Calculating Generators of Multigraded Algebras

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Let $\omega \subseteq \mathbb{Q}^d$ be a full-dimensional polyhedral cone, let $M \subseteq \mathbb{Q}^d$ be a lattice of dimension d. Consider a finitely generated normal $\mathbb{C}$-algebra

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There are two possible representations for A:

- $$A = \mathbb{C}[\underline{x}]/I$$
- $$A = \mathcal{A}(\mathcal{D}, M)$$

  with $\mathcal{D}$ a ”p-Divisor”.

Let Y be a normal semiprojective variety. Let $\omega \subseteq \mathbb{Q}^d$ be a polyhedral cone of $\dim \omega = d$. A p-Divisor is a map

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- $\mathcal{D}$ is convex, i.e. $\forall u, v \in \omega : \mathcal{D}(u) + \mathcal{D}(v) \leq \mathcal{D}(u + v)$;
- $\forall u \in \omega \cap M$: $\mathcal{D}(u)$ semiample;
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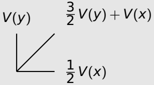
Example

Let $Y = \mathbb{A}^2$,

$\mathcal{D} =$ 

Let Y be a normal semiprojective variety, let $\mathcal{D}$ be a p-divisor on Y

$$\mathcal{A}(\mathcal{D}, M) := \bigoplus_{u \in \omega \cap M} H^0(Y, \mathcal{D}(u))\chi^u \subseteq \mathbb{C}(Y)[M]$$

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Theorem (Altmann, Hausen)

The ring $\mathcal{A}(\mathcal{D}, M)$ associated to the p-Divisor $\mathcal{D}$ and lattice M is a finitely generated normal M-graded $\mathbb{C}$-algebra of dimension $\dim M + \dim Y$.

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Let A be a finitely generated normal $\mathbb{C}$-algebra graded by some lattice M with weight cone $\omega \subseteq M$, where $\dim \omega = \dim M$ and ω spans the lattice M. Then there exists a normal semiprojective variety Y of dimension $\dim A - \dim M$ and a p-divisor $\mathcal{D} : \omega \to \mathrm{CaDiv}_{\mathbb{Q}} Y$ such that

$$A \cong \mathcal{A}(\mathcal{D}, M).$$

Let Y be a normal semiprojective variety and let $\mathcal{D}$ be a p-Divisor on Y.

Question:

What are the generators of $\mathcal{A}(\mathcal{D}, M)$ as a subring of $\mathbb{C}(Y)(M)$?

Even better: How can one find a presentation of $\mathcal{A}(\mathcal{D}, M)$ as a quotient of a polynomial ring by an ideal?

$\dim Y = 0$: Combinatorial problem of finding a Hilbert basis of $\omega \cap \mathbb{Z}^n$.
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$\dim Y = 1$: Algorithm by N. Ilten and H. Süß.

$\dim Y \geq 0$: Our Algorithm.
Note: Requires ability to calculate global sections, i.e. generators of $H^0(Y, D)$ as a $H^0(Y, \mathcal{O}_Y)$-module for some divisor D on Y.

- Let $M' \subseteq M$ be a full-dimensional sublattice. Then

$$\overline{\mathcal{A}(\mathcal{D}, M')}^{\mathbb{C}(Y)(M)} = \mathcal{A}(\mathcal{D}, M).$$

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- Let $\omega = \omega_1 \cup \ldots \cup \omega_s$ be a finite covering of ω by subcones. Then we have

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- For the quotient field of $\mathcal{A}(\mathcal{D}, M)$ we have

$$\mathrm{Quot}(\mathcal{A}(\mathcal{D}, M)) = \mathbb{C}(Y)(M).$$

Let Y be a semiprojective variety, let $\omega = \mathbb{Q}^d_{\geq 0}$ and $M = \mathbb{Z}^d$. Let $\mathcal{D} : \omega \to \mathrm{CaDiv}_{\mathbb{Q}} Y$ be a p-Divisor on Y, such that $\mathcal{D}$ is linear on ω and $\mathcal{D}(e^i)$ is effective, integral and basepoint free for all $i = 1, \ldots, d$.

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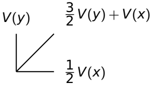
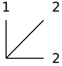
Theorem (Zariski '62)

Assume the above setting. Let R be the subring of $\mathbb{C}(Y)(M)$ generated by the collection of generators of $H^0(Y, \mathcal{D}(e^i))$ as $H^0(Y, \mathcal{O}_Y)$ modules for $i = 1, \ldots, d$ together with generators of $H^0(Y, \mathcal{O}_Y)$ as a $\mathbb{C}$-algebra. Then the integral closure $\overline{R}^{\mathbb{C}(Y)(M)}$ of R in $\mathbb{C}(Y)(M)$ equals $\mathcal{A}(\mathcal{D}, M)$.

- Use preliminaries to reduce to the setting of Zariski's theorem. Find a set of generators L.
- Add generators to L to ensure that $\mathrm{Quot}(\mathbb{C}[L]) = \mathbb{C}(Y)(M)$.
- Normalize the ring $\mathbb{C}[L]$.



STEP1: Decompose ω into cones on which $\mathcal{D}$ becomes linear. Collect the generating rays $\{\rho^i\}$.



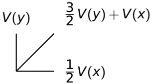
STEP2: For each ρ^i find k_i such that $\mathcal{D}(k_i\rho^i)$ is base point free and integral.

$\frac{1}{y}\chi_2, \frac{1}{y^3x^2}\chi_1^2\chi_2^2, \frac{1}{x}\chi_1^2, x, y$

STEP3: Take generators $\{s_1, \ldots, s_{r_i}\}$ of $H^0(Y, \mathcal{D}(k_i\rho^i))$ and collect the elements $s_j \cdot \chi^{k_i\rho^i}$ for all i and j in a set L. Add generators of $H^0(Y, \mathcal{O}_Y)$ as a $\mathbb{C}$-algebra to L.

REMARK: Zariski's theorem yields that $\mathcal{A}(\mathcal{D}, M)$ is the integral closure of the ring $\mathbb{C}[L]$ in $\mathbb{C}(Y)(M)$.

The algorithm



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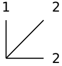

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REMARK: Zariski's theorem yields that $\mathcal{A}(\mathcal{D},M)$ is the integral closure of the ring $\mathbb{C}[L]$ in $\mathbb{C}(Y)(M)$.

STEP4: Add elements of $\mathcal{A}(\mathcal{D},M)$ to L such that the quotient field of the ring $\mathbb{C}[L]$ becomes $\mathbb{C}(Y)(M)$

$\frac{1}{y}\chi_2, \frac{1}{y^3x^2}\chi_1^2\chi_2^2, \frac{1}{x}\chi_1^2, x, y, \frac{1}{y^4x^3}\chi_1^3\chi_2^3$

STEP5: Return the normalization of $\mathbb{C}[L]$.

$\mathcal{A}(\mathcal{D},M)\cong\mathbb{C}[\underline{x}]/I$

STEP4a: Find a lattice basis $b_1, \dots, b_d$ of M that is contained in the interior of ω. For every b_i and for ascending $k \in \mathbb{Z}_{>0}$ take single sections $s \in H^0(Y, \mathcal{D}(k \cdot b_i)$ if the latter does not equal zero and add $s \cdot \chi^{kb^i}$ to L. Stop if the set of k with a non-zero section has gcd 1.

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We obtain:

$$\mathrm{Quot}(\mathbb{C}[L]) = \mathbb{C}(Y)(M) = \mathrm{Quot}(\mathcal{A}(\mathcal{D}, M)).$$

Let X be a projective variety.

$$\mathrm{Cox}(X) = \bigoplus_{D \in \mathrm{Pic}(X)} H^0(X, D)$$

Now let $X = S_5$ be the blow up of $\mathbb{P}^2$ in 4 generic points.

- Altmann, Wisniewski '09: The Cox ring of a del Pezzo surface S can be described by a p-Divisor on S (in particular, the case $S = S_5$).
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- Future: Use algorithm to calculate Cox rings for log del Pezzo surfaces.

- POLYMAKE for combinatorics.
- M2 for divisors and global sections.
- SINGULAR for normalization and reduction.

- Klaus Altmann, Jürgen Hausen: Polyhedral Divisors and Algebraic Torus Actions
- Klaus Altmann, Jaroslaw Wisniewski: Polyhedral divisors of Cox rings
- Victor V. Batyrev, Oleg N. Popov: The Cox Ring of a Del Pezzo Surface
- Oscar Zariski: The Theorem of Riemann-Roch for High Multiples of an Effective Divisor on an Algebraic Surface

The End

Thank you!