Symmetric tensor decompositions

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homogeneous of degre d. A presentation

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Can we find $r(F)$, and if not, why?

Apolarity

Sylvester et al. introduced apolarity to find decompositions.
Let $T = \mathbb{C}[y_0, \ldots, y_n]$ act on S by differentiation:

$$y_i(F) = \frac{\partial}{\partial x_i} F.$$

Then

$$l = \sum a_i x_i \text{ and } g \in T_d \Rightarrow \; g(l^d) = \lambda g(a_0, \ldots, a_n),$$

for some $\lambda \neq 0$.

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Our key object: $\quad F^\perp = \{g \in T | g(F) = 0\} \subset T.$

The quotient $T/F^\perp$ is Artinian and *Gorenstein*.

Apolarity lemma

Let $\mathbb{P}(S_1)$ and $\mathbb{P}(T_1)$ denote the projective spaces of 1-dimensional subspaces of S_1 (resp. T_1). By apolarity,

$$\mathbb{P}(S_1) = \mathbb{P}(T_1)^* \text{ and } \Gamma \subset \mathbb{P}(S_1) \Rightarrow \mathcal{I}_\Gamma \subset T.$$

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Definition

$\Gamma \subset \mathbb{P}(S_1)$ is an *apolar subscheme* to F if $\mathcal{I}_\Gamma \subset F^\perp$.

Lemma

Let $\Gamma = \{[l_1], \ldots, [l_r]\} \subset \mathbb{P}(S_1)$, *a collection of r points. Then*

$$F = \lambda_1 l_1^d + \ldots + \lambda_r l_r^d \quad \text{with } \lambda_i \in \mathbb{C}$$

if and only if

$$\mathcal{I}_\Gamma \subset F^\perp \quad \subset T.$$

Binary forms

$$F \in \mathbb{C}[x_0, x_1] \Rightarrow F^{\perp} = (g_1, g_2) \subset \mathbb{C}[y_0, y_1] \quad \text{[Sylvester]}$$

where $\deg g_1 + \deg g_2 = \deg F + 2$.

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$$F = \lambda_1 l_1^d + \ldots + \lambda_r l_r^d \quad \Leftrightarrow \quad \mathcal{I}_{\{[l_1],\ldots,[l_r]\}} = (g) \subset (g_1, g_2).$$

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Assume $\deg g_1 \leq \deg g_2$.

$$r(F) = \begin{cases} \deg g_1 & \text{when } g_1 \text{ is squarefree} \\ \deg g_2 & \text{else} \end{cases}$$

Cactus rank

Definition

The *cactus rank* (or *length*) of F is the minimal length of a 0-dimensional apolar subscheme Γ to F, i.e.

$$cr(F) := min\{\text{length } \Gamma | \dim \Gamma = 0, \mathcal{I}_\Gamma \subset F^\perp\}.$$

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Clearly

$$cr(F) \leq r(F)$$

and the inequality may be strict: If $d > 2$,

$$(x_0 x_1^{d-1})^\perp = (y_0^2, y_1^d) \Rightarrow cr(x_0 x_1^{d-1}) = 2 < r(x_0 x_1^{d-1}) = d.$$

Border rank

Another much studied rank is the *border rank*:

$$br(F) = \min\{r|F \text{ is the limit of forms of rank r}\}$$

The border rank may equivalently be defined as the minimal r such that $[F]$ lies in the $(r-1)$-th secant variety of the d-th Veronese variety

$$V_d = \{[l^d] \in \mathbb{P}(S_d)|l \in S_1\} \subset \mathbb{P}(S_d)$$

(Recent developments: [Buczynski-Ginensky-Landsberg], [Buczynska-Buczynski],[Landsberg-Ottaviani], [Raicu])).

The border rank may a priori be smaller than the cactus rank.

Bounds on the rank

The most famous bound on the rank is not a bound

Theorem (Alexander-Hirschowitz 1995)

Let $F \in \mathbb{C}[x_0, \ldots, x_n]$ be a **general** *form of degree d, then*

- $r(F) = AH(d,n) := \lceil \frac{1}{n+1}\binom{n+d}{n} \rceil$, **except**
- $r(F) = n+1$ if $d=2$,
- $r(F) = 6,10,15$ if $d=4$, $n=2,3,4$,
- $r(F) = 8$ if $d=3$, $n=4$.

Remark

Special forms may have larger rank, a sharp upper bound is only known in a few special cases

$$((n,d) = (n,2),(1,d),(2,3),(2,4),(3,3)).$$

The simplest lower bound for the rank is explained by **differentiation**. If

$$F = l_1^d + \ldots + l_r^d \text{ and } g \in \mathbb{C}[y_0, \ldots, y_n]_s$$

then

$$g(F) = \lambda_1 l_1^{d-s} + \ldots + \lambda_r l_r^{d-s} \quad \text{for some } \lambda_1, \ldots, \lambda_r \in \mathbb{C}$$

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So, if $h(F, s)$ is the **dimension** of the vector space of partials of order s of F, then

$$r(F) \geq \max\{h(F, s) | 0 < s < d\}$$

The same lower bound is valid for the cactus rank and the border rank.

Landsberg and Teitler have given a very nice improvement of this lower bound for the rank of F, depending on the partials of F and the **singular locus** of the hypersurface $V(F)$.

Let $d(F, s)$ be the **dimension** of the locus of points on $V(F)$ of multiplicity at least s.

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Theorem (Landsberg-Teitler 2009)

Let $F \in \mathbb{C}[x_0,\dots,x_n]_d$. Assume that $V(F)$ is not a cone, and let $0 < s < d$. Then

$$r(F) \geq h(F,s) + d(F,s) + 1.$$

The proof uses apolarity in a very essential way.

Bounds on the cactus rank

For $n > 2$ and $d > 6$ the cactus rank of a general form is smaller than the rank:

Proposition

Let $F \in \mathbb{C}[x_0, \dots, x_n]_d$ be any form of degree d, then

$$cr(F) \leq N(d,n) := \begin{cases} 2\binom{n+k}{n} & \text{when } d = 2k+1 \\ \binom{n+k}{n} + \binom{n+k+1}{n} & \text{when } d = 2k+2 \end{cases}$$

Notice that $N(d,n) \approx O((d/2)^n)$, while $AH(d,n) \approx O(d^n)$.

Question

Is this bound sharp???

The proof of the Proposition uses:

Theorem (Emsalem 1978)

Let $\Gamma \subset \mathbb{C}^n$ be a local 0-dimensional scheme with $\mathcal{I}_\Gamma \subset (y_1, \ldots, y_n)$.

$$\Gamma \text{ is Gorenstein} \iff \exists f \in \mathbb{C}[x_1, \ldots, x_n] \text{ s.t. } \mathcal{I}_\Gamma = f^\perp.$$

Furthermore, in this case,

$$\text{length}\,\Gamma = \dim_{\mathbb{C}} D(f)$$

where $D(f) \subset \mathbb{C}[x_1, \ldots, x_n]$ is the space of partial derivatives of f of all orders.

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This is relevant for the cactus rank:

Lemma (Buczynska-Buczynski, Brachat et al.)

If Γ is apolar to F and $cr(F) = \text{length}\Gamma$, then every component of Γ is a local Gorenstein scheme.

Proof of Proposition

We construct a natural local Gorenstein scheme Γ_{x_0} for F supported on $[x_0] \in \mathbb{P}(S_1)$. Let $f = F(1, x_1, \ldots, x_n)$ be the dehomogenization. Then

$$f^{\perp} \subset \mathbb{C}[y_1, \ldots, y_n]$$

defines a local Gorenstein scheme $\Gamma_{x_0} \subset \mathbb{C}^n = \{y_0 \neq 0\} \subset \mathbb{P}(S_1)$, which is apolar to F and has

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Remark

For any linear form $l \in S_1$, the homogeneous ideal obtained by saturation of the degree d part of the annihilator $(l^{d+1}F)^{\perp}$ defines a local Gorenstein scheme Γ_l of length bounded above by $N(d, n)$ and supported at $[l] \in \mathbb{P}(S_1)$.

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On the other hand, one may show that $r(F) = 7$.

Bounds for monomials

Proposition (R-Schreyer 2011)

Let $1 \leq d_0 \leq d_1 \ldots \leq d_n$, *then*

$$cr(x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n}) = (d_0 + 1) \cdots (d_{n-1} + 1)$$

and

$$r(x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n}) \leq (d_1 + 1) \cdots (d_n + 1).$$

In particular

$$cr((x_0 x_1 \cdots x_n)^d) = r((x_0 x_1 \cdots x_n)^d) = (d+1)^n.$$

Proof of Proposition

Lemma

Let $F \in \mathbb{C}[x_0, \ldots, x_n]_d$ be a form whose annihilator ideal is generated in degree δ. Then

$$cr(F) \geq \frac{1}{\delta} \dim_{\mathbb{C}} T/F^{\perp}.$$

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The annihilator ideal

$$F^{\perp} = (x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n})^{\perp} = (y_0^{d_0+1}, \ldots, y_n^{d_n+1})$$

is a *complete intersection*, so

$$\dim_{\mathbb{C}} T/F^{\perp} = (d_0 + 1) \cdots (d_n + 1)$$

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and the bound for $cr(F)$ follows from the lemma.
For the second part, choose general forms $g_1, \ldots, g_n \subset F^{\perp}$, with $\deg g_i = d_i + 1$, then, by Bertini, $V(g_1, \ldots, g_n)$ is smooth of degree as stated.

In how many ways?

Given r, the set of Waring decompositions

$$\{\{[l_1],\ldots,[l_r]\} | F = l_1^d + \ldots + l_r^d\} \subset \mathrm{Hilb}_r\mathbb{P}(S_1)$$

is a subscheme of the Hilbert scheme. Its closure is called the V(ariety) of S(ums) of P(owers), and denoted $VSP(F, r)$.

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For some F and r this variety is a single point:

Theorem (Sylvester 1851, Chiantini-Ciliberto, Mella, Ballico 2002-2005)

For general $F \in \mathbb{C}[x_0, \dots, x_n]_d$ of rank r **smaller** *than the generic one, the decomposition is unique, except if*

$$(n, d, r) = (n, 2, r), (2, 4, 5), (3, 4, 9), (4, 4, 14), (4, 3, 7)$$

where there are infinitely many decompositions, or

$$(n, d, r) = (2, 6, 9), (3, 4, 8)$$

where there are two decompositions.

For a general F and $r = r(F)$,

$$\dim VSP(F, r) \underset{((n,d) \neq (n,2),(4,3),(2,4),(3,4),(4,4))}{=} AH(n, d)(n + 1) - \binom{n+d}{n}.$$

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Theorem (Sylvester, Hilbert, Palatini, Richmond, 1851-1902, Mukai, Dolgachev-Kanev, Schreyer, Iliev, R 1989-2000)

If F is general and $r = r(F)$, then $VSP(F,r)$ is

- *a point if* $(n,d) = (1, 2r-1), (2,5), (3,3)$,
- $\mathbb{P}^1$ *if* $(n,d) = (1, 2r-2)$,
- $\mathbb{P}^2$ *when* $(n,d) = (2,3)$,
- *a K3 surface when* $(n,d) = (2,6)$,
- *a Fano threefold when* $(n,d) = (2,4)$,
- *a Fano fivefold when* $(n,d) = (4,3)$,
- *a Hyperkähler fourfold when* $(n,d) = (5,3)$.

Various methods have been developed, building on apolarity, to find Waring decompositions of F.

(cf. [Brachat et al.] and [Oeding-Ottaviani])

For small n and d or when $r(F) << AH(n,d)$, then these methods are effective.

In computations, one normally works over $\mathbb{Q}$ or a finite field. For general F, the **first obstruction** is therefore to find a point on the variety $VSP(F,r)$ with the additional property that each l is defined over the ground field.

Question

$VSP(x_0^3x_1 + x_1^3x_2 + x_2^3x_0, 6)$ *is Fano threefold. Does there exist* $l_1, \ldots, l_6 \in \mathbb{Q}[x_0, x_1, x_2]$ *and rational numbers* $\lambda_1, \ldots, \lambda_6$ *such that*

$$x_0^3x_1 + x_1^3x_2 + x_2^3x_0 = \lambda_1 l_1^4 + \ldots + \lambda_6 l_6^4?$$

VSP and VAPS

$VSP(F,r)$ is a natural subscheme of

$$VAPS(F,r) = \{\Gamma \subset \mathbb{P}(S_1)) | \mathcal{I}_\Gamma \subset F^\perp\} \subset \mathrm{Hilb}_r \mathbb{P}(S_1).$$

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The difference $VASP(F,r) \setminus VSP(F,r)$ is a **second obstruction** to finding a decomposition of F.

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$VSP(F,r) \subset VAPS(F,r)$ is the closure of the set of smooth apolar subschemes.
In general $VPS(F,r)$ is a proper subscheme of $VAPS(F,r)$, in particular when $r(F) > N(n,d) \geq cr(F)$.
The difference $VASP(F,r) \setminus VSP(F,r)$ is a **second obstruction** to finding a decomposition of F.

Even when F is a quadric of rank n,

$$VAPS(F,n) \setminus VSP(F,n) \neq \emptyset \text{ when } n >> 0$$

Quadrics

Let $Q \in \mathbb{C}[x_0, \ldots, x_n]$ be a quadric of maximal rank $n+1$. If

$$Q = l_0^2 + \ldots + l_n^2, \text{ then } \{l_0 \cdots l_n = 0\} \subset \mathbb{P}(T_1)$$

is isomorphic to the standard coordinate simplex. Furthermore, each hyperplane $\{l_i = 0\}$ is the polar of the intersection point of the remaining ones with respect to the quadric. It is classically known as a **polar simplex**.

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Theorem (R-Schreyer 2011)

Let $Q \in \mathbb{C}[x_0, \ldots, x_n]$ be a quadric of rank $n+1$.

- *$VSP(Q, n+1)$ is rational variety of dimension $\binom{n+1}{2}$.*
- *It is a smooth Fano variety of index 2 with Picard group isomorphic to $\mathbb{Z}$ if $n < 5$.*
- *$VSP(Q, n+1)$ is singular if $n \geq 5$.*

$$[\Gamma] \in VAPS(Q, n+1) \Rightarrow \dim_{\mathbb{C}} (\mathcal{I}_\Gamma)_2 = \binom{n+1}{2}$$

while $\dim_{\mathbb{C}}(Q^\perp)_2 = \binom{n+2}{2} - 1$. Therefore

$$VAPS(F, n+1) \hookrightarrow \mathbb{G}(\binom{n+1}{2}, \binom{n+2}{2} - 1) = \mathbb{G}(n, \binom{n+2}{2} - 1).$$

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Similarly, the Gauss map of

$$v_2(Q) \subset \mathbb{P}^{\binom{n+2}{2}-2}$$

under the Veronese embedding, maps

$$v_2(Q) \hookrightarrow \mathbb{G}(n, \binom{n+2}{2} - 1).$$

Denote the image by $\mathrm{Gauss}_{v_2}(Q)$.

Theorem (R-Schreyer 2011)

Let $Q \in \mathbb{C}[x_0, \ldots, x_n]$ *be a quadric of rank* $n+1$*, then there are natural inclusions*

$$\mathrm{Gauss}_{v_2}(Q) \subset VSP(Q, n+1) \subset VAPS(Q, n+1) \subset \mathbb{G}(n, \binom{n+2}{2} - 1).$$

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$$\subset \mathbb{G}(n, \binom{n+2}{2} - 1).$$

Furthermore, when $n \geq 23$, $VAPS(Q, n+1)$ is reducible and

$$VAPS(Q, n+1) = < \mathrm{Gauss}_{v_2}(Q) > \ \cap \mathbb{G}(n, \binom{n+2}{2} - 1)$$

in the Plücker embedding.

Thank You!

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