Outline

1. Introduction
   - Definition of multiplier ideals
   - Motivation

2. Resolution of singularities
   - Overview
   - First package

3. Monomial ideals
   - Second package
   - A question of Lazarsfeld
What is a multiplier ideal?

Ideal sheaf associated to a singularity.

- Captures subtle information about singularity
- Useful in geometry, especially birational geometry
- Defined in terms of resolution of singularities
- Usually hard to compute
Definition

Given ideal $I$ on $X = \mathbb{C}^n$, real number $t \geq 0$

- $f : Y \rightarrow X$ resolution of singularities
- $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$ total transform
- $K = K_{Y/X} = V(\det df)$ relative canonical divisor
- The $t$'th multiplier ideal is $\mathcal{J}(I^t) = f_*\mathcal{O}_Y(K - \lfloor tF \rfloor)$
Example

$I = (y^2 - x^3)$. Resolution:

\[
K = E_1 + 2E_2 + 4E_3, \quad F = 2E_1 + 3E_2 + 6E_3 + C'
\]

\[
\mathcal{J}(I^{5/6}) = f_*\mathcal{O}_Y((1 - \lfloor \frac{5}{6}2\rfloor)E_1 + (2 - \lfloor \frac{5}{6}3\rfloor)E_2 + (4 - \lfloor \frac{5}{6}6\rfloor)E_3)
\]
\[
= f_*\mathcal{O}_Y((1 - 1)E_1 + (2 - 2)E_2 + (4 - 5)E_3)
\]
\[
= f_*\mathcal{O}_Y(-E_3)
\]
\[
= (x, y)
\]
Why study multiplier ideals?

- Deformation invariance of plurigenera
- Existence of flips
- Asymptotic base loci of linear series
- Chow stability criterion
- Uniform results in commutative algebra
Why compute multiplier ideals?

- Study jumping numbers: values of $t$ at which $\mathcal{J}(I^t)$ changes.
  E.g., diagonal arrangement in $(\mathbb{C}^n)^\otimes k$

- Characterize essential exceptional divisors:
  which components of $K - \lfloor tF \rfloor$ are needed?

- Conjectural strong bounds on symbolic powers:
  strengthen results of the form $I^{(m)} \subseteq \mathcal{J} \subseteq I^r$
Computing multiplier ideals

An existing implementation:

- Dmodules (Berkesch, Leykin)

Alternative approach: via resolution of singularities.

- desing.lib by G. Bodnár and J. Schicho
- resolve.lib by A. Frühbis–Krüger and G. Pfister
Outline of computation

1. Find resolution of singularities.
2. In each chart,
   1. find equations for exceptional divisors $E_i$,
   2. write total transform $F = \sum a_i E_i$,
   3. write relative canonical divisor $K = \sum b_i E_i$,
   4. form divisor $K - \lfloor tF \rfloor$,
   5. push this ideal down to the root chart.
3. Intersect these pushdowns.
First package: multiplierideals.lib

Work in progress, but partially working:

> LIB "multiplierideals.lib";
> multiplieridealinit();
> ring R = 0,(x(1..2)),dp;
> ideal I = x(1)^3,x(2)^2;
> desing(I); numericaldataallcharts(I);
> lct(I);  
5/6
> multiplierideal(I,5/6);
  \[1\]=x(2)
  \[2\]=x(1)
\(J(I^{5/6}) = (x_1, x_2)\)
> multiplierideal(I,11/6);
  \[1\]=x(2)^3
  \[2\]=x(1)*x(2)^2
  \[3\]=x(1)^2*x(2)
  \[4\]=x(1)^4
\(J(I^{11/6}) = (x_1^4, x_1^2x_2, x_1x_2^2, x_2^3)\)
Second package: MonomialMultiplierIdeals.m2

- Howald’s theorem: multiplier ideal of monomial ideal
- Combinatorial description via Newton polyhedron
- $\mathcal{J}(I^t)$ is the monomial ideal

$$x^v \in \mathcal{J}(I^t) \iff v + (1, \ldots, 1) \in \text{Int}(t \cdot \text{Newt}(I))$$

- Use Normaliz (Bruns et al) to manipulate Newton polyhedron
- MonomialMultiplierIdeals package for Macaulay2 now available.
A question of Lazarsfeld

What do multiplier ideals of $\text{gin}(I)$ say about $V(I)$?

Example: $I =$ ideal of curve parametrized by $(t^a, t^b, t^c)$
- Ideal of curve $(t^4, t^5, t^{11})$ is $I = (y^3 - xz, x^4 - yz, x^3y^2 - z^2)$
- $J = \text{gin}(I) = (x^3, x^2y^2, xy^4, y^5)$
- Can find log canonical threshold $\text{lct}(J) = 8/15$
Example

i1 : loadPackage "MonomialMultiplierIdeals";
   loadPackage "GenericInitialIdeal";

i2 : S = QQ[t]; R = QQ[x,y,z];

i3 : curveIdeal = (a,b,c) -> ker map(S,R,{t^a,t^b,t^c});
   -- ideal of curve parametrized by (t^a,t^b,t^c)

i4 : curveGinLCT = (a,b,c) ->
   monomialLCT monomialIdeal gin curveIdeal(a,b,c);

i5 : (5..10) / (i -> curveGinLCT(4,i,11)) / toString
   o5 = (8/15, 8/15, 9/14, 2/3, 17/30, 6/11)
Outline of computation

To compute

\[ x^v \in J(I^t) \iff v + (1, \ldots, 1) \in \text{Int}(t \cdot \text{Newt}(I)) \]

1. Find defining inequalities for \( \text{Newt}(I) \), \( Av \geq b \).
2. Scale by \( t = p/q \), yielding \( qAv \geq pb \).
3. Add 1 to entries of \( pb \) corresponding to non-coordinate faces \( \sim \tilde{p}b \).
4. Find monomial ideal generated by lattice points in \( qAv \geq \tilde{p}b \).
5. Compute ideal quotient by \( x^{(1, \ldots, 1)} = x_1 \cdots x_n \).
Thank you!