Minkowski Length of 2D and 3D Lattice Polytopes

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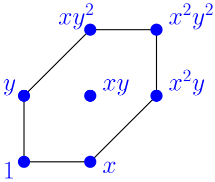
References

1. "Toric surface codes and Minkowski length of polygons" (with I. Soprunov), SIAM J. Discrete Math. 23 (2009), no. 1, 384–400
2. "Minkowski length of 3D lattice polytopes" (with O. Beckwith, M. Grimm, B. Weaver)

Newton Polytope

The Newton Polytope P_f of a polynomial f is the convex hull of the exponent vectors in f.

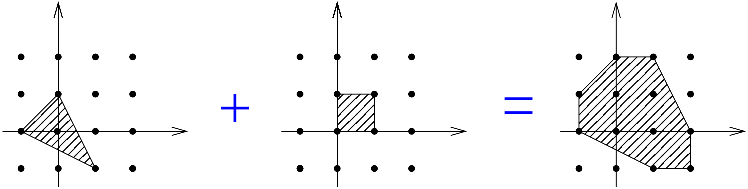
Example: $f = 1 + 2x + 3y + 4x^2y + 5xy^2 + 6x^2y^2$.



Minkowski Sum

P, Q polytopes, their Minkowski sum:

$$P + Q := \{p + q \mid p \in P, q \in Q\}$$



P_{f_1}, P_{f_2}- Newton polytopes of f_1, f_2
If $f = f_1 f_2$, then $P_f = P_{f_1} + P_{f_2}$

Minkowski Length

Let P be a convex lattice polytope

Define the *Minkowski length*

$L(P) :=$ largest number of non-trivial convex lattice polytopes whose Minkowski sum is in P

$=$ largest number of primitive lattice segments whose Minkowski sum is in P.

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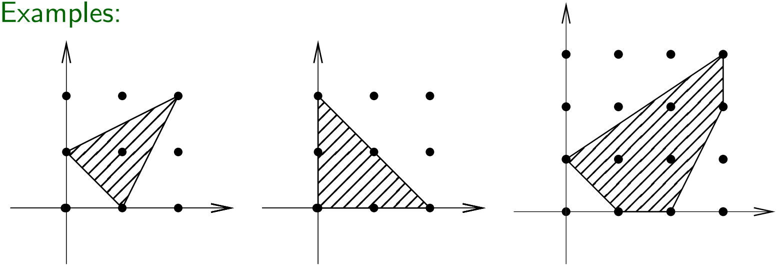

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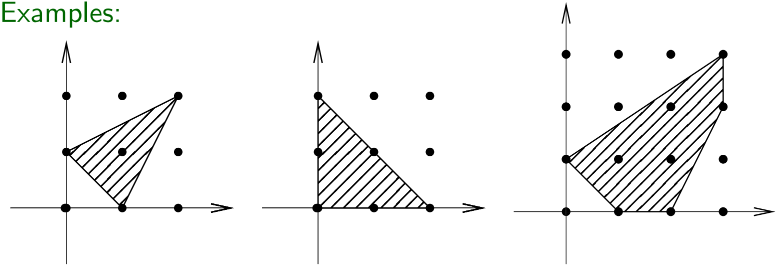

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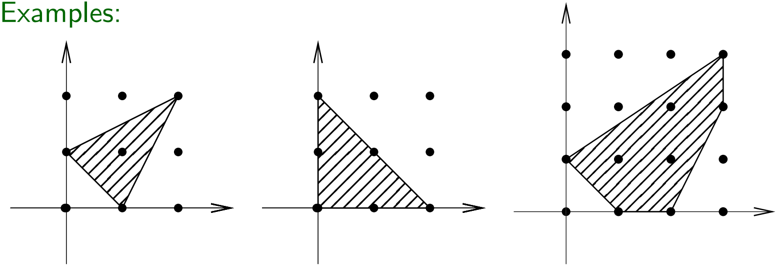

$L(T_0) = 1$ $\quad$ $L(P_1) = 2$

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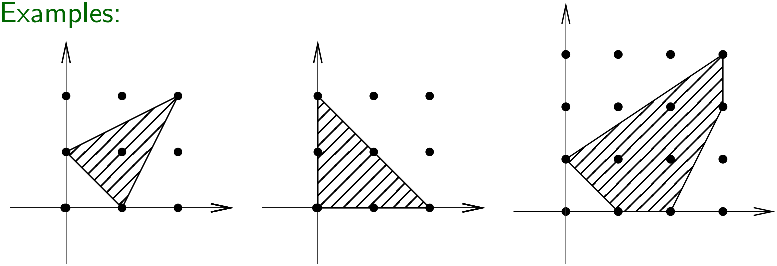
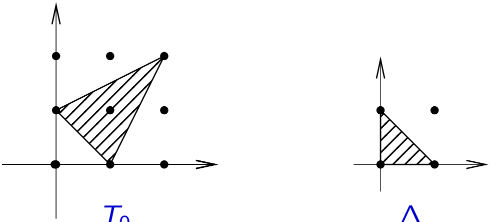


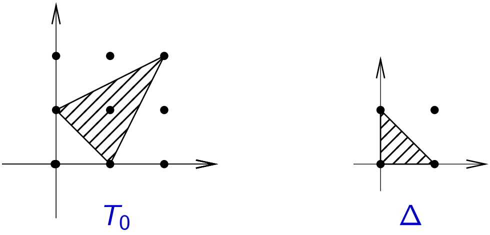
$L(T_0) = 1$ $L(P_1) = 2$ $L(P_2) = 3$

Convex lattice polygons with $L(P) = 1$?



T_0 Δ

Convex lattice polygons with $L(P) = 1$?



Theorem (S-S)

If $L(P) = 1$ then P is $AGL(2, \mathbb{Z})$-equivalent to a primitive segment, a unit triangle Δ, or an exceptional triangle T_0.

Questions

- How to compute $L(P)$?
- $P \supseteq P_1 + \cdots + P_{L(P)}$ - *maximal decomposition*. What do the summands P_i look like?

Motivation

P convex lattice polytope

Problem: Consider $\mathcal{L}_P = \{f \mid P_f \subseteq P\}$, what is the largest number of factors $f \in \mathcal{L}_P$ could have? What do the factors look like?

Little and Schenck: Let $f \in \mathbb{F}_q[x, y]$. If q is large enough, then the more absolutely irreducible factors f has, the more $\mathbb{F}_q$ zeroes it has. (Based on the Hasse-Weil formula.)

This leads to bounds on the maximum number of $\mathbb{F}_q$ zeroes of nonzero polynomials in $\mathcal{L}_P$ and on the minimum distance of toric surface codes that involve $L(P)$ (S-S).

How to find $L(P)$

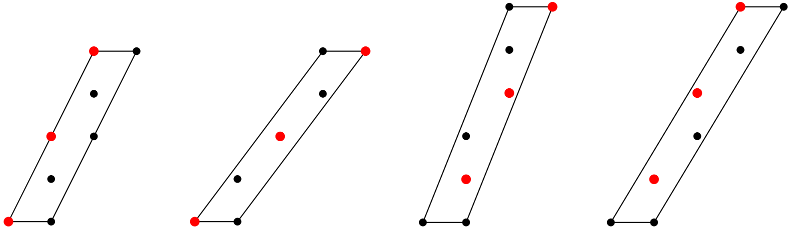
Proposition (B-G-S-W)

Let $P \subseteq \mathbb{R}^2$ be a lattice polygon. Consider a maximal decomposition Z in P of smallest possible area:

$$P \supseteq Z = E_1 + \cdots + E_{L(P)}.$$

Then $\text{Area}(E_i + E_j) \leq 1$ for any two segments E_i, E_j in Z.

Proof: If not, can assume $\text{Area}(E_1 + E_2) \geq 2$;
$v_1 = [1, 0]$, $v_2 = [a, b]$ with $0 \leq a < b$ - vectors along E_1 and E_2.
Then $\text{Area} = b \Rightarrow b \geq 2$.



a, b even a odd, b even a even, b odd a, b odd

One can always pass to a maximal decomposition of smaller area. Hence $\text{Area}(E_i + E_j) \leq 1$ for any two segments E_i, E_j in Z.

How to find $L(P)$ in 2D

Theorem (S-S)

Let $P \subseteq \mathbb{R}^2$ be a lattice polygon. Then there exists a maximal decomposition in P which is $AGL(2, \mathbb{Z})$-equivalent to

$$m_1[0, e_1] + m_2[0, e_2] + m_3[0, e_1 + e_2].$$

Here e_1, e_2 are standard basis vectors and $m_1 + m_2 + m_3 = L(P)$.

This leads to an algorithm of finding $L(P)$, polynomial in the number of lattice points in P.

Let P be a 3D convex lattice polytope.

Proposition (B-G-S-W) Consider a maximal decomposition Z in P of smallest possible volume:

$$P \supseteq Z = E_1 + \cdots + E_{L(P)}.$$

Then $\mathrm{Vol}(E_i + E_j + E_k) \leq 2$ for any segments E_i, E_j, E_k in Z.

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Theorem (B-G-S-W) There exists a maximal decomposition in P which is $AGL(3, \mathbb{Z})$-equivalent to

$$\begin{aligned} & m_1[0, e_1] + m_2[0, e_2] + m_3[0, e_3] + m_4[0, e_1 + e_2 + e_3] \\ + \quad & m_5[0, e_1 \pm e_2] + m_6[0, e_1 + e_3] + m_7[0, e_2 + e_3]. \end{aligned}$$

This leads to an algorithm of finding $L(P)$, polynomial in the number of lattice points in P.

What do summands in a max decomposition look like?

Let $P' = P_1 + \cdots + P_{L(P)} \subseteq P$ be a maximal decomposition. Each P_i is equivalent to a primitive segment, Δ, or T_0. Any other restrictions?

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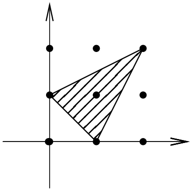
Theorem (S-S)

Let $P \supseteq P' = P_1 + \cdots + P_{L(P)}$ be a maximal decomposition. If P_1 is equivalent to T_0, then P' is equivalent to

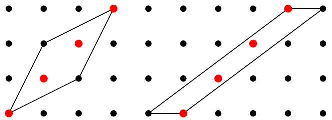
$$T_0 + m_1[0, e_1] + m_2[0, e_2] + m_3[0, e_3].$$



Proof (B-G-S-W): Reduce segments in $\mathbb{Z}^2$ modulo 3. Get four classes: $[1,0],[1,1],[1,2],[0,1] \in \mathbb{Z}_3\mathbb{P}^1$.

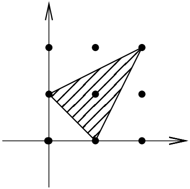
Observation 1: If each of P, Q has a segment from the same class, and those segments are not the same, then $L(P+Q) \geq 3$.

Why:

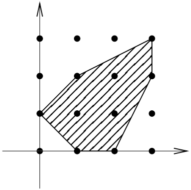


Observation 2: If P has two non-equal segments from the same class, and Q also has a segment from that class, then $L(P+Q) \geq 3$.

Segments in T_0: $[1,2], [2,1], [-1,1]$ - all from class $[1,2]$;
$[1,0], [0,1], [1,1]$ - three distinct classes



Hence the only segments that can occur in a max decomposition together with T_0 are $[1,0], [0,1], [1,1]$. One can form a triangle Δ using these three segments, but $L(T_0 + \Delta) = 3$.



Length 1 lattice polytopes in 3D

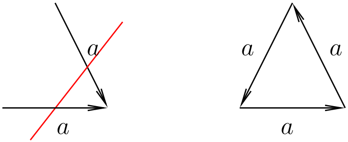
- If $L(P) = 1$, then P has at most 8 lattice points.
- This bound is sharp (Gaudinez, Outing, Vega, MSRI-UP' 09).
- There exists a lattice tetrahedron P with 8 lattice points (B-G-S-W).
- Kasprzyk: Classification of Fano tetrahedra (lattice tetrahedra with one lattice point strictly inside).
- Classification of length 1 3D polytopes? Not feasible.

Mod 3 classification of 3D length 1 polytopes

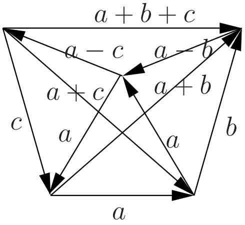
$\mathbb{Z}_3\mathbb{P}^2$ has 13 classes;

$\#P = 5$	$\#P = 6$
4+2+2+2	6+3+3+3
3+3+2+2	4+4+4+3
3+(7)	4+2+2+2+(5)
2+2+(6)	3+3+2+2+(5)
(10)	3+3+(9)
	3+2+2+(8)
	3+(12)
	3+2+2+2+(6)
	2+2+2+2+(7)
	2+2+(11)

How can segments from the same class fit together?



Example: 3+(7)



$P \supseteq P' = P_1 + \cdots + P_{L(P)}$ - max decomposition
$I(P_i) :=$ number of lattice points strictly inside P_i
$I := \sum I(P_i)$ total number of lattice points in all P_i in P'

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Theorem (S-S) If $P \subset \mathbb{R}^2$ then $I \leq 1$.

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Theorem (S-S) If $P \subset \mathbb{R}^2$ then $I \leq 1$.

Theorem (B-G-S-W) If $P \subset \mathbb{R}^3$ then $I \leq 4$.
Proof: mod 3 classification plus Kasprzyk's classification of Fano tetrahedra.