-b(S-T)}]^2[1-e^{-2b(T-t)}] \\ & + 2\rho\frac{\sigma\nu}{ab(a+b)}[1-e^{-a(S-T)}][1-e^{-b(S-T)}][1-e^{-(a+b)(T-t)}] \end{aligned}$$

and F denotes the standard normal cumulative distribution function.

The price of a corresponding put option is given by:

$$\begin{aligned} ZBP(t,T,S,N,K) =& - NP(t,S)F\left[\frac{\ln\frac{KP(t,T)}{NP(t,S)}}{\Sigma(t,T,S)} - \frac{1}{2}\Sigma(t,T,S)\right] \\ & + KP(t,T)F\left[\frac{\ln\frac{KP(t,T)}{NP(t,S)}}{\Sigma(t,T,S)} + \frac{1}{2}\Sigma(t,T,S)\right] \end{aligned} \tag{5.19}$$

5.4.1 Caps and floors

The price at time t of a cap with strike X, nominal value N, set of times $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$ and year fractions $\tau = \{\tau_0, \tau_1, \ldots, \tau_n\}$ is given by:

$$\begin{aligned} Cap(t,\mathcal{T},\tau,N,X) =& \sum_{i=1}^{n} -N(1+X\tau_i)P(t,T_i)F\left(\frac{\ln\frac{P(t,T_{i-1})}{(1+X\tau_i)P(t,T_i)}}{\Sigma(t,T_i,T_{i-1})} - \frac{1}{2}\Sigma(t,T_i,T_{i-1})\right) \\ & + \sum_{i=1}^{n} NP(t,T_{i-1})F\left(\frac{\ln\frac{P(t,T_{i-1})}{(1+X\tau_i)P(t,T_i)}}{\Sigma(t,T_i,T_{i-1})} + \frac{1}{2}\Sigma(t,T_i,T_{i-1})\right) \end{aligned} \tag{5.20}$$

and the price of the corresponding floor is

$$\begin{aligned} Flr(t, \mathcal{T}, \tau, N, X) = & \sum_{i=1}^{n} N(1 + X\tau_i)P(t, T_i)F\left(\frac{\ln \frac{(1+X\tau_i)P(t,T_i)}{P(t,T_{i-1})}}{\Sigma(t, T_i, T_{i-1})} + \frac{1}{2}\Sigma(t, T_i, T_{i-1})\right) \\ & + \sum_{i=1}^{n} -NP(t, T_{i-1})F\left(\frac{\ln \frac{(1+X\tau_i)P(t,T_i)}{P(t,T_{i-1})}}{\Sigma(t, T_i, T_{i-1})} - \frac{1}{2}\Sigma(t, T_i, T_{i-1})\right) \end{aligned} \tag{5.21}$$

5.5 Swaptions

In this section, we give the analytical formula for pricing European swaptions. The formula expression is quite complicated, but it is important to write it as it is used for the calibration of the G2++ model parameters in the following section.

We consider a European swaption with strike rate X, maturity T and nominal value N, which gives the holder the right to enter at time $t_0 = T$ an interest rate swap with payment times $\mathcal{T} = \{t_1, t_2, \ldots, t_n\}, t_1 > T$ where he pays at the fixed rate X and receives LIBOR set in arrears (at the start of each swap period). We define τ_i the year fraction from t_{i-1} and t_i, $i = 1, 2, \ldots, n$.

$$\begin{aligned} & ES(0, T, \mathcal{T}, N, X) = \\ & NP(0, T) \int_{-\infty}^{+\infty} \frac{e^{-(\frac{x-\mu_x}{\sigma_x})^2/2)}}{\sigma_x\sqrt{2\pi}} \left[F(-h_1(x)) - \sum_{i=1}^{n} \lambda_i(x)e^{k_i(x)}F(-h_2(x))\right] dx, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} h_1(x) =& \frac{y^* - \mu_y}{\sigma_y\sqrt{1 - \rho_{xy}^2}} - \frac{\rho_{xy}(x - \mu_x)}{\sigma_y\sqrt{1 - \rho_{xy}^2}} \\ h_2(x) =& h_1(x) + B(b, T, t_i)\sigma_y\sqrt{1 - \rho_{xy}^2} \\ \lambda_i(x) =& c_i A(T, t_i)e^{-B(a,T,t_i)x} \\ k_i(x) =& -B(b, T, t_i)\left[\mu_y - \frac{1}{2}(1 - \rho_{xy}^2)\sigma_y^2 B(b, T, t_i) + \rho_{xy}\sigma_y\frac{x - \mu_x}{\sigma_x}\right] \\ \sigma_x =& \sigma\frac{1 - e^{-2aT}}{2a} \\ \sigma_y =& \nu\frac{1 - e^{-2bT}}{2b} \\ \rho_{xy} =& \frac{\rho\sigma\nu}{(a + b)\sigma_x\sigma_y}\left[1 - e^{-(a+b)T}\right] \end{aligned}$$

with $c_i = X\tau_i$ for $i = 1, \dots, n-1$, $c_n = 1 + X\tau_n$ and the deterministic functions A and B are defined in Equations (5.15) and (5.16), and

$$\begin{aligned}
\mu_x = &-(\frac{\sigma^2}{a^2} + \rho\frac{\sigma\nu}{ab})(1 - e^{-aT}) + \frac{\sigma^2}{2a^2}(1 - e^{-2aT}) \\
&+ \rho\frac{\sigma\nu}{b(a+b)}(1 - e^{-(a+b)T}) \\
\mu_y = &-(\frac{\nu^2}{b^2} + \rho\frac{\sigma\nu}{ab})(1 - e^{-bT}) + \frac{\nu^2}{2b^2}(1 - e^{-2bT}) \\
&+ \rho\frac{\nu\sigma}{a(a+b)}(1 - e^{-(a+b)T})
\end{aligned}$$

Finally, $y^* = y^*(x)$ denotes the unique solution of the following equation:

$$\sum_{i=1}^{n} c_i A(T, t_i) e^{-B(a,T,t_i)x - B(b,T,t_i)y^*} = 1$$

5.6 Simulation Scheme

Given a time grid $t_0, t_1, t_2, ..., t_N$, we can simulate the short rate using the following equations:

$$x_{k+1} = e^{-a\Delta t_k} x_k + \sigma\sqrt{\frac{1 - e^{-2a\Delta t_k}}{2a}} z^1_{k+1} \tag{5.23}$$

$$y_{k+1} = e^{-b\Delta t_k} y_k + \sigma\sqrt{\frac{1 - e^{-2b\Delta t_k}}{2b}} (\rho z^1_{k+1} + \sqrt{1-\rho^2} z^2_{k+1}) \tag{5.24}$$

$$\phi_{k+1} = f^M(0, t_{k+1}) + \frac{\sigma^2}{2a^2}(1 - e^{-at_{k+1}})^2 + \frac{\nu^2}{2b^2}(1 - e^{-bt_{k+1}})^2 + \rho\frac{\sigma\nu}{ab}(1 - e^{-at_{k+1}})(1 - e^{-bt_{k+1}}) \tag{5.25}$$

and finally

$$r_{k+1} = x_{k+1} + y_{k+1} + \phi_{k+1} \tag{5.26}$$

with $z^1_k \sim N(0, 1)$, and $z^2_k \sim N(0, 1)$ two independent random variables and $\Delta t_k = t_{k+1} - t_k$ for $0 \leq k \leq N$.

5.7 Calibration

To calibrate the G2++ Interest rate model, we choose a calibration portfolio composed of N instruments such as swaptions or caps (we choose 50 swaptions in our tests) and we try to minimize the following *cost function*:

$$f(a, b, \sigma, \nu, \rho) = \sum_{i=1}^{N} (V_i^{G2++}(a, b, \sigma, \nu, \rho) - V_i^{Market})^2 \tag{5.27}$$

where

- a, b, σ, ν, ρ are the G2++ parameters to calibrate.
- V_i^{G2++} is the value of the product i calculated with the G2++ model (for the swaptions, we use formula (5.22)).
- V_i^{Market} is the value of the product i obtained using market data.

In our tests, we used the Black volatilities given by Bloomberg in order to compute the market price of the swaptions.

As we one see directly from the figure 5.1, the Black volatility is relatively higher for small exercise date (namely maturity of swaption) and small tenor (of underlying swap). Therefore, we will use more swaptions with small exercise date and small tenor in the portfolio for calibration to have a better fit for the surface.

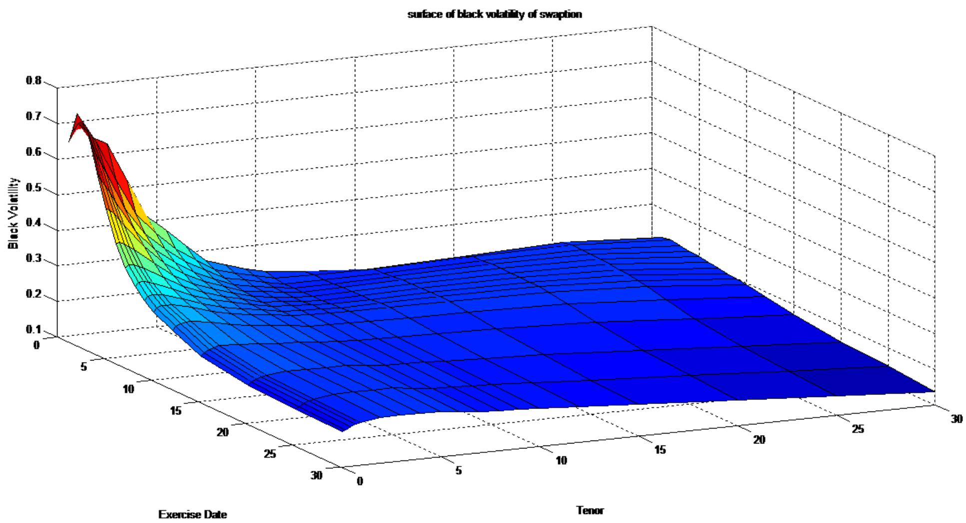

Figure 5.1: Black volatility surface

5.8 Numerical results

In order to test our implementation of the G2++ model, we calibrated the G2++ parameters using the method described in the previous section and then, we compares the pricing results for swaptions obtained by Monte-Carlo simulations based on the scheme presented in section 5.6, with the results given by the closed formulas presented in section 5.3.

Calibration results.

The calibration portfolio contained 50 swaptions from different maturity and expiry dates. We tried to minimize the cost function stated in 5.27 using the Levenberg-Marquardt algorithm. We find the following parameters :

```
a = 0.59499, sigma = 0.00429, b = 0.15408, eta = 0.00196, rho = -0.97238
```

Figure 5.2 presents the market price and the calibrated G2++ prices of a set of swaptions contained in the calibration portfolio. As we can see from the figure, the two swaption prices are very close.

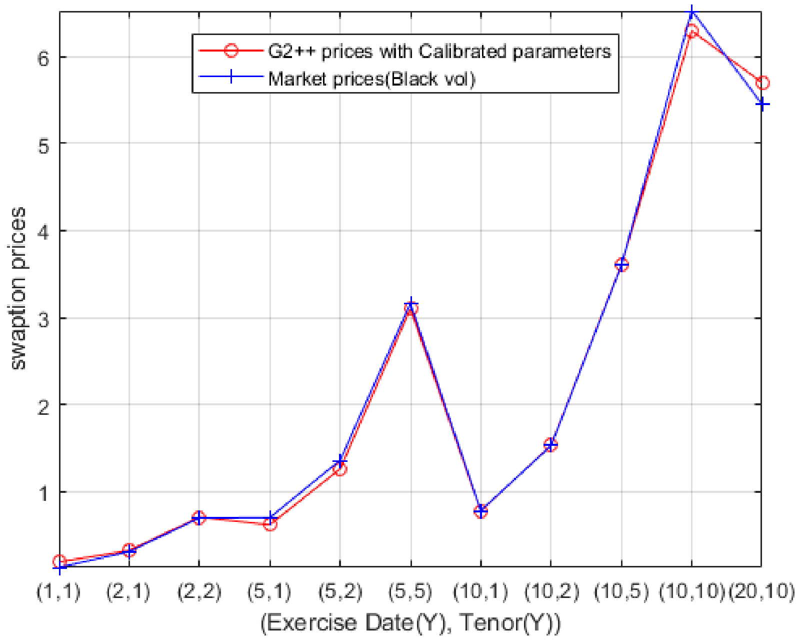

Figure 5.2: Market prices and the calibrated G2++ prices of swaptions

Figure 5.3 shows 50 simulated paths of the short-rate r of the G2++ model using the calibrated parameters.

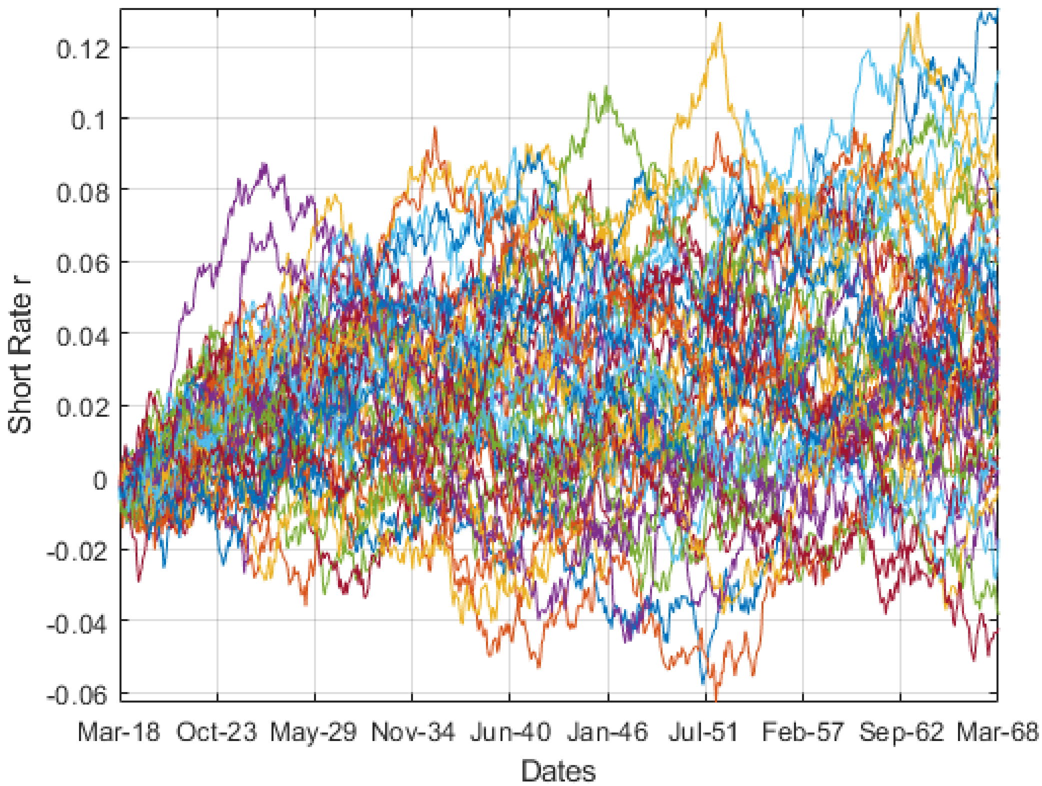

Figure 5.3: 50 simulated paths of the short-rate r of the G2++ model

Swaption pricing

We used the Monte-Carlo method to price swaptions of different strike rates by simulating 10000 paths of the short interest rate. Figure 5.4 shows the difference between Monte Carlo pricing and the one using the swaption closed formula.

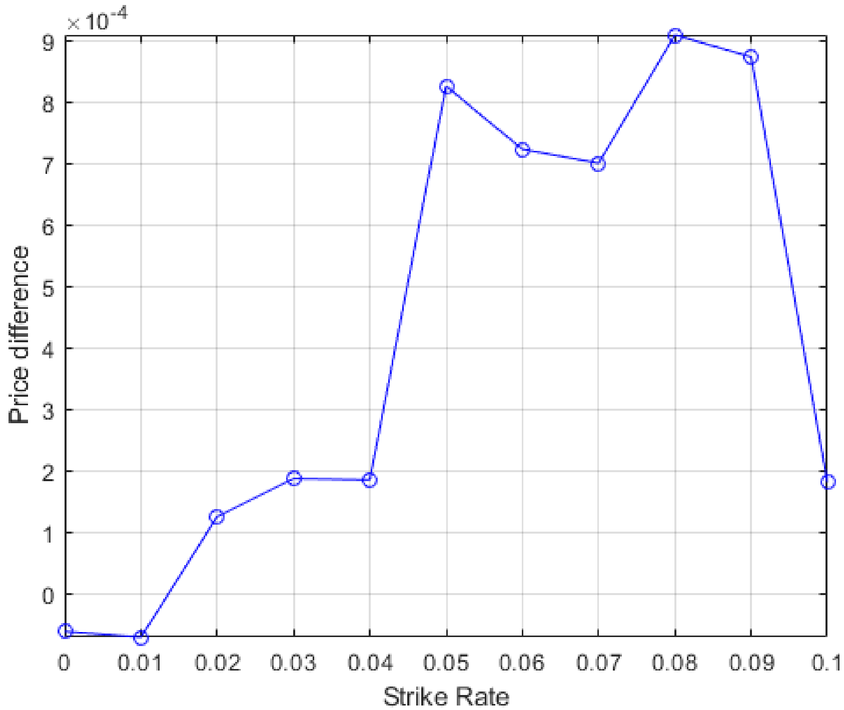

Figure 5.4: Difference between Monte Carlo and the closed formula pricing for swaptions with different strike rates

We computed also the prices of swaptions with different payment frequencies using the two methods *i.e.* closed formula and Monte Carlo methods. The results are presented in figure 5.5.

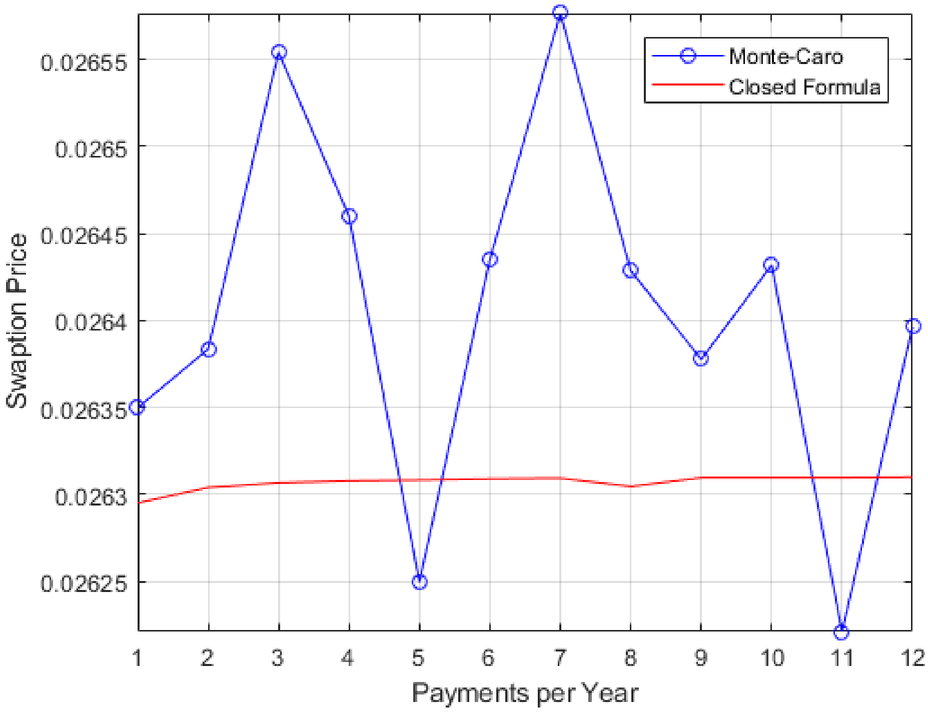

Figure 5.5: Difference between Monte Carlo and the closed formula pricing for swaptions with different frequencies

Chapter 6

Expected Exposure

6.1 Introduction

As stated in chapter 4, the method that we used to calculate the CVA is based on a Monte Carlo simulation approach. To do so, we used the CVA approximation formula presented previously :

$$CVA(0,T) = (1-R)\sum_{t=0}^{n-1}(\Phi(t_i) - \Phi(t_{i+1}))E^Q[D(0,t_i)\max(V(t_i,T),0)] \tag{6.1}$$

where Φ_t is the cumulative distribution function of the default time τ of the counterparty.

We presented the calculation methods for the loss given default and the default probability of a counterparty. The last term left to compute is the expected future exposure $E^Q[\max(V(t_i,T),0)]$. However, the difficulty of the exposure calculation depends highly on the complexity of the derivatives that compose the portfolio. In this section, we present two methods that we used to compute the expected exposure.

6.2 Direct approach

The first method is the traditional Monte Carlo approach which is based on:

Step 1: Simulate the risk factors paths (short rate r for interest rate derivatives) using a fixed time grid $t_0 = 0, t_1, t_2, \dots, t_n = T$.

Step 2: Value the term $V^k(t_i, T)$ at each time step t_i and for each simulated paths k, $0 \leq k < N$.

Step 3: Calculate the expected exposure (and expected positive exposure respectively) at each time step t_i by averaging the terms $V^k(t_i, T)$ (resp. $\max(V^k(t_i, T), 0)$) for all $k \in [0, N-1]$.

This direct method works well with portfolios containing derivatives with available analytical pricing formulas. For example, the expected exposure of a simple interest rate swap can be calculated by valuing the swap at each time step for each path using the swap valuation formula (*i.e.* finding the value $\max(V^k(t_i, T), 0)$ at each time step t_i and for each simulated paths k) and then, averaging the exposures. Another example is the equity European call option. In this case, we can use the Black-Scholes formula to compute the exposure at each time step and simulated path and then, average the exposures to obtain the expected exposure at each time step. We can also compute potential future exposure (PFE) (defined as the maximum expected exposure over a specified period of time calculated at some given level of confidence) by calculating the appropriate quantile of the exposures.

However, the traditional direct Monte Carlo approach does not work very well with complex derivatives. For example, let us consider a derivative which does not have an available analytical pricing formula. The valuation will require a numerical method like PDE or Monte carlo method which will require more computing resources and execution time. We can illustrate this difficulty by considering a Monte Carlo valuation method. To calculate the CVA in this case, we will, first, simulate N paths for the risk factors and then we will need to perform a Monte Carlo Simulation at each time step and for each path in order to estimate $V^k(t_i, T)$ which leads to the so-called “Monte Carlo on Monte Carlo”, and this can make it impossible to compute the CVA in a reasonable time frame.

In order to solve this problem, we will introduce an alternative method to compute the CVA based on the American Monte Carlo Method.

6.3 American Monte Carlo (AMC)

6.3.1 Introduction

The American Monte Carlo, also known as Least-Squares Monte Carlo or Regression Monte Carlo, is a simulation method that combines Monte Carlo simulation with least-squares regression. The first application of this method (as it was introduced by Longstaff and Schwartz in 2001 [8]) was to price path-dependent derivatives such as American and Bermudian options. Later, Antonov *et al.* [1] and Cesari *et al.* [4] showed that the regression-based method can be extended to the calculation of expected exposures and CVA.

6.3.2 Example: Pricing of American options

Problem formulation:

The objective of this example is to evaluate a path-dependent derivative (for example an American option) on an underlying S *i.e.* to compute:

$$V_0(S_0) = \sup_{\tau \in [0;T]} E^Q[D(0,\tau)h(S_\tau)] \tag{6.2}$$

where h is the payoff function of the derivative.

The first approximation is to transform the problem into a discrete optimal stopping problem. We consider a set of possible stopping times $t_0 = 0, t_1, t_2, \ldots, t_M = T$. We then have:

$$V_0(S_0) = \sup_{\tau \in \{t_0, t_1, t_2, \ldots, t_M\}} E^Q[D(0,\tau)h(S_\tau)] \tag{6.3}$$

The algorithm relies on the dynamic programming principle going backward:

- for $t = T$
$$V_T(S_T) = h(S_T) \tag{6.4}$$
- for $0 \leq t_m \leq t_{M-1}$
$$\begin{aligned} U_{t_m}(S_{t_m}) &= E^Q[D(t_m, t_{m+1})V_{t_{m+1}}(S_{t_{m+1}})|F_{t_m}] \\ V_{t_m}(S_{t_m}) &= \max(h(S_{t_m}), U_{t_m}(S_{t_m})) \end{aligned}$$

where $U_{t_m}(S_{t_m})$ denotes the continuation value *i.e.* the value of holding on to the option at time t_m.

Going backward, we find the value $V_0(S_0)$

$$V_0(S_0) = E^Q[D(0,\tau^*)h(S_{\tau^*})] \tag{6.5}$$

where $\tau^* = \inf\{t_m \in \{t_0,\dots,t_M\} : U_{t_m}(S_{t_m}) \leq h(S_{t_m})\}$

Longstaff & Schwartz algorithm principle

The algorithm can be summarized in the following steps:

Step 1: Generate the grid points $(S_{t_m}(n))$ with $n = 1,\dots,N; m = 0,\dots,M$ and N is the number of paths

Step 2: At time $t_M = T$ compute the option value $V_{t_M}(S_{t_M}) = h(S_T)$

Step 3: For time step $t_m \leq t_{M-1}$:

- Compute an approximation of the continuation value $U_{t_m}^{approx}(S_{t_m}(i))$ for each path $i \in [0,N]$
- Evaluate the cash-flow for each path i

$$V_{t_m}^{approx}(S_{t_m}(i)) = \begin{cases} h(S_{t_m}) & \text{if} \quad U_{t_m}^{approx}(S_{t_m}(i)) \leq h(S_{t_m}(i)) \\ D(t_m,t_{m+1})V_{t_{m+1}}(S_{t_{m+1}}(i)) & \text{else} \end{cases}$$

Step 4: For each path we consider

$$\tau^* = \inf\left\{t_m \in \{t_0,\dots,t_M\} : U_{t_m}^{approx}(S_{t_m}) \leq h(S_{t_m})\right\}$$

Step 5: The price estimator is given by

$$V_0^{approx}(S_0) = \frac{\sum_{i=1}^{N} D(0,\tau^*(i))h(S_{\tau^*(i)}(i))}{N}$$

Conditional expectation approximation

The main difficulty of the algorithm is the approximation of the continuation value $U_{t_m}^{approx}(S_{t_m}(i))$ in step 3 (which is a conditional expectation). The method proposed by Longstaff & Schwartz is based on the assumption that the conditional expectation

$$E^Q[D(t_m, t_{m+1})V_{t_{m+1}}(S_{t_{m+1}})|F_{t_m}]$$

can be represented as a linear combination of a countable set of F_{t_m}-measurable basis functions. A typical approximation is realized using least-square regression to state variables. We then have:

$$E^Q[D(t,T)V_t(S_T)|F_t] \approx \sum_i \alpha_i L_i(x_1(t), x_2(t), \ldots)$$

where

- α_i are regression coefficients computed by solving a classical least-square minimization.
- $x_1(t), x_2(t), \ldots$ are model states at time t.
- L_i are the basis functions.

6.3.3 Application to CVA Calculation

As stated in the introduction to this chapter, we need to compute the expectation

$$E^Q[\max(V(t_i, T), 0)]$$

for each time step t_i for $i = 1, \ldots, n$, in order to calculate the CVA.

To do this, we generate N forward paths, and we use the approximation:

$$E^Q[\max(V(t_i, T), 0)] = \sum_{k=1}^{N} \frac{\max(V^k(t_i, T), 0)}{N}$$

where $V^k(t_i, T)$ is the value of the derivatives at time t_i for the path k. This value can be expressed as follows:

$$V^k(t_i, T) = E^Q[\nu(t_i, T)|\boldsymbol{x_{t_i}} = x^k_{t_i}]$$

where $\nu(t_i, T)$ is the cash flows in the time interval $[t_i, T]$ of the positions and $x^k_{t_i}$ is the value of the state variable generated at of the state variables $x^k_{t_i}$. (In the case of interest rate derivatives, the state variable is the spot interest rate r).This expectation can be difficult to estimate if there is no available pricing formula. In this case, we will need an inner Monte Carlo in order to estimate $V^k(t_i, T)$.

Instead, we can use the same approximation method as in the Longstaff-Schwartz algorithm, and compute the conditional expectation using regression. In fact, we can obtain the positions value $\nu^k(t_i, T)$ at time point t_i along path k using a backward induction. Then, we estimate $V^k(t_i, T)$ for each backward step by regressing the values $\nu^k(t_i, T)$ against the state variables $x_{t_i}^k$ for $k \in [1, N]$

Broadie and Glasserman [3] observed that using the same set of simulated sample paths for estimating the conditional expectation function (*i.e.* finding the value of the regression coefficient for each time step) and computing the expected exposure, can lead to a bias in the calculations. Therefore, we decide to add a preliminary simulation in order to estimate the regression coefficient before the principal simulation where the continuation values (and the exposures) are calculated. As a consequence, the American Monte Carlo consists of three main steps:

- **Forward Phase 1**: simulation of the underlying asset/model diffusion with N paths and computing the state variables $x_{t_i}^k$ for each path k and time point t_i.
- **Backward Phase**: calculation of the values $\nu^k(t_i, T)$ and computing the coefficients associated to the basis functions by performing a regression of $\nu^k(t_i, T)$ against $x_{t_i}^k$.
- **Forward Phase 2**: second simulation of P paths using the diffusion model. Calculation of $V^k(t_i, T)$ using the regression coefficients obtained in the backward phase and taking the average of the values of all paths to approximate the exposure.

6.4 Numerical Results

6.4.1 Swap exposure

First, we consider a interest rate payer swap. The swap characteristics are given in the appendix B.1. We note that we in this calculations used a Multi-Curve framework as we worked with two different interest rate curves, a discount based on the OIS curve (Overnight indexed swap) and a forecast one based on EURIBOR.

We use the direct approach to compute the exposure profiles using the calibrated parameters presented in the calibration section. Figure 6.1 shows the swap expected positive exposure (EPE) and potential future exposures (PFE) for different confidence levels.

We also compute the expected positive exposure using the American Monte Carlo (AMC) approach. Figure 6.2 shows the difference between the expected exposure results obtained using the direct approach and the expected exposure calculated using the AMC with different regression functions:

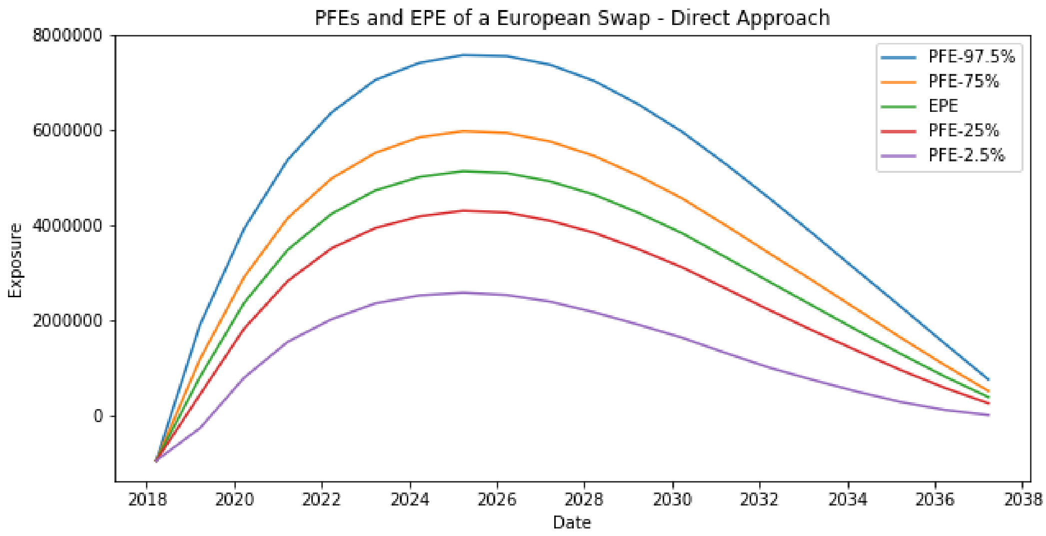

Figure 6.1: Expected positive exposure (EPE) and Potential future exposures (PFE) for different confidence levels

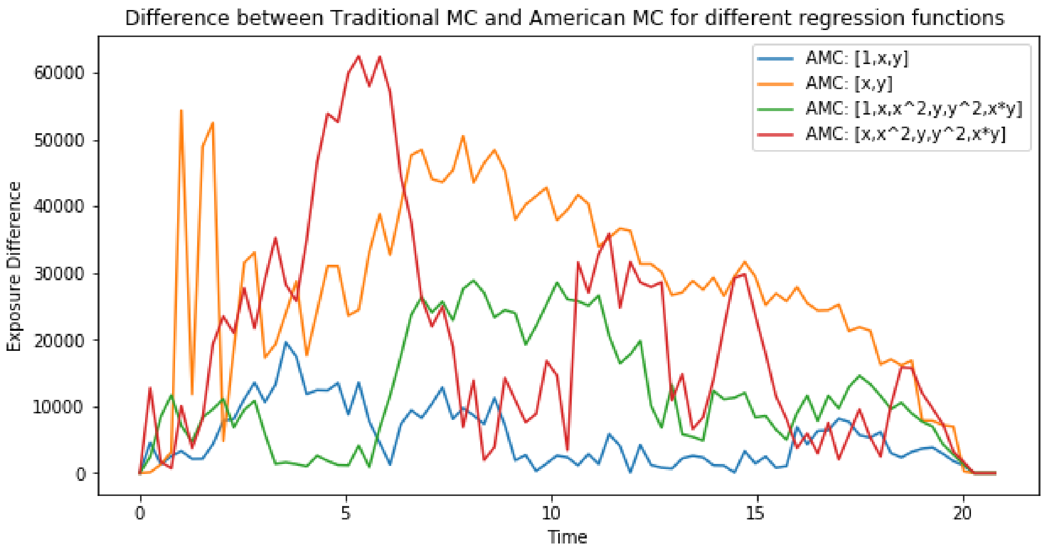

Figure 6.2: Difference between the American Monte Carlo method and the traditional Monte Carlo. x (resp. y) represent the instantaneous short-rate r of the discount (resp. forecast) interest rate model

We can see that, for the swap exposures calculations, the American Monte Carlo approach produces different results depending on the choice of the regression functions. In fact, we observe that the calculations that include the constant coefficient α_0 give results closer to the traditional Monte Carlo than the other choices of regression functions.

6.4.2 European Swaption exposure

In this section, we consider an European swaption (Call) on an underlying receiver swap. The properties of the swaption and the underlying swap are given in the appendix B.2 .

In this case, we can not use the direct approach to estimate the future exposures because there is no available valuation formula, and performing a nested Monte Carlo is very time-consuming.

Using the same test as for the swaps, we compute the expected exposures for different regression functions. The results are shown in Figure 6.3.

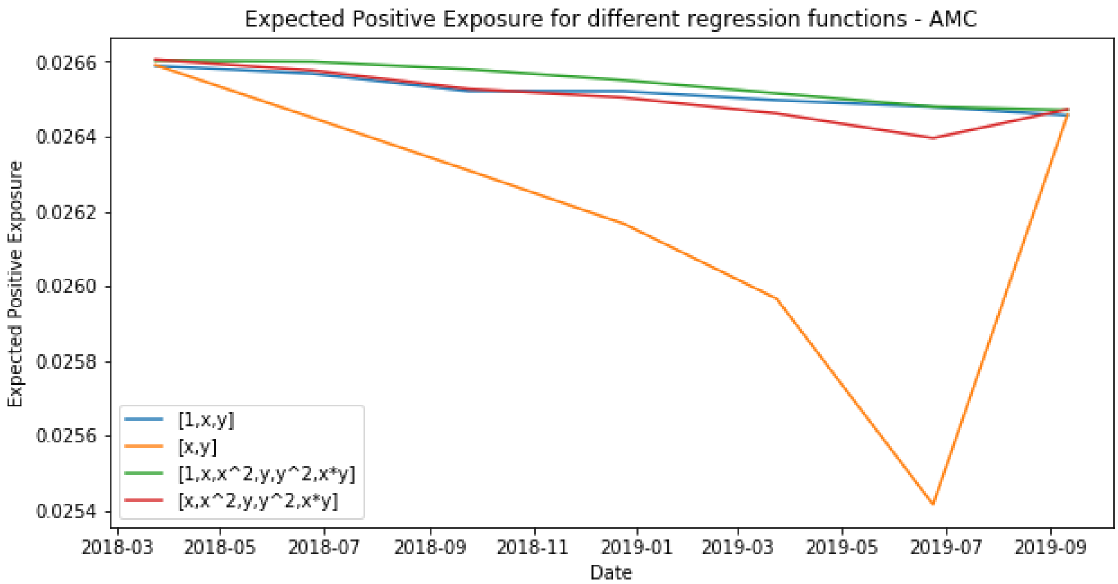

Figure 6.3: The swaption expected future exposures for different regression functions. x (resp. y) represent the instantaneous short-rate r of the discount (resp. forecast) interest rate model

We observe that the difference between the exposure calculation that used only 2 regression functions and the other regressions is relatively significant. In fact, we can say that the number of regression functions is an important factor for the accuracy and the performance of the American Monte Carlo method as we need to choose a sufficient number of functions in order to get accurate results. However, choosing a high number of basis functions can lead to an overfitting and a decrease in the performance of the method.

Figure 6.4 shows the potential future exposures curves using the first set of basis functions $(1,r_{discount},r_{forecast})$.

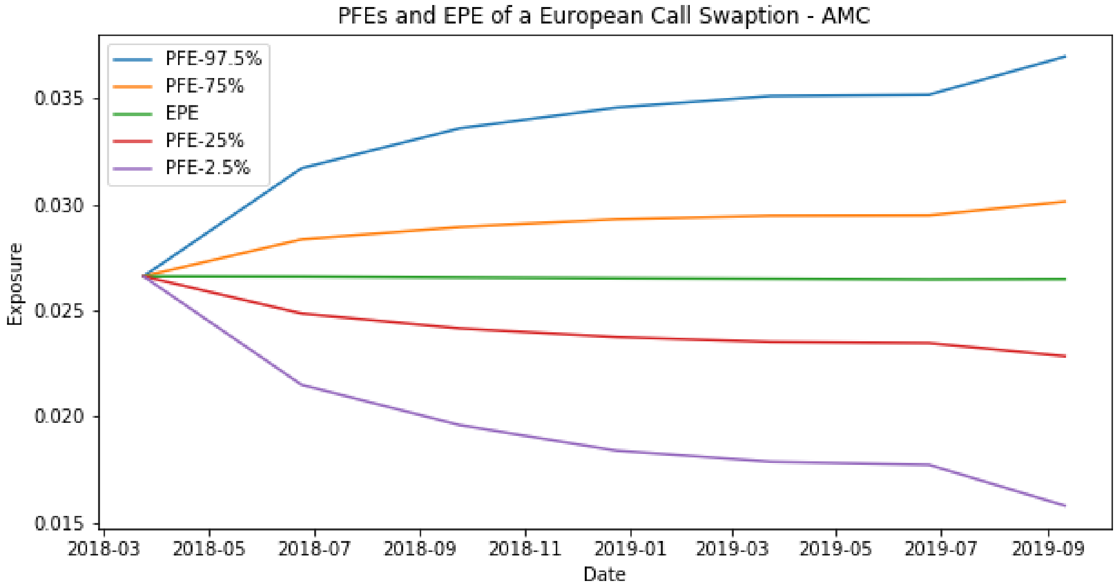

Figure 6.4: The swaption expected positive exposure (EPE) and Potential future exposures (PFE) for different confidence levels

6.4.3 G2++ model parameters

In this section, we see hohw the parameters of th g2++ model effect the expected exposure calculations.

To do so, we compute the EPE of the swap, described in the first test, using different settings of the interest rate model parameters.

First, we study the impact of the correlation ρ of the G2++ model. we consider 3 different sets of parameters:

- Set 1: a = 0.5, σ = 0.1, b = 0.8, η = 0.1, ρ = -0.9
- Set 2: a = 0.5, σ = 0.1, b = 0.8, η = 0.1, ρ = 0.9
- Set 2: a = 0.5, σ = 0.1, b = 0.8, η = 0.1, ρ = 0

Figure 6.5 shows the calculated EPE for these sets.

We study also the impact of the parameters σ and η of the G2++ model. Figure 6.6 shows the calculated EPE the following sets of parameters:

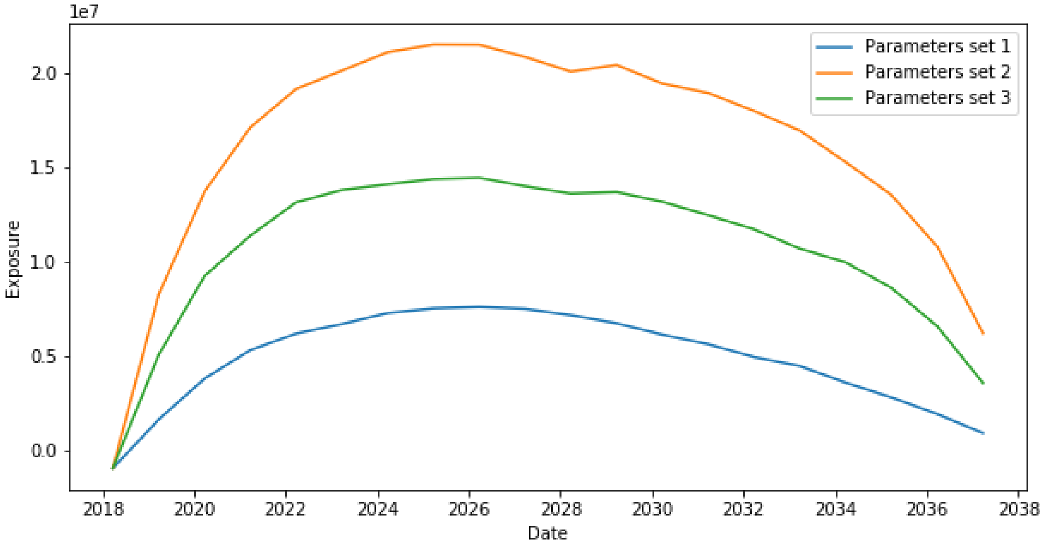

Figure 6.5: EPE for sets 1,2 and 3

- Set 4: a = 0.5, σ = 0.1, b = 0.8, η = 0.1, ρ = -0.9
- Set 5: a = 0.5, σ = 0.3, b = 0.8, η = 0.3, ρ = -0.9
- Set 6: a = 0.5, σ = 0.5, b = 0.8, η = 0.5, ρ = -0.9

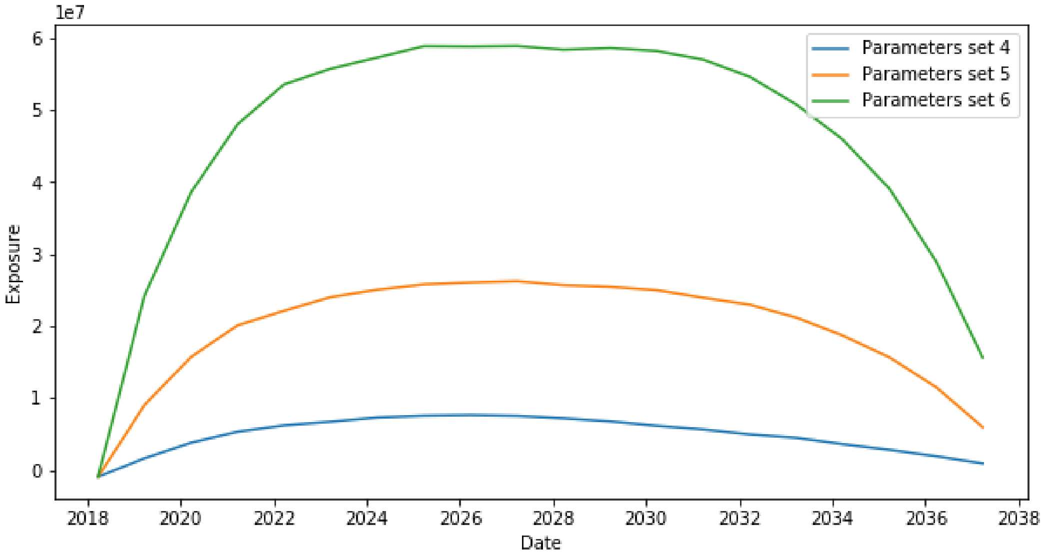

Figure 6.6: EPE for sets 4,5 and 6

Chapter 7

Conclusion

In this report, we introduced the definitions and expressions for the X-value adjustments and we investigated the different approaches to compute the CVA, as an application of the XVA calculations. We presented the American Monte Carlo (AMC) used initially for pricing American options, and we showed that it can be a good alternative to solve the Nested Monte Carlo problem that we face when we use a simulation approach to compute the CVA. In fact, the AMC generates the swaption exposure profile in a short time (3.5 seconds), while the traditional direct MC takes about 1h to compute the exposures. However, the accuracy and performances of the AMC depends highly on the choice of the basis functions used in the regression step, as shown in the numerical experiments. For example, there are some regression functions that work well with some models and payoffs, but are not adapted to other types of derivatives.

In addition, the number of basis functions needed, in order to reach a given level of accuracy, increases for problems with higher dimension. Therefore, the choice of the set of basic functions will become more difficult.

More sophisticated methods based on regression have been developed in order to improve the accuracy and performance of the American Monte Carlo such as localized regression and the Stochastic Grid Bundling Method (SGBM) introduced by Jain and Oosterlee [7], which propose a bundling approach *i.e.* partitioning the state space into several non-overlapping groups in order to reduce the approximation error.

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Appendix A

A derivation for CVA and DVA expressions

The CVA and DVA derivations presented here are inspired by the presentation in [9].

The fair value of a financial product is given by the expectation of the present value of its future cash flows under the risk-neutral measure Q :

$$V_0 = \mathrm{E}^Q(\sum_i PV(CashFlow_i)) \tag{A.1}$$

Let's consider a derivative that have a future cash flow $x_t dt$ between t and $t + dt$, then

$$\sum_i PV(CashFlow_i) = \int_0^T e^{-\int_0^t r_u du} x_t dt \tag{A.2}$$

with r is the instantaneous risk-free interest rate and T is the maturity of the derivative. We have then,

$$V_0 = \mathrm{E}^Q(\int_0^T e^{-\int_0^t r_u du} x_t dt) \tag{A.3}$$

To introduce the counterparty risk, we suppose that we have a bilateral contract on this derivative with a counterparty. If we assume that both counterparties have survived up to the time point t, there are four events that can happen in the time interval from t to $t + dt$:

- Event 1: The two counterparties survive up to $t + dt$.
- Event 2: We survive up to $t + dt$, but our counterparty defaults.

- Event 3: Our counterparty survives up to $t + dt$, but we default.
- Event 4: Both counterparties default during the time interval between t and $t + dt$.

To compute the probabilities of these events, we suppose that there exists a deterministic default intensity λ. The default probability of a counterparty in the interval $[t, t + dt]$ is given by

$$\lambda_t dt$$

and the survival probability for the time interval $[t, t + dt]$, if the entity survived up to the time point t is given by

$$e^{-\int_t^{t+dt} \lambda_u du}$$

Consequently, the probabilities of the four events are given in the following table,

Event	Probability in $[t, t + dt]$
Event 1	$e^{-\int_t^{t+dt} \lambda_u^{our} du} e^{-\int_t^{t+dt} \lambda_u^{cpty} du}$
Event 2	$e^{-\int_t^{t+dt} \lambda_u^{our} du} \lambda_t^{cpty} dt$
Event 3	$e^{-\int_t^{t+dt} \lambda_u^{cpty} du} \lambda_t^{our} dt$
Event 4	$\lambda_t^{our} \lambda_t^{cpty} dt$

In order to obtain the final probabilities, the values in the previous table should be multiplied by the probability of the two entities having survived up to time t given by $e^{-\int_0^t (\lambda_u^{our} + \lambda_u^{cpty}) du}$. We note that if we consider an infinitesimal time step dt, we have $e^{-\int_t^{t+dt} \lambda_u du} \simeq 1$.

We denote by R the deterministic recovery rate obtained by the surviving counterparty if a default occurs. The cash flows for each event are given in the following table,

Event	Cash flow
Event 1	$x_t dt$
Event 2	$-(1 - R^{cpty}) \max(V_t, 0)$
Event 3	$-(1 - R^{our}) \min(V_t, 0)$
Event 4	$-(1 - R^{cpty}) \max(V_t, 0) - (1 - R^{our}) \min(V_t, 0))$

We know that the expectation of a random variable Z can be written in the following way:

$$\mathrm{E}(Z) = \sum_i P_i Z_i \tag{A.4}$$

with P_i is the probability of the event i and Z_i is the value of Z if the event i takes place. Using this formula for the expectation of the present value and the approximation $e^{-\int_t^{t+dt}\lambda_u du} \simeq 1$, we obtain,

$$\begin{aligned} V_0 =& E^Q[\int_0^T D_{0,t}^{risky} x_t dt] - \\ & E^Q[\int_0^T D^{risky}(0,t)\lambda_t^{cpty}(1-R^{cpty})\max(V_t,0)dt] - \\ & E^Q[\int_0^T D^{risky}(0,t)\lambda_t^{our}(1-R^{our})\min(V_t,0)dt] - \\ & E^Q[\int_0^T D^{risky}(0,t)\lambda_t^{cpty}\lambda_t^{our}((1-R^{cpty})\max(V_t,0)-(1-R^{our})\min(V_t,0))dt] \end{aligned} \tag{A.5}$$

with the risky discount factor $D^{risky}(0,t) = e^{-\int_0^t (r_u+\lambda_u^{our}+\lambda_u^{cpty})du}$

The second term of the equation is known as the CVA (Credit Valuation Adjustment), while the third term is known as DVA (Debit Valuation Adjustment).

$$CVA_0 = E^Q[\int_0^T D^{risky}(0,t)\lambda_t^{cpty}(1-R^{cpty})\max(V_t,0)dt] \tag{A.6}$$

$$DVA_0 = E^Q[\int_0^T D^{risky}(0,t)\lambda_t^{our}(1-R^{our})\min(V_t,0)dt] \tag{A.7}$$

It is often assumed that the fourth event has a negligible probability i.e joint default probability is nearly zero if the correlation between our default event and the counterparty default is not relevant (it will become relevant if we have two similar institutions in the same country for example).

Assuming that the discount factors and V_t are independent of the default events, we can simplify the equations to:

$$CVA_0 = (1-R^{cpty})\int_0^T \lambda_t^{cpty} E^Q[D^{risky}(0,t)\max(V_t,0)]dt \tag{A.8}$$

$$DVA_0 = (1-R^{our})\int_0^T \lambda_t^{our} E^Q[D^{risky}(0,t)\min(V_t,0)]dt \tag{A.9}$$

We can also neglect the riskiness of the discount factors if the counterparties have good credit ratings. We have, then, the following formulas

$$CVA_0 = (1 - R^{cpty}) \int_0^T \lambda_t^{cpty} EPE_t dt \tag{A.10}$$

$$DVA_0 = (1 - R^{our}) \int_0^T \lambda_t^{our} ENE_t dt \tag{A.11}$$

with

- $EPE_t = E^Q[D(0,t) \max(V_t, 0)]$ is the Expected Positive Exposure
- $ENE_t = E^Q[D(0,t) \min(V_t, 0)]$ is the Expected Negative Exposure.

Finally, we can approximate $(1 - R)\lambda_t$ by using the credit spread of the Credir Default Swap (CDS) of the entity $spread_t$.

$$CVA_0 = \int_0^T spread_t^{cpty} EPE_t dt \tag{A.12}$$

$$DVA_0 = \int_0^T spread_t^{our} ENE_t dt \tag{A.13}$$

Appendix B

Properties of the derivatives used in the numerical experiments

The valuation date of the derivatives is: March 22, 2018

B.1 Interest Rate Swap

The properties of the swap are:

- Start date: March 22, 2018
- 20 years maturity
- 1 year frequency for the fixed leg
- 3 months EURIBOR payment frequency
- Notional of 100000000
- Fixed rate of 1.5%

B.2 European Call Swaption

The properties of the underlying receiver swap are:

- Start date: September 12, 2019
- 10 years maturity
- 1 year frequency for the fixed leg
- 3 months EURIBOR payment frequency
- Nominal of 100000000
- Fixed rate of 1.5%

The exercise date of European swaption is September 9, 2019