Secant degree of toric surfaces and delightful planar toric degenerations

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1. k-secant varieties
2. Secants of toric surfaces: a combinatorial approach
3. Results

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Secant varieties

Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective surface.

Definition

The k-th secant variety of X is $\mathcal{S}_k(X) := \overline{\bigcup \langle x_1, \ldots, x_k \rangle} \subseteq \mathbb{P}^r$.

$$\dim(\mathcal{S}_k(X)) \leq \min\{3k-1, r\}.$$

Assume $\dim(\mathcal{S}_k(X)) = 3k-1 \leq r$.

Question

What is the number $\nu_k(X)$ of k-secant $\mathbb{P}^{k-1}$'s intersecting a general $\mathbb{P}^{r-(3k-1)}$?

If $\mu_k(X)$ is the k-secant order of X, i.e. the number of k-secant $\mathbb{P}^{k-1}$ passing through the general point of $\mathcal{S}_k(X)$, then it is

$$\nu_k(X) = \mu_k(X) \cdot \deg(\mathcal{S}_k(X)).$$

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Secant ideals

Let $\mathcal{I} \subseteq \mathbb{C}[x_0, \ldots, x_r]$ be the homogeneous prime ideal of $X \subseteq \mathbb{P}^r$.
Let $\mathcal{I}^{\{k\}} = \mathcal{I} * \cdots * \mathcal{I}$ be the *k-secant* ideal of $\mathcal{I}$.

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Compute $\deg \mathcal{I}^{\{k\}}$.

Theorem [Simis-Ulrich '00, Sturmfels-Sullivant '06]

$$\mathrm{in}_{\prec}(\mathcal{I}^{\{k\}}) \subseteq (\mathrm{in}_{\prec}(\mathcal{I}))^{\{k\}} \qquad (\star)$$

In particular, if $\dim \mathbb{C}[x]/(\mathrm{in}_{\prec}(\mathcal{I}^{\{k\}})) = \dim \mathbb{C}[x]/(\mathrm{in}_{\prec}(\mathcal{I}))^{\{k\}})$, then

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If equality holds in $(\star)$, then $\prec$ is said to be *k-delightful* for the ideal $\mathcal{I}$.

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Projective toric surfaces.
Let $P \subseteq \mathbb{R}^2$ be a convex lattice polytope.
Take $P \cap \mathbb{Z}^2 = \{m_0, \dots, m_r\}$ and define the morphism

$$\phi_P : x = (x_1, x_2) \in (\mathbb{C}^*)^2 \to [x^{m_0}, \dots, x^{m_r}] \in \mathbb{P}^r$$

with $x^{m_i} = x_1^{m_{i1}} \cdot x_2^{m_{i2}}$.

$$X_P = \overline{\mathrm{Im}(\phi_P)} \subseteq \mathbb{P}^r.$$

Example. The 3-Veronese embedding V_3 of $\mathbb{P}^2$ in $\mathbb{P}^9$

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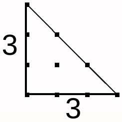
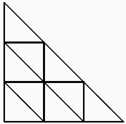
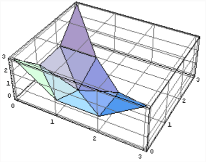


Toric degenerations: regular subdivisions

Pick a subdivision $\mathcal{D} = \{Q_i\}_{i \in I}$ of P and a lifting function $F_{\mathcal{D}}$



$\longleftarrow$



$$\begin{array}{rccc} \Phi_{\mathcal{D}}: & (\mathbb{C}^*)^3 & \to & \mathbb{P}^r \times \mathbb{C} \\ & (x,t) & \mapsto & ([\cdots : t^{F_{\mathcal{D}}(m_i)} x^{m_i}, \ldots], t). \end{array}$$

defines a degeneration of X_P to X_0, whose fibers are

- $X_t := \overline{(\Phi_{\mathcal{D}}((\mathbb{C}^*)^2 \times \{t\}))}$ projectively equivalent to X_P, $t \neq 0$
- $X_0 := \lim_{t \to 0} X_t = \bigcup_{i \in I} X_{Q_i}$.

If $\mathcal{D}$ is a regular unimodular triangulation of P, then X_0 is a reduced union of planes. This corresponds to taking the initial ideal $\mathrm{in}_{\prec}(\mathcal{I}_X)$, for some term order.

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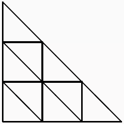

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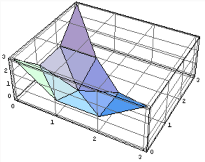

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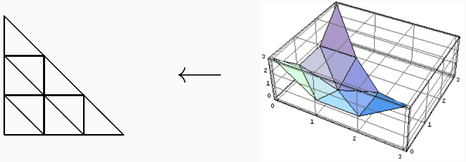

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Computing $\nu_k(X)$ combinatorially

Definition

Given $X = X_P \in \mathbb{P}^r$ toric variety and $\mathcal{D}$ triangulation of P, we define

$$\mathcal{N}_k(\mathcal{D}) := \{\text{skew k-sets}\};$$

$$N_k(\mathcal{D}) := \#(\mathcal{N}_k(\mathcal{D})).$$

Example.

$V_3 \subseteq \mathbb{P}^9$, $k = 3$

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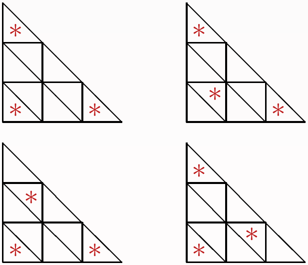

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Theorem [Sturmfels-Sullivant '06]

If there exists $\mathcal{D}$ with $N_k(\mathcal{D}) \geq 1$, then $\mathcal{S}_k(X)$ has the expected dimension $3k-1$. Moreover

$$\nu_k(X) \geq N_k(\mathcal{D}).$$

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Pick $(T_1, \ldots, T_k) \in \mathcal{N}_k(\mathcal{D})$ and $\pi := \langle X_{T_1}, \ldots, X_{T_k} \rangle \cong \mathbb{P}^{3k-1}$. The flat limit $\lim_{\mathcal{D}} \mathcal{S}_k(X)$ contains the subspace $\pi \subseteq \mathbb{P}^r$. Then
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All those π's are linear components of $\lim_{\mathcal{D}} \mathcal{S}_k(X)$ of dimension $3k-1$ thus the conclusion follows. □

Remarks:

- The Theorem holds also if $n \geq 3$.
- A similar proof appears in [CDM '07] for Veronese varieties.
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We will call k-delightful a degeneration $\mathcal{D}$ such that $\nu_k(X) = N_k(\mathcal{D})$, according to [SS] and [CDM].

Example 1.

$V_3 \subseteq \mathbb{P}^9$,
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$N_3(\mathcal{D}) = 4 = \deg(\mathcal{S}_3(V_3)) = \nu_3(V_3)$: $\mathcal{D}$ is 3-delightful!

Example 2.

Del Pezzo surface $S_6 \subseteq \mathbb{P}^6$,
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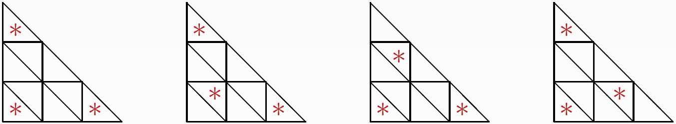

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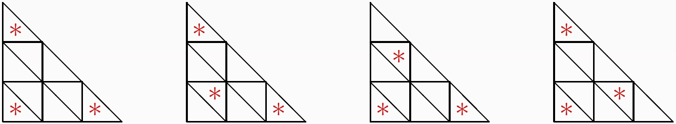

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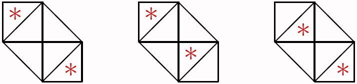

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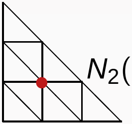

$N_2(\mathcal{D}) = 12$

$\deg(\mathcal{S}_2(V_3)) = \nu_2(V_3) = 15.$

Question: how does the presence of multiple points • in the configuration explain the k-delightfulness defect?

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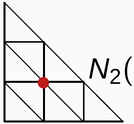

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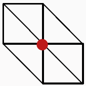

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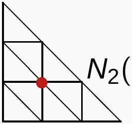

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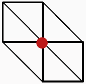

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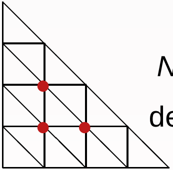
Example 3. $V_4 \subseteq \mathbb{P}^{14}$



$N_2(\mathcal{D}) = 66$

$\deg(\mathcal{S}_2(V_4)) = \nu_2(V_4) = 75.$

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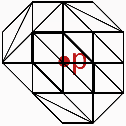
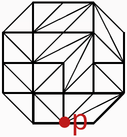
Towards an answer

$k = 2$.

Let $X = X_P$ be a toric surface with $\dim \mathcal{S}_2(X) = 5$ and let $\mathcal{D}$ be a regular unimodular triangulation of X.

- Let p be a "red" point of $\mathcal{D}$ with

  $g = 1$ e.g. 

  or $g = 0$ e.g. 

- Let Q_p be the subpolytope of P given by the union of all triangles having a vertex in p.
- Define a toric surface from Q_p: $Y_p \subseteq \mathbb{P}^{\#Q_p \cap \mathbb{Z}^2 - 1}$
- Assume that $\dim(\mathcal{S}_2(Y_p)) = 5$.

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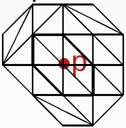
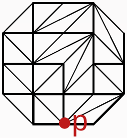
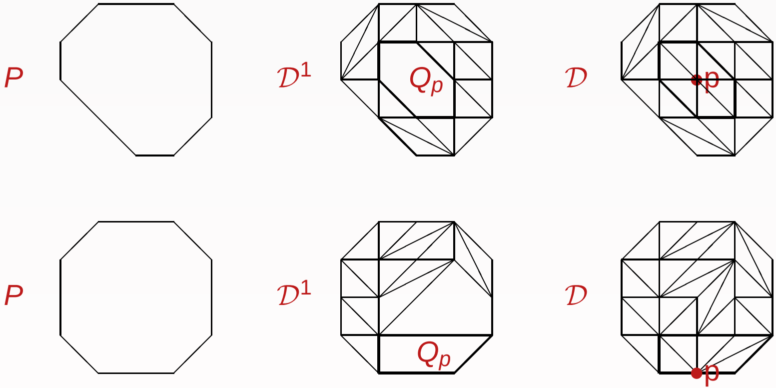


  or $g = 0$ e.g.



- Let Q_p be the subpolytope of P given by the union of all triangles having a vertex in p.
- Define a toric surface from Q_p: $Y_p \subseteq \mathbb{P}^{\#Q_p \cap \mathbb{Z}^2 - 1}$
- Assume that $\dim(\mathcal{S}_2(Y_p)) = 5$.

- Suppose that there exists an intermediate regular subdivision $\mathcal{D}^1$ of P containing Q_p and coarsening $\mathcal{D}$.



Theorem, Part I

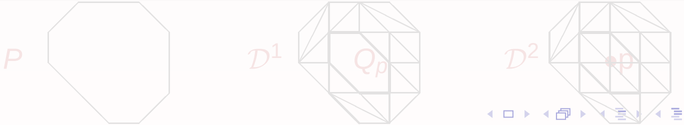
X, $\mathcal{D}$, p and Y_p as above. $\nu_2(X) \geq N_2(\mathcal{D}) + \nu_2(Y_p)$.

Proof.

Let $\mathcal{D}^1$ and $\mathcal{D}^2$ be subsequent subdivisions of P from which one gets, as composition, $\mathcal{D}$.

1. The flat limit $\lim_{\mathcal{D}^1} \mathcal{S}_2(X)$ contains
   - (a) $\mathcal{S}_2(Y_p)$,
   - (b) all secant varieties and joins of components of X_0.
2. the flat limit $\lim_{\mathcal{D}} \mathcal{S}(X) = \lim_{\mathcal{D}^2} \lim_{\mathcal{D}^1} \mathcal{S}(X)$ contains the flat limits, via $\mathcal{D}^2$, of all components of (a) and (b): in particular
   - (a') $\lim_{\mathcal{D}^2} \mathcal{S}_2(Y_p)$,
   - (b') the limits of (b) (including $\mathcal{N}_2(\mathcal{D})$).

□



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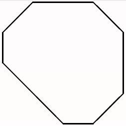
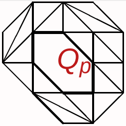
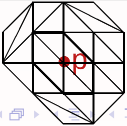
P



$\mathcal{D}^1$

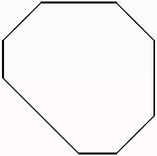


$\mathcal{D}^2$

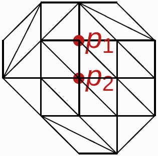


Let $\{p_i\}_i$ be the set of "red" points of $\mathcal{D}$. We want to consider simultaneously all contributions $\nu_2(Y_{p_i})$ to $\nu_2(X)$.

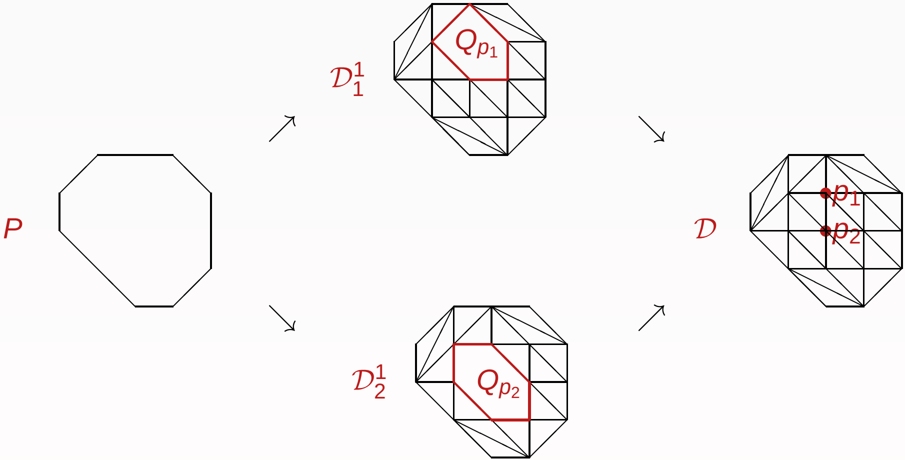
P



$\mathcal{D}$



Let $\{p_i\}_i$ be the set of "red" points of $\mathcal{D}$. We want to consider simultaneously all contributions $\nu_2(Y_{p_i})$ to $\nu_2(X)$.
To this end, for each i take a subdivision D_i^1 (if it exists).



Theorem, Part II

The contributions in terms of degree given by these components can be summed up:

$$\nu_2(X) \geq N_2(\mathcal{D}) + \sum_i \nu_2(Y_{p_i}).$$

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Proof.

Let $\mathbb{P}_i \subseteq \mathbb{P}^r$ be the projective subspace where Y_{p_i}, $\mathcal{S}_2(Y_{p_i})$ and its limits live, namely the space whose coordinate are given by the lattice points of Q_{p_i}:

$$(\lim_{D_i^2} \mathcal{S}_2(Y_{p_i})) \cap (\lim_{D_i^2} \mathcal{S}_2(Y_{p_i})) \subseteq \mathbb{P}_i \cap \mathbb{P}_j$$

Since $\dim(\mathbb{P}_i \cap \mathbb{P}_j) \leq 3$, we conclude. □

Theorem, Part II

The contributions in terms of degree given by these components can be summed up:

$$\nu_2(X) \geq N_2(\mathcal{D}) + \sum_i \nu_2(Y_{p_i}).$$

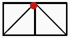
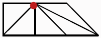
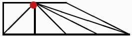
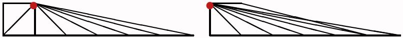
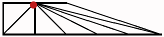
Corollary

If $\mathcal{J}_i := \mathcal{I}_{Y_{p_i}} \subseteq \mathbb{C}[x_0, \ldots, x_r]$ is the toric ideal of $Y_{p_i} \subseteq \mathbb{P}_i \subseteq \mathbb{P}^r$ then

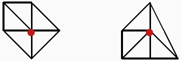
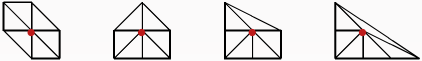
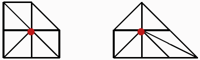
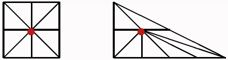
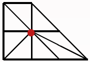
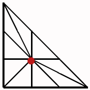
$$in_{\prec}(\mathcal{I}_X^{\{k\}}) \subseteq in_{\prec}(\mathcal{I}_X)^{\{k\}} \cap \bigcap_i in_{\prec}(\mathcal{J}_i^{\{k\}}) \subseteq in_{\prec}(\mathcal{I}_X)^{\{k\}}$$

Remark: The result can be generalized for $k \geq 2$, but this gives interesting info only if $k = 2, 3$ (cfr. tables).

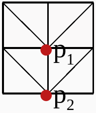
The $g = 0$ case

$\mathcal{D}_{\|Q_p}$	$\deg(Y_p)$	$\nu_2(Y_p)$	$\nu_3(Y_p)$
	4	1	/
	4	1	/
	5	3	/
	5	3	/
	6	6	/
	6	6	/
	$\delta \geq 7$	$\binom{\delta-2}{2}$	/
	$\delta \geq 7$	$\binom{\delta-2}{2}$	$\binom{\delta-4}{3}$

The $g = 1$ case

$\mathcal{D}_{\mid Q_p}$	$\deg(Y_p)$	$\nu_2(Y_p)$	$\nu_3(Y_p)$
	5	1	/
	6	3	/
	7	6	/
	8	10	/
	8	10	1
	9	15	4

Example.



$$\nu_2(X) = N_2(\mathcal{D}) + \nu_2(Y_1) + \nu_2(Y_2) = 6 + 3 + 1 = \underline{10}$$

x_0	x_1	x_2
x_3	x_4	x_5
x_6	x_7	x_8

$\mathrm{in}_\prec(I)^{\{2\}} = \langle x_0x_2x_6, \ldots, x_4x_6x_8 \rangle$ $\quad$ $\deg = 6 = N_2(D)$

$\mathrm{in}_\prec(I^{\{2\}}) = ?$ $\quad$ $\deg = \underline{10}$

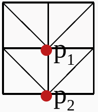
$\mathrm{in}_\prec(J_1^{\{2\}}) = \langle x_6, x_8, x_0x_2x_7 \rangle$ $\quad$ $\deg = 3 = \nu_2(Y_1)$

$\mathrm{in}_\prec(J_2^{\{2\}}) = \langle x_0, x_1, x_2 \rangle$ $\quad$ $\deg = 1 = \nu_2(Y_2)$

$\mathrm{in}_\prec(I)^{\{2\}} \cap \mathrm{in}_\prec(J_1^{\{2\}}) \cap \mathrm{in}_\prec(J_1^{\{2\}})$ $\quad$ $\deg = \underline{10}$

$\Rightarrow \quad \mathrm{in}_\prec(I^{\{2\}}) = \mathrm{in}_\prec(I)^{\{2\}} \cap \mathrm{in}_\prec(J_1^{\{2\}}) \cap \mathrm{in}_\prec(J_1^{\{2\}})$

Example.



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$$\Rightarrow \quad \mathrm{in}_{\prec}(I^{\{2\}}) = \mathrm{in}_{\prec}(I)^{\{2\}} \cap \mathrm{in}_{\prec}(J_1^{\{2\}}) \cap \mathrm{in}_{\prec}(J_1^{\{2\}})$$

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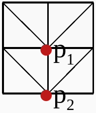

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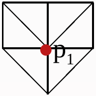

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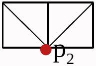

$$\mathrm{in}_{\prec}(J_2^{\{2\}}) = \langle x_0, x_1, x_2 \rangle \qquad \deg = 1 = \nu_2(Y_2)$$

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$$\Rightarrow \quad \mathrm{in}_{\prec}(I^{\{2\}}) = \mathrm{in}_{\prec}(I)^{\{2\}} \cap \mathrm{in}_{\prec}(J_1^{\{2\}}) \cap \mathrm{in}_{\prec}(J_1^{\{2\}})$$

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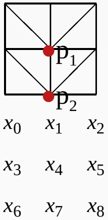

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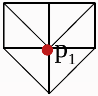

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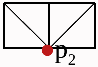

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$$\Rightarrow \qquad \mathrm{in}_{\prec}(I^{\{2\}}) = \mathrm{in}_{\prec}(I)^{\{2\}} \cap \mathrm{in}_{\prec}(J_1^{\{2\}}) \cap \mathrm{in}_{\prec}(J_1^{\{2\}})$$

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