Flat deformation of points

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Exploit the notion of stability in algebra

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1. Border bases in 0-dim rings
2. Flatness criterion for deformation of finite set of points.
3. A Newton-type algorithm

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Border bases in 0-dim rings $\mathcal{A} = K[x_1, ..., x_n]/I$

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In this case, we look for

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From these matrices you can get the multiplicty of the roots and its numerical approximation by computing eigenvalues (Auzinger-Stetter)

Border relations of zero-dim algebras

- Assume $\mathcal{A}$ is a quotient algebra with basis a set B of *monomials* in $x = (x_1, \ldots, x_n)$, of cardinal μ. We identify B with a set of $\mathbb{N}^n$. We assume it is stable by division, or connected to 1 ($x^\alpha \in B$ hence, there is $i : x^\alpha / x_i \in B$

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- Denote $B^+ = x_1 B \cup \cdots \cup x_n B \cup B$ and $\partial B = B^+ - B$.
- For any $\alpha \in \partial B$, the monomial x^α is a linear combination in $\mathcal{A}$ of the monomials of B: there exists $z_{\alpha,\beta} \in \mathbb{K}$ ($\beta \in B$) $h^{\mathbf{z}}_\alpha(x) := x^\alpha - \sum_{\beta \in B} z_{\alpha,\beta} x^\beta \equiv 0$ in $\mathcal{A}$. The equations $h^{\mathbf{z}}_\alpha(x)$ are called **border relations** of B in $\mathcal{A}$.

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- *Aplications:* **Contructive approach of Hilbert Scheme of points**: Huibregste '06 , Laksov et alt. 2005, **Stability**: Kreuzer-Robbiano, and Mourrain-Trebuchet (2008), **Equations of degree two for the Hilbert Scheme of points in $\mathbb{P}^n$**: A-B-M, MEGA 2009.

Normal form

- Given the border relations, we define a **normal formal**

$N^z : \langle B^+ \rangle \to \langle B \rangle$ by:

- $N^z(x^\beta) = x^\beta$ if $x^\beta \in B$,
- $N^z(x^\alpha) = x^\alpha - h^z_\alpha(x) = \sum_{\beta \in B} z_{\alpha,\beta}\, x^\beta$ if $\alpha \in \partial B$.

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- The **multiplication matrices** $M^z_{x_i} : \langle B \rangle \to \langle B \rangle$:

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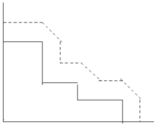
- The coefficients of the matrix of $M^{\mathbf{z}}_{x_i}$ in the basis B are linear in the parameters $\mathbf{z}$.

▶ More generally, a monomial m can be reduced modulo polynomials of the form $(h^{\mathbf{z}}_{\alpha}(x))_{\alpha\in\partial B}$ to a linear combination of monomials in B, as follows: decompose $m = x_{i_1}\cdots x_{i_l}$ and compute $N^z(m) = M^{\mathbf{z}}_{i_1}\circ\cdots\circ M^{\mathbf{z}}_{i_l}(1)$. We easily check that $m - N^z(m) \in (h^{\mathbf{z}}_{\alpha}(x))_{\alpha\in\partial B}$.



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A quotient algebra $\mathcal{A}$ with basis B is entirely described by the coefficients $\mathbf{z} = (\mathbf{z}_{\alpha,\beta}) \in \bar{\mathbb{K}}^{\partial B \times B}$ of the border relations. So, given equations

$$h_\alpha^{\mathbf{z}}(\mathbf{x}) := \mathbf{x}^\alpha - \sum_{\beta \in B} \mathbf{z}_{\alpha,\beta}\, \mathbf{x}^\beta, \quad \alpha \in \partial B.$$

. Let $\mathcal{A}^{\mathbf{z}}$ the corresponding quotient ring.

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Theorem

The polynomials $h^{\mathbf{z}}_\alpha(\mathbf{x})$ are the border relations of a quotient algebra $\mathcal{A}^{\mathbf{z}}$ iff $M^{\mathbf{z}}_{x_i} \circ M^{\mathbf{z}}_{x_j} - M^{\mathbf{z}}_{x_j} \circ M^{\mathbf{z}}_{x_i} = 0 \ {}_{(1 \leqslant i < j \leqslant n)}$.

$$\mathcal{H}_B := \{\mathbf{z} = (z_{\alpha,\beta}) \in \mathbb{K}^{\partial B \times B}; M^{\mathbf{z}}_{x_i} \circ M^{\mathbf{z}}_{x_j} - M^{\mathbf{z}}_{x_j} \circ M^{\mathbf{z}}_{x_i} = 0 \ {}_{1 \leqslant i < j \leqslant n}\}$$

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Hilbert scheme of μ points in $\mathbb{P}^n$:= variety parametrizating the set of homogeneous saturated ideals I which defines a quotient $\mathbb{K}[x_0, \ldots, x_n]/I$ with constant Hilbert function μ.

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GOAL: *An effective criteria (in many cases) for a flat deformation, in terms of the deformed equations and the border basis of the initial equations.*

Perturbed systems

- Given a pertubed system $\mathbf{f}^{\varepsilon} = \mathbf{f}^{0} + \varepsilon\, \mathbf{f}^{1} + \cdots$. Consider $\mathcal{A}^{\varepsilon} := R^{\varepsilon}/\mathsf{I}^{\varepsilon}$ Where $R^{\varepsilon} = \mathbb{K}[[\varepsilon]][\mathbf{x}]$ and, $(\mathbf{f}^{\varepsilon}) = \mathsf{I}^{\varepsilon}$ with I^{0} describing the initial finite zero-set.

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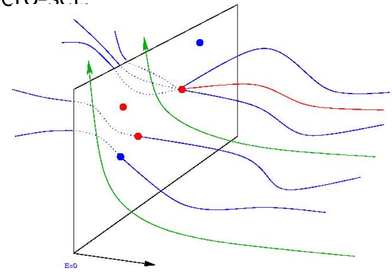

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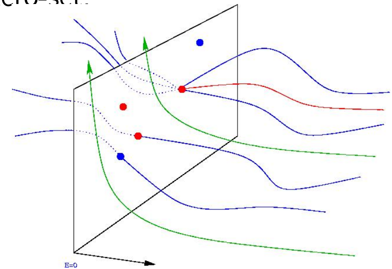

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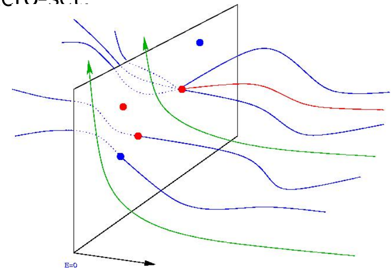

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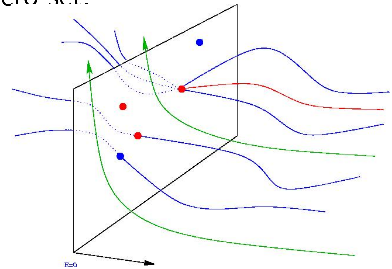

- **Obstructions for flatness of $\mathbb{K}[[\varepsilon]] \rightarrow \mathcal{A}^{\varepsilon}$:** isolated, embedded points, "points going to infinite"
- $\mathrm{I}^{\varepsilon} = \mathrm{I}_{\mathrm{a}}^{\varepsilon} \cap \mathrm{I}_{\infty}^{\varepsilon}$, primmary components that do not intersect (resp. do intersect $\varepsilon = 0$)
- flatness $\Leftrightarrow$ the monomial basis B is still a basis of $\mathcal{A}^{\varepsilon}$ as $\mathbb{K}[[\varepsilon]]$ module

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The zero-set $\mathcal{Z}(\mathsf{I}^{\varepsilon}_{\mathsf{a}})$ "represents" the zero-subset of the deformation close to the initial zero-set of I^{0}.

Effective criterion for flatness

Proposition

Given a pertubed system $f_k^\varepsilon = f_k^0 + \varepsilon f_k^1 + \cdots \in R^\varepsilon$, $(k = 1 \ldots m)$, *s.t.* $(h_\alpha^0)_{\alpha \in \partial B}$ *are* **the border relations** *of* $\mathcal{A}^0$ *for* B.

1. *One can compute a rewritting family* $\mathbf{h}^\varepsilon := (h_\alpha^\varepsilon)_{\alpha \in \partial B} \subset \mathbf{I}_\mathbf{a}^\varepsilon$, *of the form* $h_\alpha^\varepsilon = h_\alpha^0 + \varepsilon \sum_{\beta \in B} u_{\alpha,\beta}(\varepsilon) \mathbf{x}^\beta$ *with* $u_{\alpha,\beta}(\varepsilon) \in \mathbb{K}[[\varepsilon]]$ *(algebraic) analytic in* ε.
2. *One can decide if* $\mathcal{A}_\mathbf{a}^\varepsilon$ *is a flat* $\mathbb{K}[[\varepsilon]]$*- module, and if yes, the formulas of 1) give the border relations of* $\mathcal{A}_\mathbf{a}^\varepsilon$.

Proof of 1.

- **Formal border relations**

$$h_\alpha^{\varepsilon,\mathbf{v}} = h_\alpha^0 + \varepsilon \sum_{\beta \in B} v_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta$$

for $(\alpha \in \partial B)$, where $v_{\alpha,\beta}$ are **unknowns**
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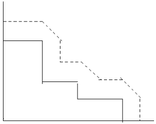
- **Lifting the border relations**: for every $\alpha \in \partial B$,

$$\tilde{h}_\alpha^\varepsilon := \sum_j p_{\alpha,j}^0 f_j^\varepsilon = \sum_j p_{\alpha,j}^0 (f_j^0 + \varepsilon f_j^1 + \cdots) = h_\alpha^0 + \varepsilon \sum_j p_{\alpha,j}^0 f_j^1 + \cdots$$

▶ Make the obvious reduction of the polynomial $\tilde{h}_{\alpha}^{\varepsilon}$ (as polynomial in the $x's$), wrt. the *formal border basis*. Starting with the monomials closer to the boundary ∂B, and iterate the reduction by h_{α}^{ε} until all monomials of the reduced polynomial are in B.



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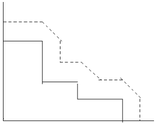

We have

$$\tilde{h}^{\varepsilon}_{\alpha} - h^{\varepsilon,\mathbf{v}}_{\alpha} = -\varepsilon\,\mathbf{v}_{\alpha}(\underline{\mathbf{x}}) + \varepsilon \sum_j p^0_{\alpha j} f^1_j + \cdots$$

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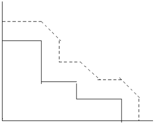

We have

$$\tilde{h}_\alpha^\varepsilon - h_\alpha^{\varepsilon,\mathbf{v}} = -\varepsilon\, \mathbf{v}_\alpha(\underline{\mathbf{x}}) + \varepsilon \sum_j p_{\alpha j}^0 f_j^1 + \cdots$$

N.B. $-\varepsilon \mathbf{v}_\alpha(\underline{\mathbf{x}}) = -\varepsilon \sum_{\beta\in B} v_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta$ **has its support in** B **and will not be reduced by the border relations** $\{{h_\alpha}^{\varepsilon,\mathbf{v}} : \alpha \in \partial B\}$

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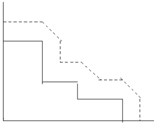

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- Hence we get

$$\tilde{h}^{\varepsilon}_{\alpha} - h^{\varepsilon,\mathbf{v}}_{\alpha} = -\varepsilon\, \mathbf{v}_{\alpha}(\underline{\mathbf{x}}^{\beta}) + \sum_{\alpha' \in \partial B} Q_{\alpha,\alpha'}(\varepsilon, \mathbf{v}, \underline{\mathbf{x}})\, h^{\varepsilon}_{\alpha'} + \varepsilon \sum_{\beta \in B} R_{\alpha,\beta}(\varepsilon, \mathbf{v})\, \underline{\mathbf{x}}^{\beta}$$

where $Q_{\alpha,\alpha'}$ and $R_{\alpha,\beta}$ are polynomial functions in $\mathbf{v}$ and ε, and degree ≥ 1 in ε.

Setting the remainder equals to zero

- By the **IFT**, there is an analytic solution $\mathbf{u}$, to the equation in $\mathbf{v}$:

$$\mathbf{\Phi}_{\alpha,\beta}(\varepsilon, \mathbf{v}) := \mathbf{v}_{\alpha,\beta} - \mathbf{R}_{\alpha,\beta}(\varepsilon, \mathbf{v}) = \mathbf{0}, \alpha \in \partial \mathbf{B}, \beta \in \mathbf{B}; [*] \qquad (1)$$

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There exist $u_{\alpha,\beta} \in \mathbb{K}[[\varepsilon]]$, analytic in ε, such that $h_\alpha^\varepsilon := h_\alpha^0 + \varepsilon \sum_{\beta\in B} u_{\alpha,\beta}\mathbf{x}^\beta$, allow to reduce $\tilde{h}_\alpha^\varepsilon$ to 0

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- Hence, the ideals $\mathrm{H}^\varepsilon := (\mathrm{h}_\alpha^\varepsilon)$, and $\widetilde{\mathrm{H}^\varepsilon} := (\tilde{h}_\alpha^\varepsilon)$, verify $S^{-1}\widetilde{\mathrm{H}}^\varepsilon = S^{-1}\mathrm{H}^\varepsilon$

Proof of **2.**

Let B be a basis of $\mathcal{A}^0 = R/(f_k^0)$, and $(h_\alpha^0)_{\alpha\in\partial B}$ a border basis of B for (f_k^0) and $f_k^\varepsilon \in \mathbb{K}[\mathbf{x}, \varepsilon]$ a deformation of f_k^0

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- **Reduce the f_k^ε with the formal border relations h_α^ε**, obtaining $f_i^\varepsilon = \sum_{\beta\in B} \rho_{i,\beta}(v)x^\beta + < h_\beta >$ **[**]**

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Claim

Then $\mathcal{A}_a^\varepsilon = R^\varepsilon / \mathrm{I}_a^\varepsilon$ is a flat $\mathbb{K}[[\varepsilon]]$-module with basis B iff

- *the coefficients of the commutation expression with the formal matrices: $M_{x_i}^{\mathbf{v}} \circ M_{x_j}^{\mathbf{v}} - M_{x_j}^{\mathbf{v}} \circ M_{x_i}^{\mathbf{v}}$, $1 \leqslant i < j \leqslant n$, and*
- *the polynomials $\rho_{i,\beta}(\mathbf{v})$, $\beta \in B, i = 1, \ldots, k$ in [**],*

reduces to 0 by the relations $\Phi_{\alpha,\beta}(\varepsilon, \mathbf{v})$ ([*]), in the local ring $\mathbb{K}[\varepsilon, \mathbf{v}]_{(\varepsilon, \mathbf{v})}$.

Examples

The two above conditions are independent

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Example 1. The perturbed system $f_1^\varepsilon = x^2 - \varepsilon x$, $f_2^\varepsilon = xy - \varepsilon x$, $f_3^\varepsilon = xy - \varepsilon y$, $f_4^\varepsilon = y^2 - \varepsilon y$, $f_5^\varepsilon = \varepsilon x - \varepsilon^2$, $f_6^\varepsilon = \varepsilon y - \varepsilon^2$.

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- $\mathrm{I}^\varepsilon = (\mathrm{x}, \mathrm{y}, \varepsilon) \cdot (\mathrm{x} - \varepsilon, \mathrm{y} - \varepsilon) = \mathrm{I}_\mathrm{a}^\varepsilon$, and $\mathrm{I}^0 = (\mathrm{x}^2, \mathrm{xy}, \mathrm{y}^2)$

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- The set $B = \{1, x, y\}$ is a basis of R/I^0, and the border relations are $h_{x^2}^0 = x^2$, $h_{xy}^0 = xy$, $h_{y^2}^0 = y^2$. As $h_{x^2}^0 = f_1^0$, $h_{xy}^0 = f_2^0$, $h_{y^2}^0 = f_4^0$, these border relations lift to

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$$\tilde{h}^\varepsilon_{x^2} = f_1^\varepsilon = x^2 - \varepsilon x,\ \tilde{h}^\varepsilon_{xy} = f_2^\varepsilon = xy - \varepsilon x,\ \tilde{h}^\varepsilon_{y^2} = f_4^\varepsilon = y^2 - \varepsilon y$$

After reduction by the formal border relations and resolution of the corresponding (linear) system, we have
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- **The [******] equations do not reduce to zero** : $f^{\varepsilon}_3, f^{\varepsilon}_5, f^{\varepsilon}_6$ do not reduce to zero, so they are not border basis for $\mathcal{A}_{\varepsilon} = R^{\varepsilon}/\mathrm{I}^{\varepsilon}_{\mathrm{a}}$

Example 2. We consider almost the same deformation, but we rewritte $x^2 = f_1^0 + f_3^0 - f_2^0$ and $y^2 = f_4^0 + f_2^0 - f_3^0$ which lift in $\widetilde{h}^{\varepsilon}_{x^2} = f_1^{\varepsilon} + f_3^{\varepsilon} - f_2^{\varepsilon}$ and $\widetilde{h}^{\varepsilon}_{y^2} = f_4^{\varepsilon} + f_2^{\varepsilon} - f_3^{\varepsilon}$. The other relation h^0_{xy} is lift as in the previous example. After reduction by the formal border relations and resolution of the corresponding (linear) system, we obtain the following rewritting relations:

- $h^{\varepsilon}_{x^2} = x^2 - \varepsilon y$,
- $h^{\varepsilon}_{xy} = xy - \varepsilon x$,
- $h^{\varepsilon}_{y^2} = y^2 - \varepsilon x$.

We easily check that the matrix of multiplication M_x, M_y are not commuting, since $M_x \circ M_y(x) = \varepsilon^2 y$ and $M_y \circ M_x(x) = \varepsilon^2 x$. The polynomials $h^{\varepsilon}_{x^2}, h^{\varepsilon}_{xy}, h^{\varepsilon}_{y^2}$ **are not a border basis for** B.

Tangent space to a deformation

- *Assume now we have a perturbation that is known to be flat We describe a Newton-like iteration which allows us to moved tangentially to the Hilbert scheme and to improve the numerical quality of our quotient representation.*

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- Operator *multiplication by x_i* : $M_{x_i}^\varepsilon$ decomposes:

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The commutation implies

$$M_{x_i}^\varepsilon \circ M_{x_j}^\varepsilon - M_{x_j}^\varepsilon \circ M_{x_i}^\varepsilon = (M_{x_i}^0 \circ M_{x_j}^0 - M_{x_j}^0 \circ M_{x_i}^0) +$$

$$+\varepsilon(M_{x_i}^1 \circ M_{x_j}^0 + M_{x_i}^0 \circ M_{x_j}^1 - M_{x_j}^1 \circ M_{x_i}^0 - M_{x_j}^0 \circ M_{x_i}^1) + \mathcal{O}(\varepsilon^2)$$

$$= \varepsilon(M_{x_i}^1 \circ M_{x_j}^0 + M_{x_i}^0 \circ M_{x_j}^1 - M_{x_j}^1 \circ M_{x_i}^0 - M_{x_j}^0 \circ M_{x_i}^1) + \mathcal{O}(\varepsilon^2)$$

- We deduce the linear equations in $\mathbf{h}^1$

$$M_{x_i}^1 \circ M_{x_j}^0 + M_{x_i}^0 \circ M_{x_j}^1 - M_{x_j}^1 \circ M_{x_i}^0 - M_{x_j}^0 \circ M_{x_i}^1 = 0(1 \leqslant i < j \leqslant n)[***]$$

The above is the equation of the Tangent space T_{I_0} to the variety $\mathcal{H}_{\mathcal{B}}$ at the point I_0 whose border relations are $(h_\alpha^0)_\alpha$

Iterative method

Theorem

Assume $\mathcal{A}^0 = R/(f_0^0, \dots, f_m^0)$ *admits* B *as a basis. Let* $f_k^\varepsilon = f_k^0 + \varepsilon f_k^1 + \cdots$ $(k = 1 \dots m)$ *be a flat deformation* $\mathcal{A}^\varepsilon = R^\varepsilon/(f_k^\varepsilon)$ *of* $\mathcal{A}^0$. *Then the border relations of* $\mathcal{A}^\varepsilon$ *are of the form* $h_\alpha^\varepsilon = h_\alpha^0 + \varepsilon h_\alpha^1 \sum_{\beta \in B} h_{\alpha,\beta}^1 \mathbf{x}^\beta + \mathcal{O}(\varepsilon^2)$ *where* $\mathbf{h}^0 = (h_\alpha^0)$ *are the border relations of* $\mathcal{A}^0$ *for* B, $h_\alpha^1 := \sum_{\beta \in B} h_{\alpha,\beta}^1 \mathbf{x}^\beta$ *and* $\mathbf{h}^1 = (h_{\alpha,\beta}^1)_{\alpha \in \partial B, \beta \in B}$ *is the unique solution of the linear equations below (*** and ****)*

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- Let $f_k^0 = \sum q_{k,\alpha}^0 h_\alpha^0$
- From the hypothesis we have $\mathrm{I}^\varepsilon = (f_k^\varepsilon)$ reduce to zero with $\mathrm{H}^\varepsilon = (h_\alpha^\varepsilon)$:

$$f_k^0 + \varepsilon f_k^1 + \mathcal{O}(\varepsilon^2) = \sum_{\alpha\in\partial B} q_k^\varepsilon h_\alpha^\varepsilon = \sum_{\alpha\in\partial B} q_k^0 h_\alpha^0 + \varepsilon(\sum_{\alpha\in\partial B} q_{k,\alpha}^1 h_\alpha^0 + q_{k,\alpha}^0 h_\alpha^1) + \mathcal{O}(\varepsilon^2)$$

- Taking normalforms ("N") wrt $(h_\alpha^0 : \alpha)$:

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$$N^0(f_k^1) - \sum N^0(q_{k,\alpha}^0, h_\alpha^1) = 0[****]$$

Procedure for following a path: Predictor step

Given $f_0^t, \ldots, f_k^t \in R[t]$ such that $\mathfrak{I}^t := (\mathbf{f}^t) = (f_0^t, \ldots, f_k^t)$ stays on $\mathbf{Hilb}^{\mu}(\mathbb{P}^n)$. **Predictor step: moving in the tangent direction,** as follows:

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- Compute the border basis $\mathbf{h}^0$ of $\mathbf{f}^{t_0}$ for a set B connected to 1 and let N^0 be the projection on $\langle B \rangle$ along $\mathfrak{I}^{t_0}$,

Procedure for following a path: Predictor step

Given $f_0^t, \ldots, f_k^t \in R[t]$ such that $\mathfrak{I}^t := (\mathbf{f}^t) = (f_0^t, \ldots, f_k^t)$ stays on $\mathbf{Hilb}^{\mu}(\mathbb{P}^n)$. **Predictor step: moving in the tangent direction,** as follows:

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- Compute the linear equations in $\mathbf{h}^1$:

$$N^0(f_k^1) - \sum_{\alpha\in\partial B} N^0(q^0_{k,\alpha} h^1_\alpha) = 0,$$

obtained by reduction with respect to the border basis $\mathbf{h}^0$;

- For each $\beta \in B, 1 \leqslant i < j \leqslant n$, compute the linear equations in $\mathbf{h}^1$

$$N^1(x_i\, N^0(x_j\, \underline{\mathbf{x}}^{\beta})) + N^0(x_i\, N^1(x_j\, \underline{\mathbf{x}}^{\beta}))$$
$$-N^1(x_j\, N^0(x_i\, \underline{\mathbf{x}}^{\beta})) - N^0(x_j\, N^1(x_i\, \underline{\mathbf{x}}^{\beta})) = 0$$

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- Solve all these linear equations in $\mathbf{h}^1$ and replace $\mathbf{h}^0$ by $\tilde{\mathbf{h}} := \mathbf{h}^0 + s\,\mathbf{h}^1$ where **s is the step size from t_0 to t_1.**

Corrector step

Projecting back onto the variety $H^B_{x_0}$:

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Projecting back onto the variety $H^B_{x_0}$:

- Compute the projection operator $\tilde{N} : \langle B^+ \rangle \to \langle B \rangle$ associated to the border prebasis $\tilde{\mathbf{h}} = \mathbf{h}^0 + s\,\mathbf{h}^1$ such that $\tilde{N}(\underline{\mathbf{x}}^\alpha) = -h_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $\tilde{N}(\underline{\mathbf{x}}^\beta) = \underline{\mathbf{x}}^\beta$ for $\beta \in B$.

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- Take **variables** $\mathbf{h}^1 = (h^1_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B}$ and define the **formal polynomials** $h^1_\alpha(\underline{\mathbf{x}}) := \sum_{\beta\in B} h^1_{\alpha,\beta}\,\underline{\mathbf{x}}^\beta$ for $\alpha \in \partial B$ and $N^1 : \langle B^+ \rangle \to \langle B \rangle$ such that $N^1(\underline{\mathbf{x}}^\alpha) = h^1_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $N^1(\underline{\mathbf{x}}^\beta) = 0$ for $\beta \in B$.

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- Take **variables** $\mathbf{h}^1 = (h^1_{\alpha,\beta})_{\alpha \in \partial B, \beta \in B}$ and define the **formal polynomials** $h^1_\alpha(\underline{\mathbf{x}}) := \sum_{\beta \in B} h^1_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta$ for $\alpha \in \partial B$ and $N^1 : \langle B^+ \rangle \rightarrow \langle B \rangle$ such that $N^1(\underline{\mathbf{x}}^\alpha) = h^1_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $N^1(\underline{\mathbf{x}}^\beta) = 0$ for $\beta \in B$.
- For each $\beta \in B, 1 \leqslant i < j \leqslant n$, compute the linear equations in $\mathbf{h}^1$

Corrector step

Projecting back onto the variety $H_{x_0}^B$:

- Compute the projection operator $\tilde{N} : \langle B^+ \rangle \to \langle B \rangle$ associated to the border prebasis $\tilde{\mathbf{h}} = \mathbf{h}^0 + s\, \mathbf{h}^1$ such that $\tilde{N}(\underline{\mathbf{x}}^\alpha) = -h_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $\tilde{N}(\underline{\mathbf{x}}^\beta) = \underline{\mathbf{x}}^\beta$ for $\beta \in B$.
- Take **variables** $\mathbf{h}^1 = (h^1_{\alpha,\beta})_{\alpha \in \partial B, \beta \in B}$ and define the **formal polynomials** $h^1_\alpha(\underline{\mathbf{x}}) := \sum_{\beta \in B} h^1_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta$ for $\alpha \in \partial B$ and $N^1 : \langle B^+ \rangle \to \langle B \rangle$ such that $N^1(\underline{\mathbf{x}}^\alpha) = h^1_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $N^1(\underline{\mathbf{x}}^\beta) = 0$ for $\beta \in B$.
- For each $\beta \in B, 1 \leqslant i < j \leqslant n$, compute the linear equations in $\mathbf{h}^1$ **Trying to force the commutativity relations for**

Corrector step

Projecting back onto the variety $H^B_{x_0}$:

- Compute the projection operator $\tilde{N} : \langle B^+\rangle \to \langle B\rangle$ associated to the border prebasis $\tilde{\mathbf{h}} = \mathbf{h}^0 + s\,\mathbf{h}^1$ such that $\tilde{N}(\underline{\mathbf{x}}^\alpha) = -h_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $\tilde{N}(\underline{\mathbf{x}}^\beta) = \underline{\mathbf{x}}^\beta$ for $\beta \in B$.
- Take **variables** $\mathbf{h}^1 = (h^1_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B}$ and define the **formal polynomials** $h^1_\alpha(\underline{\mathbf{x}}) := \sum_{\beta\in B} h^1_{\alpha,\beta}\,\underline{\mathbf{x}}^\beta$ for $\alpha \in \partial B$ and $N^1 : \langle B^+\rangle \to \langle B\rangle$ such that $N^1(\underline{\mathbf{x}}^\alpha) = h^1_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $N^1(\underline{\mathbf{x}}^\beta) = 0$ for $\beta \in B$.
- For each $\beta \in B, 1 \leqslant i < j \leqslant n$, compute the linear equations in $\mathbf{h}^1$

$$\begin{aligned} & N^1(x_i\,\tilde{N}(x_j\,\underline{\mathbf{x}}^\beta)) + \tilde{N}(x_i\,N^1(x_j\,\underline{\mathbf{x}}^\beta)) \\ & -N^1(x_j\,\tilde{N}(x_i\,\underline{\mathbf{x}}^\beta)) - \tilde{N}(x_j\,N^1(x_i\,\underline{\mathbf{x}}^\beta)) \\ & =: -\tilde{N}(x_i\,\tilde{N}(x_j\,\underline{\mathbf{x}}^\beta)) + \tilde{N}(x_j\,\tilde{N}(x_i\,\underline{\mathbf{x}}^\beta)) \end{aligned}$$

Corrector step

Projecting back onto the variety $H_{x_0}^B$:

- Compute the projection operator $\tilde{N} : \langle B^+ \rangle \to \langle B \rangle$ associated to the border prebasis $\tilde{\mathbf{h}} = \mathbf{h}^0 + s\,\mathbf{h}^1$ such that $\tilde{N}(\underline{\mathbf{x}}^\alpha) = -h_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $\tilde{N}(\underline{\mathbf{x}}^\beta) = \underline{\mathbf{x}}^\beta$ for $\beta \in B$.
- Take **variables** $\mathbf{h}^1 = (h^1_{\alpha,\beta})_{\alpha \in \partial B, \beta \in B}$ and define the **formal polynomials** $h^1_\alpha(\underline{\mathbf{x}}) := \sum_{\beta \in B} h^1_{\alpha,\beta}\, \underline{\mathbf{x}}^\beta$ for $\alpha \in \partial B$ and $N^1 : \langle B^+ \rangle \to \langle B \rangle$ such that $N^1(\underline{\mathbf{x}}^\alpha) = h^1_\alpha(\underline{\mathbf{x}})$ for $\alpha \in \partial B$ and $N^1(\underline{\mathbf{x}}^\beta) = 0$ for $\beta \in B$.
- For each $\beta \in B, 1 \leqslant i < j \leqslant n$, compute the linear equations in $\mathbf{h}^1$
- Solve this system in $\mathbf{h}^1$, replace $\tilde{\mathbf{h}}$ by $\tilde{\mathbf{h}} := \tilde{\mathbf{h}} + \mathbf{h}^1$ and iterate this loop until $\mathbf{h}^1$ is small enough.

THANK YOU !