Monge Parametrizations and Integration of Rectangular Linear Differential Systems

Thomas Cluzeau

University of Limoges ; CNRS ; XLIM (France)

Joint work with **M. Barkatou, C. El Bacha** (University of Limoges ; CNRS ; XLIM) and **A. Quadrat** (INRIA Saclay)

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Introduction / Motivation

◇ Starting points:

- Most algorithms for computing local or global solutions of linear differential systems handle only square systems;
- Systems appearing in many applications such as control theory are in general non-square.

◇ Contribution: reduce the integration of a rectangular system to that of a square one.

◇ Main tool: constructive algebraic analysis techniques, Monge parametrizations

Outline of the talk

1. Constructive algebraic analysis approach to systems theory
2. Monge parametrizations of linear systems
3. Main result
4. Extensions and perspectives

I

Constructive algebraic analysis approach to systems theory

Methodology

1. A linear system is defined by a matrix R with coefficients in a ring D of functional operators:

$$R\,y = 0. \quad (\star)$$

2. To $(\star)$ we associate a left D-module M (finitely presented).
3. There exists a dictionary between the properties of $(\star)$ and M.
4. Homological algebra allows to check the properties of M.
5. Effective algebra (non-commutative Gröbner/Janet bases) gives algorithms.
6. Implementation : Maple (OreModules, OreMorphisms), Singular:Plural, GAP4 (homalg), ...

Example: Wind tunnel model

◇ The wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a\, x_1(t) - k\, a\, x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\, x_2(t) + 2\,\zeta\,\omega\, x_3(t) - \omega^2\, u(t) = 0. \end{cases}$$

◇ Let us consider $D = \mathbb{Q}(a, k, \omega, \zeta)[\partial, \delta]$

◇ The system can then be written as $R\, y = 0$ with

$$R = \begin{pmatrix} \partial + a & -k\, a\, \delta & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D^{3 \times 4}.$$

The left D-module M

◇ D ring of functional operators, $R \in D^{q \times p}$ and $\mathcal{F}$ a left D-module (the functional space):

$$\forall\, P_1,\, P_2 \in D,\ \forall\, \eta_1,\, \eta_2 \in \mathcal{F}:\ P_1\, \eta_1 + P_2\, \eta_2 \in \mathcal{F}.$$

◇ Consider the system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}$.

◇ As in number theory or algebraic geometry, to $\ker_{\mathcal{F}}(R.)$ we associate the left D-module:

$$M = D^{1 \times p} / (D^{1 \times q}\, R).$$

given by the finite presentation:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\, R & & & & \end{array}$$

Theorem [Malgrange]:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) = \{f : M \to \mathcal{F},\ f \text{ is left } D\text{-linear}\}.$$

Definitions of module theory

- $\diamond$ $M' \xrightarrow{f} M \xrightarrow{g} M''$ is called a complex if $\mathrm{im}\, f \subseteq \ker\, g$.

- $\diamond$ $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said exact if $\mathrm{im}\, f = \ker\, g$.

Definition

1. M is free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^r$.
2. M is projective if $\exists\, r \in \mathbb{Z}_+$ and a D-module P such that:
$$M \oplus P \cong D^r.$$
3. M is torsion-free if:
$$t(M) = \{m \in M \mid \exists\, 0 \neq d \in D : d\, m = 0\} = 0.$$

(**Ex.**: $\dot{x}_1 = x_2 + u$, $\dot{x}_2 = x_1 + u$ defines a torsion module since $z = x_1 - x_2$ satisfies $\dot{z} + z = 0$ so that $t(M) \neq 0$)

Syzygy module computations

◇ Let D be a ring of functional operators which is a noetherian domain and $R \in D^{q \times p}$.

◇ There exist

- $P \in D^{r \times q}$ such that $\ker_D(.R) = D^{1 \times r}\, P$:

$$D^{1\times r} \xrightarrow{.P} D^{1\times q} \xrightarrow{.R} D^{1\times p} \text{ is exact.}$$

- $Q \in D^{p \times m}$ such that $\ker_D(R.) = Q\, D^m$:

$$D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m \qquad \text{is exact,} \quad R\, Q = 0.$$

◇ Such matrices P and Q can be computed by means of non-commutative Gröbner/Janet bases computations.

◇ Implementation in the Maple library OreModules available at

http://wwwb.math.rwth-aachen.de/OreModules/

Injective modules

Let $\mathcal{F}$ be a left D-module, applying $\hom_D(., \mathcal{F})$ to the exact sequence

$$D^{1\times r} \quad \xrightarrow{.P} \quad D^{1\times q} \quad \xrightarrow{.R} \quad D^{1\times p},$$

we get the following complex with $R\,\mathcal{F}^p \subseteq \ker_{\mathcal{F}}(P.)$

$$\mathcal{F}^r \quad \xleftarrow{P.} \quad \mathcal{F}^q \quad \xleftarrow{R.} \quad \mathcal{F}^p. \tag{1}$$

Definition: A left D-module $\mathcal{F}$ is injective if for every injective left D-homomorphism $f : A \to B$ from a left D-module A to a left D-module B and for every $\psi \in \hom_D(A, \mathcal{F})$, there exists a left D-homomorphism $\phi \in \hom_D(B, \mathcal{F})$ such that $\psi = \phi \circ f$.

Theorem: $\mathcal{F}$ is injective iff the functor $\hom_D(., \mathcal{F})$ is exact.

◇ If $P \in D^{r\times q}$ is such that $\ker_D(.R) = D^{1\times r}\,P$ and $\mathcal{F}$ is an injective left D-module, then $R\,\mathcal{F}^p = \ker_{\mathcal{F}}(P.)$, *i.e.*, (1) is exact.

II

Monge parametrizations

Monge parametrization of linear functional systems

⋄ Let D be a ring of functional operators which is a noetherian domain.

⋄ Let $\mathcal{F}$ be a left D-module.

⋄ Let us consider $R \in D^{q \times p}$ and the linear functional system:

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

⋄ Quadrat-Robertz' algorithm: compute, if they exist, 4 matrices $Q \in D^{p \times m}$, $R'' \in D^{q \times q'}$, $T \in D^{r' \times q'}$ and $S \in D^{p \times q'}$ such that:

$$\ker_{\mathcal{F}}(R.) = \{Q\,\mu + S\,\zeta \mid \mu \in \mathcal{F}^m, \quad \begin{pmatrix} R'' \\ T \end{pmatrix} \zeta = 0\}.$$

⋄ Key point: integrating $R\,\eta = 0 \Longleftrightarrow$ integrating $\begin{pmatrix} R'' \\ T \end{pmatrix} \zeta = 0.$

Monge parametrization of LFS: QR's algorithm (1)

◇ Compute $Q \in D^{p \times m}$ and $R' \in D^{q' \times p}$ such that

$$\ker_D(R.) = Q\, D^m, \quad \ker_D(.Q) = D^{1 \times q'}\, R'.$$

◇ $R\, Q = 0$ implies $D^{1 \times q} R \subseteq \ker_D(.Q) = D^{1 \times q'}\, R'$
$\Rightarrow$ there exists $R'' \in D^{q \times q'}$ such that $R = R''\, R'$.

◇ Compute $T \in D^{r' \times q'}$ such that $\ker_D(.R') = D^{1 \times r'}\, T$.
$\Rightarrow$ we thus have $R'\, \mathcal{F}^p \subseteq \ker_{\mathcal{F}}(T.)$.

◇ We have:

$$R\, \eta = 0 \Longleftrightarrow R''\, R'\, \eta = 0 \Longleftrightarrow \left\{ \begin{array}{lcl} R''\, \zeta & = & 0, \\ T\, \zeta & = & 0, \\ R'\, \eta & = & \zeta. \end{array} \right.$$

◇ Links with $t(M)$:

$$M/t(M) = D^{1 \times p}/(D^{1 \times q'}\, R'),$$

$$t(M) = (D^{1 \times q'}\, R')/(D^{1 \times q}\, R) \cong D^{1 \times q'} / \left(D^{1 \times (q+r')}\, (R''^T \quad T^T)^T \right).$$

Monge parametrization of LFS: QR's algorithm (2)

◇ We now consider the system: $R' \, \eta = \zeta, \quad \zeta \in \ker_{\mathcal{F}} \left(\left(\begin{array}{c} R'' \\ T \end{array} \right) . \right)$.

◇ Since $\ker_D(.Q) = D^{1 \times q'} \, R'$, if $\mathcal{F}$ is an injective left D-module, then $\ker_{\mathcal{F}}(R'.) = Q \, \mathcal{F}^m$.

$\Rightarrow$ General solution of the homogeneous system $R' \, \eta = 0$ given by $Q \, \mu$, for all $\mu \in \mathcal{F}^m$

◇ Quadrat-Robertz: particular solution given by $S \, \zeta$ if there exists $S \in D^{p \times q'}$ satisfying that there exists $V \in D^{q' \times q}$ such that $R' - R' \, S \, R' = V \, R$.

- S is called a *generalized inverse* of R' modulo $D^{1 \times q} \, R$;
- Existence $\Leftrightarrow M \cong t(M) \oplus M/t(M)$ (ok if $M/t(M)$ projective);
- Algorithm implemented in OreModules.

Monge parametrization: QR's result

Theorem: If $\mathcal{F}$ is an injective left D-module and $M \cong t(M) \oplus M/t(M)$, then we can compute 4 matrices $Q \in D^{p\times m}$, $R'' \in D^{q\times q'}$, $T \in D^{r'\times q'}$ and $S \in D^{p\times q'}$ such that:

$$\ker_{\mathcal{F}}(R.) = \{Q\,\mu + S\,\zeta \mid \mu \in \mathcal{F}^m, \quad \begin{pmatrix} R'' \\ T \end{pmatrix} \zeta = 0\}.$$

Remark 1: $t(M) \cong D^{1\times q'} / \left(D^{1\times(q+r')}\,({R''}^T \quad T^T)^T\right)$. The system to be integrated is (over)determined: algorithms exist.

Remark 2: M is torsion-free $\rightsquigarrow \ker_{\mathcal{F}}(R.) = \{Q\,\mu \mid \mu \in \mathcal{F}^m\}$.

◇ Implemented on the Maple library OreModules available at

http://wwwb.math.rwth-aachen.de/OreModules/

III

Main result

The case of a linear differential system: $D = \mathbb{C}[x][\frac{d}{dx}]$

$\diamond$ Let $R \in D^{q \times p}$: the matrix defining our linear differential system.

$\diamond$ Let $\mathcal{F}$ be a given space of functions (*e.g.*, regular formal power series, polynomial, rational or exponential functions, ...).

$\rightsquigarrow$ The solution spaces of interest are not injective left D-modules!

However

$\diamond$ The ring $D = \mathbb{C}[x][\frac{d}{dx}]$ has strong properties (hereditary ring, Stafford's theorem)

$\rightsquigarrow$ generalization of the previous Quadrat-Robertz' algorithm

Hereditary rings

Definition: A ring D is left hereditary if every left ideal is projective.

Properties:

1. D is left hereditary iff every submodule of a projective module is projective;
2. If D is left hereditary, then every torsion free left D-module is projective;
   $\rightsquigarrow$ we always have $M \cong t(M) \oplus M/t(M)$.

Main theorem

Theorem: Let $D = \mathbb{C}[x][\frac{d}{dx}]$ and $R \in D^{q \times p}$ full row-rank. Assume that $q \geq 2$ and let $\mathcal{F}$ be a given space of functions (*e.g.*, regular formal power series, polynomial, rational or exponential functions, ...) having a left D-module structure.
Then, there exist $Q \in D^{p \times m}$, a full row-rank matrix $R' \in D^{q \times p}$, a square matrix $R'' \in D^{q \times q}$ and $S \in D^{p \times q}$ satisfying

$$R = R'' R', \quad \ker_{\mathcal{F}}(R'.) = Q\, \mathcal{F}^m, \quad R' S = I_q.$$

Moreover, we have

$$\ker_{\mathcal{F}}(R.) = \{Q\, \mu + S\, \zeta \mid \mu \in \mathcal{F}^m, \quad R''\, \zeta = 0\}.$$

$\Rightarrow$ Integrating the rectangular linear differential system $R\, \eta = 0$ is reduced to integrating the square linear differential system

$$R''\, \zeta = 0.$$

Sketch of the proof

- Follow Quadrat-Robertz' method in our case:

1. Compute $Q \in D^{p \times m}$ such that $\ker_D(R.) = Q\, D^m$;
2. D hereditary ring & R full row rank $\Rightarrow \ker_D(.Q)$ projective of rank q;
3. $q \geq 2$ & Stafford's theorem $\Rightarrow \ker_D(.Q)$ free of rank q;
4. Compute a basis: $R' \in D^{q \times p}$ full row-rank such that $\ker_D(.Q) = D^{1 \times q}\, R'$;
   $\Rightarrow T = 0$ and R'' square of size q such that $R = R''\, R'$;
5. D hereditary $\Rightarrow \ker_{\mathcal{F}}(R'.) = Q\, \mathcal{F}^m$;
6. D hereditary & R' full row rank $\Rightarrow$ R' admits a right inverse $S \in D^{p \times q}$.

- Algorithm has been implemented in Maple using OreModules.

Example (1)

◇ Let $D = \mathbb{C}[t][d]$ with $d = \frac{d}{dt}$ and consider the rectangular linear differential system given by

$$R = \begin{pmatrix} 1 & t^3 d^2 - 2\,t^3 d + t^3 & 0 & 1 \\ t^2 & 2\,td + t^2 d - 2\,t - t^2 & 1+t & -t \\ 0 & 0 & -1 & t \end{pmatrix} \in D^{3\times 4}.$$

◇ Using OreModules, we compute

$$Q = \begin{pmatrix} -1 & 0 & t & 1 \end{pmatrix}^T$$

that satisfies $\ker_D(R.) = Q\,D$ and

$$R' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & t \end{pmatrix}, \quad R'' = \begin{pmatrix} 1 & t^3 d^2 - 2\,t^3 d + t^3 & 0 \\ t^2 & 2\,td + t^2 d - 2\,t - t^2 & -t-1 \\ 0 & 0 & 1 \end{pmatrix}$$

so that $\ker_D(.Q) = D^{1\times 3}\,R'$ and $R = R''\,R'$.

Example (2)

◇ Computing a right inverse S of R', *i.e.*, $R'\,S = I_3$ we find:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

◇ Theorem $\Rightarrow$ Solutions of $R\,\eta = 0$ in $\mathcal{F}$ given by $Q\,\mu + S\,\zeta$ where $\mu \in \mathcal{F}$ and $\zeta \in \mathcal{F}^3$ satisfies the square linear differential system $R''\,\zeta = 0$.

Example (3)

◇ Regular formal solutions (use algorithms developed in C. El Bacha's PhD for square systems) of $R\,\eta = 0$:

$$\eta(t) = S\,\zeta(t) + Q\,\mu(t) = \begin{pmatrix} -\mu(t) \\ 1 + t + \frac{1}{2}\,t^2 + \frac{1}{6}\,t^3 + \frac{1}{24}\,t^4 + O(t^5) \\ t\mu(t) \\ \mu(t) \end{pmatrix},$$

for all $\mu(t) = t^{\lambda_0}\,z(t)$ where $\lambda_0 \in \mathbb{C}$, $z(t) \in \mathbb{C}[\log(t)][[t]]$.

IV

Extensions and perspectives

Extensions / Perspectives

$\diamond$ In our theorem, we can replace $D = \mathbb{C}[x][\frac{d}{dx}]$ by $D = k[\![x]\!]\langle\partial\rangle$, where k is a field of characteristic zero, or by $D = k\{x\}\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$.

Indeed, Quadrat-Robertz proved recently that those rings have the needed algebraic properties.

$\diamond$ In the future:

- Gather our implementation with those of algorithms computing global or local solutions;
  (*e.g.*, ISOLDE - `http://isolde.sourceforge.net/`)
- Comparisons to Jacobson normal form, B.-El B.-Pfluegel and Grigoriev: singularities may be introduced!
- Can this be extended to other rings D?