Computing tropical linear spaces

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Definition

The field $\mathbb{C}\{\{t\}\}$ of Puiseux series on the variable t is the set of formal series of the form

$$f = c_0 t^{r_0} + c_1 t^{r_1} + c_2 t^{r_2} + \cdots ,$$

where $c_i \in \mathbb{C}$, $c_0 \neq 0$, and $r_0 < r_1 < r_2 < \cdots$ are rational numbers with a common denominator.

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The field of Puiseux series is algebraically closed, and it comes equipped with a valuation map val : $\mathbb{C}\{\{t\}\} - 0 \to \mathbb{Q}$ defined as $\text{val}(f) = r_0$.

Tropicalizing varieties

Suppose $I \subseteq \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is an ideal and $V \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$ its corresponding variety. The tropicalization of V is the set

$$\mathcal{T}(V) := \{(\mathrm{val}(f_1), \ldots, \mathrm{val}(f_n)) \mid (f_1, \ldots, f_n) \in V\} \subseteq \mathbb{Q}^n.$$

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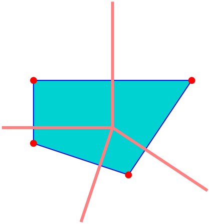
If V is the hypersurface defined by some polynomial p then $\mathcal{T}(V)$ is the (negative of the) codimension 1 skeleton of the normal fan of the Newton polytope of p.



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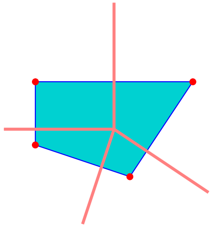

One can recover the Newton polytope of p from $\mathcal{T}(V)$ using a ray shooting algorithm.

Tropical linear spaces

Suppose L is the rowspace over $\mathbb{C}\{\{t\}\}$ of an $m \times n$ integer matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition

$C \subseteq \{1, \ldots, n\}$ is a circuit of A $\iff$ $\{\mathbf{a}_i \mid i \in C\}$ is minimally dependent.

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Theorem

The tropicalization $\mathcal{T}(L)$ depends only on the set of circuits of A (its matroid). In fact,

$$w \in \mathcal{T}(L) \iff \textit{for any circuit } C, \; \min(w_i \mid i \in C) \textit{ is attained at least twice.}$$

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Why should we be interested in computing tropical linear spaces?

A-discriminants

Suppose A is an $m \times n$ integer matrix. The columns of A give rise to Laurent monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n}$ in the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Let $\mathbb{C}^A$ be the space of all Laurent polynomials of the form $p(\mathbf{x}) = \sum_{i=1}^{n} c_i \cdot \mathbf{x}^{\mathbf{a}_i}$.

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The discriminantal variety ∇_A is the Zariski closure of the set of polynomials p in $\mathbb{C}^A$ for which there exists a $\mathbf{z} \in (\mathbb{C}^*)^m$ satisfying

$$p(\mathbf{z}) = \frac{\partial p}{\partial x_i}(\mathbf{z}) = 0 \quad \text{for all } i = 1, \ldots, m.$$

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Example

Take $A = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$, so $\mathbb{C}^A = \{p = a \cdot x^2 + b \cdot x + c \mid a, b, c \in \mathbb{C}\}$. Then $\nabla_A = \{p \in \mathbb{C}^A \mid b^2 - 4ac = 0\}$.

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If ∇_A has codimension 1 then its defining polynomial Δ_A is called the *A*-discriminant.

An example

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$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

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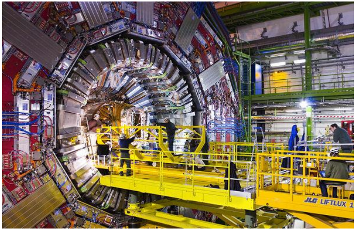

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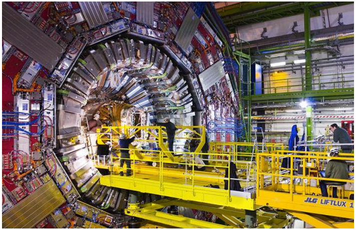

So discriminants are interesting and useful, but also large and hard to compute.

The tropical approach

Theorem (Horn uniformization)

Let T be the image of the monomial map

$$\begin{array}{rcl} (\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^n \\ \boldsymbol{t} & \longmapsto & (\boldsymbol{t}^{\boldsymbol{a}_1}, \ldots, \boldsymbol{t}^{\boldsymbol{a}_n}). \end{array}$$

The discriminantal variety ∇_A *is the Zariski closure of the coordinate-wise product between* $\ker(A)$ *and* T*, i.e.,*

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The tropical discriminantal variety can be described as

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This can be used to compute $\mathcal{T}(\nabla_A)$, if we can effectively compute $\mathcal{T}(\ker(A))$.

A local criterion

Suppose A is an $m \times n$ integer matrix of rank m with columns labeled by the set $\{1, \ldots, n\}$, and let $L = \text{rowspace}(A)$. Suppose the columns in $B \subseteq \{1, \ldots, n\}$ form a basis for $\mathbb{R}^m$.

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Theorem (Feichtner-Sturmfels)

Suppose $w \in \Sigma_B$. Then w is in $\mathcal{T}(L)$ if and only if $\min\{w_i \mid i \in C\}$ is attained twice for any fundamental circuit C over the basis B.

A nice fan structure

A generic vector $w \in \Sigma_B \cap \mathcal{T}(L)$ induces a function $p : B^c \to B$ in the following way:

1. Define the total order J on B as $a <_J b \iff w_a < w_b$.
2. For any $k \notin B$ define $p(k) =$ "J-smallest element of $C(k, B) - \{k\}$".

Note that $\min\{w_i \mid i \in C(k, B)\} = w_k = w_{p(k)}$, so p encodes which coordinates attain the minimum in each fundamental circuit.

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We will say that the pair (p, L) is the compatible pair induced by w.

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Proposition

Let (p, L) be a compatible pair. The set of vectors $v \in \Sigma_B \cap \mathcal{T}(L)$ that induce (p, L) is an m-dimensional polyhedral cone $\Gamma(p, L)$ in $\mathbb{R}^n$ whose extremal rays can all be taken to be $0/1$ vectors.

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Proof by example.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$C(5, B) = \{2, 4, 5\}, \quad C(6, B) = \{1, 2, 4, 6\}, \quad C(7, B) = \{1, 2, 3, 7\}$$

$$p: \quad 5 \mapsto 4, \quad 6 \mapsto 1, \quad 7 \mapsto 1 \qquad\qquad L: \quad 1 <_L 4$$

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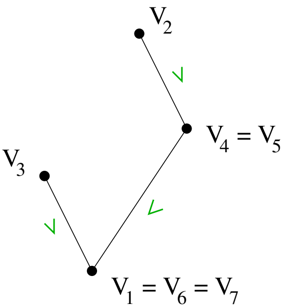
A vector v induces this compatible pair iff it satisfies $\longrightarrow$



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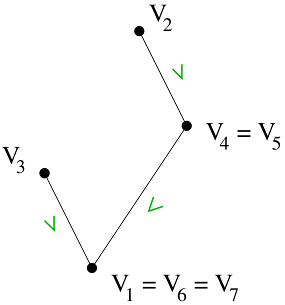

The cone $\Gamma(p, L)$ is generated by the rays

$e_2, \quad e_2 + e_4 + e_5, \quad e_3, \quad \pm(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7).$

□

Computing $\mathcal{T}(L)$

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Key Fact

There is an effective algorithm for computing compatible pairs, which builds up both p and L at the same time.

C++ implementations

This algorithm for computing $\mathcal{T}(L)$ has been implemented in C++.

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Example

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & -2 & -1 & 0 & -3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -2 & 0 & -1 & -2 & -3 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A^{\perp} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 & -6 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -1 \end{pmatrix}$$

The matrix $A^{\perp}$ has size 9×13. A Maple implementation for computing tropical linear spaces locally using their nested fan structure (already much faster than Gfan) takes almost 19 hours to compute $\mathcal{T}(L^{\perp})$.

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This algorithm for computing $\mathcal{T}(L)$ has been implemented in C++.

Example

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & -2 & -1 & 0 & -3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -2 & 0 & -1 & -2 & -3 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A^\perp = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 & -6 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -1 \end{pmatrix}$$

The matrix $A^\perp$ has size 9×13. A Maple implementation for computing tropical linear spaces locally using their nested fan structure (already much faster than Gfan) takes almost 19 hours to compute $\mathcal{T}(L^\perp)$.

Our code takes 1 second!

C++ implementations

Example

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The matrix $A^\perp$ has size 15×20.

C++ implementations

Example

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The matrix $A^{\perp}$ has size 15×20.

Our code computes $\mathcal{T}(L^{\perp})$ as fan with 172 rays and 475 722 maximal cones. All the computation takes just a little more than 60 seconds!

C++ implementations

Example

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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A C++ implementation for computing vertices of the Newton polytope of an A-discriminant is also available.

C++ implementations

Example

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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Thank you!

Ooops...