Decompositions of binomial ideals in Macaulay 2

Thomas Kahle

EPDI @ Institut-Mittag-Leffler
Djursholm, Sweden

June 1st, 2011

Binomial ideals in Macaulay 2

Binomials.m2 is a package for binomial ideals

An ideal $I \subseteq \Bbbk[x_1, \dots, x_n] =: \Bbbk[\mathbf{x}]$ is a binomial ideal if it is generated by binomials

$$\mathbf{x}^u - \lambda \mathbf{x}^v, \quad \lambda \in \Bbbk$$

Effective methods for binomial ideals

- needed in applications
  - Conditional independence ideals in algebraic statistics
  - Binomial D-modules
  - Combinatorial game theory
- Binomial ideals are sufficiently combinatorial to save some of the theory of monomial ideals.
- General idea for computational speed-ups

Use binomial-friendly operations!

Computations with binomial ideals

Binomial friendly

- Gröbner bases are binomial. (S-pairs are binomial friendly)
- If $I \subseteq \mathbb{k}[\mathbf{x}]$ is binomial, and $m \in \mathbb{k}[\mathbf{x}]$ a monomial, then

$$(I : m) = \langle f \in \mathbb{k}[\mathbf{x}] : mf \in I \rangle \quad \text{is binomial.}$$

If $\mathbb{k}$ is algebraically closed:

- Associated primes and the radical are binomial.
- Primary components can be chosen binomial.

Not binomial friendly

- $(I : M)$ where M is a monomial ideal.
- $(I : b)$ where b is a binomial.
- $I_1 \cap I_2$ for binomial ideals I_1, I_2.

"Binomial Ideals" (Eisenbud / Sturmfels, 1996).

Binomial primary decomposition - Algorithm

Algorithm

Input: A pure difference binomial ideal $I \subseteq \mathbb{Q}[\mathbf{x}]$.

Output: A binomial primary decomposition.

1. Compute a decomposition into cellular binomial ideals.
2. For each cellular component:
    1. Determine the binomial associated primes.
    2. For each associated prime, compute the primary component.
3. Remove redundant components.

Note: Computations are done over cyclotomic fields: $S = \mathbb{Q}(\xi_l)[\mathbf{x}]$.

Binomial primary decomposition - Algorithm

Algorithm

Input: A pure difference binomial ideal $I \subseteq \mathbb{Q}[\mathbf{x}]$.

Output: A binomial primary decomposition.

1. Compute a decomposition into cellular binomial ideals.
2. For each cellular component:
    1. Determine the binomial associated primes.
    2. For each associated prime, compute the primary component.
3. Remove redundant components.

Note: Computations are done over cyclotomic fields: $S = \mathbb{Q}(\xi_l)[\mathbf{x}]$.

Cellular Ideals

Definition

- A binomial ideal $I \subseteq S$ is cellular if in S/I every *variable* is either a nonzerodivisor or nilpotent.

Example

Primary binomial ideals are cellular.

Representations of cellular binomial ideals

- $\mathcal{E}$ - set of variables that are nonzerodivisors.

A cellular ideal contains a power of $\mathfrak{m}_{\mathcal{E}} := \langle x_i : i \notin \mathcal{E} \rangle$:

$$I = \left(I + \mathfrak{m}_{\mathcal{E}}^d\right) : \left(\prod_{i \in \mathcal{E}} x_i\right)^{\infty}$$

Cellular decomposition

Algorithm (Ojeda/Sanchez)

1. If I is not cellular, choose a variable which is a zerodivisor but not nilpotent modulo I.
2. Determine the power s such that $(I : x_i^s) = (I : x_i^\infty)$.
3. Iterate with $(I : x_i^s)$ and $I + \langle x_i^s \rangle$.

Lemma

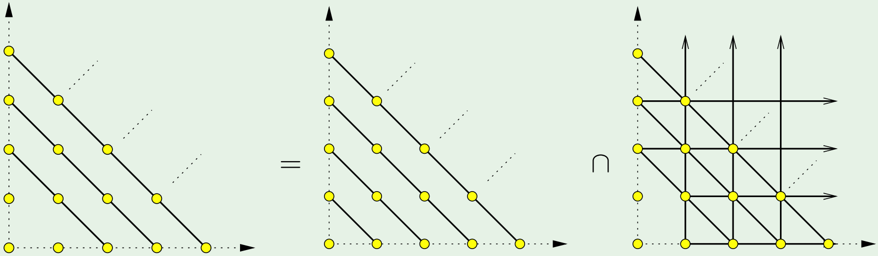
Let I be an ideal in a Noetherian ring S and $g \in S$ such that $(I : g) = (I : g^\infty)$. Then

1. $I = (I : g) \cap (I + \langle g \rangle)$.
2. $\mathrm{Ass}(S/(I : g)) \cap \mathrm{Ass}(S/(I + \langle g \rangle)) = \emptyset$.
3. *A minimal primary decomposition of I consists of the primary components of $(I : g)$ and those primary components of $I + \langle g \rangle$ that correspond to associated primes of I.*

Cellular Decompositon

Celullar decomposition

$$I = \langle x^2 - xy, xy - y^2 \rangle = \langle x - y \rangle \cap \langle x, y^2 \rangle$$



$$\langle x^2 - xy, xy - y^2 \rangle \qquad = \qquad (I : x) = \langle x - y \rangle \qquad \cap \qquad I + \langle x \rangle = \langle x, y^2 \rangle$$

Binomial primary decomposition - Algorithm

Algorithm

Input: A pure difference binomial ideal $I \subseteq \mathbb{Q}[\mathbf{x}]$.

Output: A binomial primary decomposition.

1. Compute a decomposition into cellular binomial ideals.
2. For each cellular component:
   1. Determine the binomial associated primes.
   2. For each associated prime, compute the primary component.
3. Remove redundant components.

Note: Computations are done over cyclotomic fields: $S = \mathbb{Q}(\xi_l)[\mathbf{x}]$.

Associated Primes of cellular binomial ideals

Theorem (Associated primes of cellular ideals [ES'96])

Let $I \subseteq \Bbbk[\mathbf{x}]$ be an $\mathcal{E}$-cellular binomial ideal. The associated primes of I are exactly the associated primes of the lattice ideals

$$(I : m) + \mathfrak{m}_{\mathcal{E}}$$

where m ranges over the nilpotent monomials.

Remarks

- There are finitely many ideals $(I : m)$ to be inspected.
- $(I : m)$ is a lattice ideal in the $\mathcal{E}$-variables.

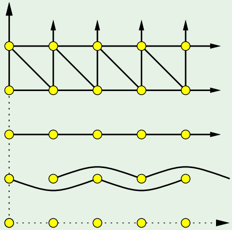
Algorithmic problem

- There is a monotonous poset map from monomials to lattice ideals.
- Problem: Efficient search for "witnesses"

Associated (Meso)primes of cellular ideals

$I = \langle y(x^2-1), y^2(x-1), y^3 \rangle$

- $(I : m)$ is I "shifted down" by m.
- e.g. $(I : y) \ni (x^2 - 1)$



Associated primes of I:

$\langle y \rangle$	1
$\langle y, x \pm 1 \rangle$	y
$\langle y, x - 1 \rangle$	y^2

Remarks

- Associated primes of lattice ideals are easy.
  - Coefficients appear only in this step.
- Mesoprimary decomposition
  - Works directly with the lattice ideals
  - Decompose a binomial ideal optimally over the original field.

Binomial primary decomposition - Algorithm

Algorithm

Input: A pure difference binomial ideal $I \subseteq \mathbb{Q}[\mathbf{x}]$.

Output: A binomial primary decomposition.

1. Compute a decomposition into cellular binomial ideals.
2. For each cellular component:
   1. Determine the binomial associated primes.
   2. For each associated prime, compute the primary component.
3. Remove redundant components.

Note: Computations are done over cyclotomic fields: $S = \mathbb{Q}(\xi_l)[\mathbf{x}]$.

Minimal primary components

Theorem [ES'96]

- A prime binomial ideal is of the form $P = P^{(b)} + \mathfrak{m}_{\mathcal{E}}$ for some $\mathcal{E} \subseteq [n]$ and $P^{(b)}$ a saturated lattice ideal in $\Bbbk[x_i : i \in \mathcal{E}]$.
- A primary binomial ideal contains $P^{(b)}$ of its associated prime.

Theorem (minimal primary component) [ES96], [DMM09]

Let $I \subseteq \mathbb{Q}[\mathbf{x}]$ be $\mathcal{E}$-cellular and $P = P^{(b)} + \mathfrak{m}_{\mathcal{E}}$ an associated prime.

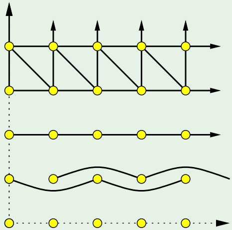
- $Q'' = I + P^{(b)}$ has a unique minimal prime.
- $Q' = Q'' : (\prod_{i \in \mathcal{E}} x_i)^{\infty}$ is cellular.
- Remove embedded components of Q':
  - $Q = Q' + M$ is a primary component for $M = \langle m | (Q' : m) \neq Q' \rangle$.

Example: Minimal primary components

$I = \langle y(x^2-1), y^2(x-1), y^3 \rangle$

- $(I : m)$ is I "shifted down" by m.
- e.g. $(I : y) \ni (x^2-1)$

Associated primes of I: $\begin{array}{c|c} \langle y \rangle & 1 \\ \langle y, x \pm 1 \rangle & y \\ \langle y, x-1 \rangle & y^2 \end{array}$



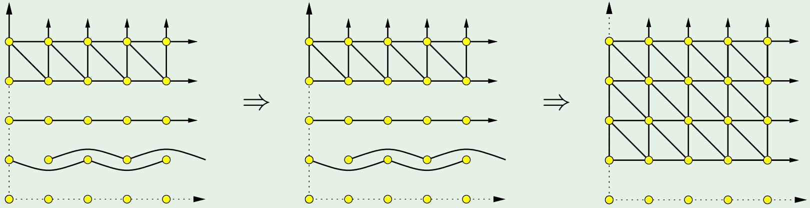
- Compute primary components over $\langle y \rangle$ and $\langle y, x-1 \rangle$

Example: Minimal primary components

Minimal primary component

- $Q'' = I + P^{(b)}$ has a unique minimal prime.
- $Q = Q' + M$ is primary when $M = \langle m | (Q' : m) \neq Q' \rangle$.

Component over $\langle y \rangle$



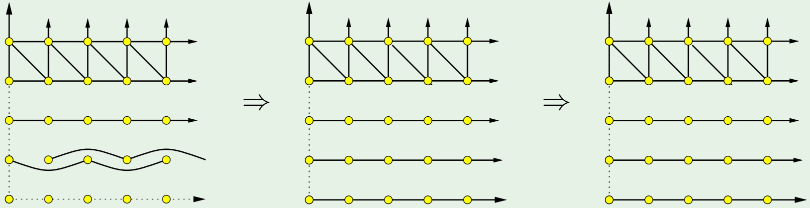
$$\langle y(x^2 - 1), y^2(x - 1), y^3 \rangle + \langle 0 \rangle + \langle y \rangle \Rightarrow \langle y \rangle$$

Example: Minimal primary components

Minimal primary component

- $Q'' = I + P^{(b)}$ has a unique minimal prime.
- $Q = Q' + M$ is primary when $M = \langle m | (Q' : m) \neq Q' \rangle$.

Component over $\langle y, x-1 \rangle$



$$\langle y(x^2-1), y^2(x-1), y^3 \rangle + \langle x-1 \rangle + \langle 0 \rangle \Rightarrow \langle y^3, x-1 \rangle$$

Overview of functions

Functions in Binomials.m2

- binomialPrimaryDecomposition
- binomialCellularDecomposition
- binomialRadical
- binomialMinimalPrimes
- binomialAssociatedPrimes
- binomialSolve (symbolic solver for zero-diml. ideals)
- binomialIsPrime
- binomialIsPrimary
- ...
- latticeBasisIdeal
- randomBinomialIdeal

A large decomposition

In

$$\Bbbk[R_{00}, U_{00}, R_{01}, D_{01}, U_{01}, R_{02}, D_{02}, U_{02}, R_{03}, D_{03}, R_{10}, L_{10}, U_{10}, R_{11}, L_{11}, D_{11}, U_{11}, R_{12}, L_{12}, D_{12}, U_{12}, R_{13}, L_{13}, D_{13}, L_{20}, U_{20}, L_{21}, D_{21}, U_{21}, L_{22}, D_{22}, U_{22}, L_{23}, D_{23}],$$

the ideal

$$\begin{aligned} I := \langle U_{00}R_{01} - R_{00}U_{10}, \quad & R_{01}D_{11} - D_{01}R_{00}, \quad D_{11}L_{10} - L_{11}D_{01}, \\ L_{10}U_{00} - U_{10}L_{11}, \quad & U_{01}R_{02} - R_{01}U_{11}, \quad R_{02}D_{12} - D_{02}R_{01}, \\ D_{12}L_{11} - L_{12}D_{02}, \quad & L_{11}U_{01} - U_{11}L_{12}, \quad U_{02}R_{03} - R_{02}U_{12}, \\ R_{03}D_{13} - D_{03}R_{02}, \quad & D_{13}L_{12} - L_{13}D_{03}, \quad L_{12}U_{02} - U_{12}L_{13}, \\ U_{10}R_{11} - R_{10}U_{20}, \quad & R_{11}D_{21} - D_{11}R_{10}, \quad D_{21}L_{20} - L_{21}D_{11}, \\ L_{20}U_{10} - U_{20}L_{21}, \quad & U_{11}R_{12} - R_{11}U_{21}, \quad R_{12}D_{22} - D_{12}R_{11}, \\ D_{22}L_{21} - L_{22}D_{12}, \quad & L_{21}U_{11} - U_{21}L_{22}, \quad U_{12}R_{13} - R_{12}U_{22}, \\ R_{13}D_{23} - D_{13}R_{12}, \quad & D_{23}L_{22} - L_{23}D_{13}, \quad L_{22}U_{12} - U_{22}L_{23}\rangle \end{aligned}$$

is not radical, but the intersection of 2638 primary components, 10 of which are not prime.

(Challenge from Evans/Sturmfels/Uhler)

Getting Binomials.m2

Download

- Comes with Macaulay 2 version ≥ 1.4
- or `http://www.thomas-kahle.de/bpd`

Please try *Binomials.m2* ...

... or send me your challenges.

Thanks!