Ideals of Curves given by Points

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Outline of talk:

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- Degree Bounds

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- Complements and Border Bases

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Corollary

If C is an irreducible projective curve of degree d which can be generated in degree at most s and R is a set of more than sd points on C, then $I(R)_{\leq s}$ generates $I(C)$.

Degrees of generators of General curves

Theorem (Gruson-Lazarsfeld-Peskine)

Let $C \subset \mathbb{P}^n(K)$ be a non-degenerate irreducible curve with $n \geq 3$ then $I(C)$ can be generated in degree at most $d - n + 2$. If C has genus $g > 1$ then $I(C)$ can be generated in degree at most $d - n + 1$.

Canonical curves

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Remark

Any non-degenerate curve of genus g and degree $2g - 2$ embedded in $\mathbb{P}^{g-1}$ is a smooth non-hyperelliptic canonical curve

Curves of high degree

Theorem (Saint-Donat)

Let $C \subset \mathbb{P}^n(K)$ be a non-degenerate irreducible curve of genus g and degree d with $n = d - g$ then:

- *If $d = 2g + 1$ then $I(C)$ can be generated in degrees 2 and 3.*
- *If $d \geq 2g + 2$ then $I(C)$ can be generated in degree 2.*

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Remark

The condition $n = d - g$ implies the curve is embedded by a complete linear series

Complements for zero dimensional ideals

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Proposition

The ideal I *is zero dimensional if and only if it has a finite complement.*

Complement and Border bases for zero dimensional ideals

Definition

If N is a complement for an ideal, let $N^+ := \bigcup_{j=1}^{n}(x_j N)$ and let $\partial N = N^+ \setminus N$ be called the border monomials of N.

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The finite set of border polynomials associated to the complement of a zero dimensional ideal are called a border basis.

Complements of homogeneous ideals

Remark

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Proposition

If $P_{k-1} = I_{k-1} \oplus < N_{k-1} >$ *and we choose a set of monomials* N_k *satisfying* $< N_{k-1}^{+} >= (I_k \cap < N_{k-1}^{+} >) \oplus < N_k >$ *then* $P_k = I_k \oplus < N_k >$ *and so* N_k *is a valid complement in degree* k.

Proposition

Let J be a proper homogeneous ideal in $\mathcal{P}$. Assume that $\mathcal{N} = \{\mathcal{N}_0, \ldots, \mathcal{N}_s\}$ is a complement of J up to degree s and let $\mathcal{B} = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_s$ be the associated border basis. Then:

1. *The ideal $L = I(\mathcal{B}, \mathcal{N}_s^+)$ is homogeneous and zero-dimensional, and $\mathcal{B} \cup \mathcal{N}_s^+$ is the border basis of L associated to its complement $\mathcal{N}$.*
2. *The module $Syz(\mathcal{B}, \mathcal{N}_s^+)$ is generated by vectors whose entries have degree at most 1.*

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Proposition

If B_k *denotes the degree* k *components of a border basis for a homogeneous ideal whose complement is connected to 1, then a generator in* B_k *is redundant if and only if there is a linear relation between elements of* B_{k-1}^+ *and* B_k.

Complexity of reduction to minimal generators

Proposition

If N_k is the degree k component of the complement of a homogeneous ideal of s points in $\mathbb{P}^n$, then $|N_k| \leq s$. If B_k is the degree k component of its border bases then $|B_k| \leq (n+1)s$

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Corollary

The complexity of computing a minimal set of generators for an ideal of s points in $\mathbb{P}^n$ up to degree k is bounded by $O(kn^4s^3)$.

Complements and generators for point ideals

- Let N_{k-1} be a complement for the ideal I of a finite point set $(R_1, \ldots, R_h)$ in $\mathbb{P}^n$
- we need to compute N_k satisfying $< N_{k-1}^+ >= (I_k \cap < N_{k-1}^+ >) \oplus < N_k >$.

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- The null space NM of M corresponds to $I_k \cap < N_{k-1}^+ >$
- With NM in reduced echelon form:
  - the non-pivot columns correspond to a complement N_k
  - the pivots correspond to border monomials
  - the null space generators correspond to border polynomials.
  - rank $M = \dim N_k$.

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6. the monomials corresponding to the columns of R_2 will be our complement

Rank of point evaluation matrix for projectively normal curves

Remark

rank(M_R) = *dim*(N_k) = *dim*(P_k) - *dim*(I_k).

Proposition

If C *is projectively normal of genus* g *and* $k \deg(C) \geq 2g - 1$ *then* $\dim N_k = k \deg(C) - g + 1$

Corollary

If C is a canonical curve of genus $g \geq 4$ then:

$$\dim N_2 = 2(2g-2) - g + 1 = 3g - 3$$

$$\dim N_3 = 3(2g-2) - g + 1 = 5g - 5$$

Corollary

If C is a bicanoncal curve of genus $g \geq 3$ then:

$$\dim N_2 = 2(4g-4) - g + 1 = 7g - 7$$

Parametric space curve example

We implemented our algorithm in Octave and tested it on the following parametric sextic space curve in $\mathbb{P}^3(\mathbb{C})$:

$$\begin{aligned}
x &= 3s^4t^2 - 9s^3t^3 - 3s^2t^4 + 12st^5 + 6t^6 \\
y &= -3s^6 + 18s^5t - 27s^4t^2 - 12s^3t^3 + 33s^2t^4 + 6st^5 - 6t^6 \\
z &= s^6 - 6s^5t + 13s^4t^2 - 16s^3t^3 + 9s^2t^4 + 14st^5 - 6t^6 \\
w &= -2s^4t^2 + 8s^3t^3 - 14s^2t^4 + 20st^5 - 6t^6.
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We chose $31 > 6 \cdot 5$ points using roots of unity for the parameters.

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- We found 11 border polynomials of degree 4
- We found 22 border polynomials of degree 5 (total time for generators .08 sec).
- The 4 border polynomials of degree 3 are a minimum set of generators (time for minimal generators .02 sec).

Approximate minimal generators for parametric curve

$$
\begin{aligned}
f_1 = \;& zyx + 0.066666666\, z^2y + z^2x + 0.068055555\, zy^2 - 0.0361111 \\
& -0.2833333\, zyw - 0.55\, zx^2 - 1.066666\, zxw + 0.01527777\, y^3 \\
& -0.091666666\, y^2x + 0.3055555\, yx^2 + 0.1833333\, x^2w, \\
f_2 = \;& yxw + 0.2\, z^2y + 0.1416666\, zy^2 - 0.4833333\, zyx - 0.1\, zyw \\
& -0.9\, zx^2 - 0.20000001\, zxw + 0.025\, y^3 - 0.15\, y^2x + 0.5\, yx^2 \\
& +0.30000001\, x^2w, \\
f_3 = \;& y^2w - 0.8\, z^2y - 0.3166666\, zy^2 - 0.5666666\, zyx + 0.4\, zyw \\
& +0.6000001\, zx^2 + 0.8000002\, zxw - 0.016666666\, y^3 \\
& +0.099999999\, y^2x - 0.3333333\, yx^2 - 0.2000002\, x^2w, \\
f_4 = \;& z^2w - 0.6666667\, z^3 - 0.162962\, z^2y + 0.060493827\, zy^2 \\
& -0.032098765\, zyx + 0.9703703\, zyw - 0.4888888\, zx^2 \\
& +0.3851851\, zxw - 0.1666667\, zw^2 + 0.013580246\, y^3 \\
& -0.081481481\, y^2x + 0.27160493\, yx^2 - 0.9444444\, yw^2 \\
& +0.1629629\, x^2w - 0.2222224\, xw^2
\end{aligned}
$$

Exact minimal generators for parametric space curve

The floating point coefficients were sufficiently accurate to allow us to recover the exact rational coefficients using continued fractions yielding minimal generators of the exact ideal of the curve over $\mathbb{Q}$.

$$f_1 = z^2x + \frac{1}{15}\, z^2y + \frac{49}{720}\, zy^2 + \frac{11}{720}\, y^3 - \frac{13}{360}\, zyx - \frac{11}{120}\, y^2x - \frac{11}{20}\, zx^2 + \frac{11}{36}\, yx^2 - \frac{17}{60}\, zyw - \frac{16}{15}\, zxw + \frac{11}{60}\, x^2w,$$

$$f_2 = yxw + \frac{1}{5}\, z^2y + \frac{17}{120}\, zy^2 + \frac{1}{40}\, y^3 - \frac{29}{60}\, zyx - \frac{3}{20}\, y^2x - \frac{9}{10}\, zx^2 \; \frac{1}{2}\, yx^2 - \frac{1}{10}\, zyw - \frac{1}{5}\, zxw + yxw + \frac{3}{10}\, x^2w,$$

$$f_3 = y^2w - \frac{4}{5}\, z^2y - \frac{19}{60}\, zy^2 - \frac{1}{60}\, y^3 - \frac{17}{30}\, zyx + \frac{1}{10}\, y^2x + \frac{3}{5}\, zx^2 - \frac{1}{3}\, yx^2 + \frac{2}{5}\, zyw + \frac{4}{5}\, zxw - \frac{1}{5}\, x^2w,$$

$$f_4 = z^2w - \frac{2}{3}\, z^3 - \frac{22}{135}\, z^2y + \frac{49}{810}\, zy^2 + \frac{11}{810}\, y^3 - \frac{13}{405}\, zyx - \frac{11}{135}\, y^2x - \frac{22}{45}\, zx^2 + \frac{22}{81}\, yx^2 + \frac{131}{135}\, zyw + \frac{52}{135}\, zxw + \frac{22}{135}\, x^2w - \frac{1}{6}\, zw^2 - \frac{17}{18}\, yw^2 - \frac{2}{9}\, xw^2.$$